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ON COMPLEX MULTIPLICATIVE DIFFERENTIATION[1]

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Abstract. In the present paper we discuss multiplicative differentiation for complex-valued functions. Some drawbacks, arising with this concept in the real case, are explained satisfactorily. Some new difficulties, coming from the complex nature of variables, are discussed and they are outreached. Multiplicative Cauchy–Riemann conditions are established. Properties of complex multiplicative derivatives are studied.

Keywords: Complex calculus, complex differentiation, multiplicative calculus, Cauchy–Riemann conditions.

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1. Introduction

Two real numbers a and b with $a \leq b$ can be distinguished by the difference $h = b - a$, saying that b is greater than a for h units. In the second half of the 17th century Isaac Newton and Gottfried Wilhelm Leibnitz created differential and integral calculus using this method.

Another way to distinguish a and b, if $0 < a \leq b$, is the ratio $r = b/a$, saying that b is r times as greater as a. This method gave rise to an alternative calculus, called multiplicative calculus, in the work of Michael Grossman and Robert Katz [12]. Further contribution to multiplicative calculus, basically to its applications and popularization, was done in Stanley [21], Bashirov et al. [3, 4, 5] and Riza et al. [17], etc. Independently, elements of stochastic multiplicative calculus are concerned in the works of Karandikar [13] and Daletskii and Teterina [8]. In the works by Volterra and Hostinsky [23], Aniszewska [2], Kasprzak et al. [14], Rybaczuk et al. [18] the term of multiplicative calculus is employed but indeed they concern bigeometric calculus in the Grossman's terminology [11]. The bigeometric calculus is also considered in Córdova-Lepe and Pinto [7] under the name of proportional calculus. The Volterra's product integral from [23] found further development in the book by Slavik [20].

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Multiplicative calculus, as a calculus applicable to positive functions, creates several questions. For example, the multiplicative derivative can also be extended to negative functions while the functions with both positive and negative values do not fit into the scope of multiplicative calculus. This and other questions suggest, that there may be an extension of multiplicative calculus, explaining this issue. Such an extension becomes a development of multiplicative calculus for complex-valued functions of real or complex variable, i.e., creation of complex multiplicative calculus. An attempt to define complex multiplicative derivative (as well as integral) has been done in Uzer [22]. But definition from [22] is not applicable to functions with values on $(-\infty, 0)$ of the complex plane. This is related to multi-valued nature of complex logarithm. In the present paper we handle this issue and define complex multiplicative derivative covering the above kind of functions as well. We interpret complex multiplicative integral geometrically, explaining satisfactorily the obstacles from the real case, and derive Cauchy–Riemann conditions for multiplicative case.

There are a lot of books on complex calculus; Ahlfors [1], Conway [6], Sarason [19], Greene and Krantz [12], Lang [15], Derrick [9] and Palka [16], which are used during this study.

One major remark about the notation is that the multiplicative versions of the concepts of Newtonian calculus will be called as *concepts, for example, a *derivative means a multiplicative derivative. We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. $\operatorname{Arg} z$ is the principal value of $\arg z$, noticing that $-\pi < \operatorname{Arg} z \leq \pi$. Always $\ln x$ refers to the natural logarithm of the real number $x > 0$ whereas $\log z$ to the same of the complex number $z \neq 0$. By $\operatorname{Log} z$ we denote the value at z of the principal branch of the complex logarithmic function, i.e., $\operatorname{Log} z = \ln |z| + i\operatorname{Arg} z$, where i is the imaginary unit and $|z|$ is the modulus of z. Under a function we always mean a single-valued function. The cases of multi-valued functions are pointed out.

Note also that in the existing literature on complex calculus the symbol e^z has two inconsistent usages. Most popularly, e^z denotes the value at z of the complex exponential function, that is a unique solution of $f'(z) = f(z)$ with $f(0) = 1$. This is a single-valued function and, by the Euler's formula, equals to $e^z = e^x(\cos y + i \sin y)$ if $z = x + iy$. The second usage is the raising e to the complex power z, that results multiple values $e^{z \log e}$ or $e^{z(1+2\pi ni)}$ for $n = 0, \pm 1, \pm 2, \dots$, in which the first usage is assumed. Later on, we will deal with raising to complex powers. In order to avoid possible ambiguities, instead of w^z (raising complex w to complex power z) we will prefer to write $e^{z \log w}$, reserving the symbol e^z for the complex exponential function. For raising real x to real power y we will still use the symbol x^y.

2. Motivation

The *derivative at t of a real-valued function f of real variable is the limit

$$f^*(t) = \lim_{h \to 0} (f(t+h)/f(t))^{1/h}, \tag{1}$$

showing how many times the value $f(t)$ changes at t. If f has pure positive values and is differentiable at t, then f^* exists and the relation between f^* and the ordinary derivative

f' is as follows (see, for example, Bashirov et al. [3]):

$$f^*(t) = e^{[\ln f]'(t)} = e^{f'(t)/f(t)}.$$

It can be easily observed that if f has pure negative values and is differentiable at t, then the limit in (1) still exists in the form

$$f^*(t) = e^{[\ln |f|]'(t)} = e^{|f|'(t)/|f|(t)}.$$

In both cases $f^*(t) > 0$ and all higher order *derivatives of f are the first order *derivatives of positive functions. Based on this, in existing literature multiplicative calculus is presented as a calculus for positive functions. This raises the following questions. Why all order *derivatives of a negative function are positive functions? Why *differentiation is not applicable to functions with both positive and negative values? What is the role of zero in *differentiation? These questions from the real case suggest that there may be an extension of *derivative, bringing an explanation to this issue. Indeed, all these questions find satisfactory answers in the complex case.

For motivation, consider a complex-valued function f of real variable on an open interval (a, b). Considering informally the limit in (1) as a *derivative of f at t, we have

$$\begin{aligned} f^*(t) &= \lim_{h\to 0}(f(t+h)/f(t))^{1/h} = \lim_{h\to 0} e^{(1/h)\log(f(t+h)/f(t))} \\ &= e^{\lim_{h\to 0}(1/h)\ln|f(t+h)/f(t)| + i\lim_{h\to 0}(1/h)(\mathrm{Arg}\,(f(t+h)/f(t))+2\pi n)}. \end{aligned}$$

If $f(t) \neq 0$ and $f'(t)$ exists, then there is no problem with the first limit. It is

$$\lim_{h\to 0}\frac{\ln|f(t+h)/f(t)|}{h} = \lim_{h\to 0}\frac{\ln|f(t+h)| - \ln|f(t)|}{h} = [\ln|f|]'(t).$$

Assuming additionally that $\mathrm{Arg}\, f(t) \neq \pi$, we have

$$\mathrm{Arg}\,(f(t+h)/f(t)) = \mathrm{Arg}\, f(t+h) - \mathrm{Arg}\, f(t)$$

for all $|h| < \varepsilon$, where ε is sufficiently small. This implies that

$$\lim_{h\to 0}\frac{\mathrm{Arg}\,(f(t+h)/f(t)) + 2\pi n}{h} = \lim_{h\to 0}\frac{\mathrm{Arg}\, f(t+h) - \mathrm{Arg}\, f(t) + 2\pi n}{h}$$

exists and equals to $[\mathrm{Arg}\, f]'(t)$ if and only if $n = 0$. Therefore, it would be reasonable to understand under $e^{(1/h)\log(f(t+h)/f(t))}$ its principal value.

Now let $\mathrm{Arg}\, f(t) = \pi$. Then for small values of h,

$$\mathrm{Arg}\,(f(t+h)/f(t)) = \begin{cases} \mathrm{Arg}\, f(t+h) - \mathrm{Arg}\, f(t) & \text{if } \mathrm{Arg}\, f(t+h) \geq 0, \\ \mathrm{Arg}\, f(t+h) - \mathrm{Arg}\, f(t) + 2\pi & \text{if } \mathrm{Arg}\, f(t+h) < 0. \end{cases}$$

Therefore, the selection of the principal value of $e^{(1/h)\log(f(t+h)/f(t))}$ does not guarantee the equality

$$\lim_{h\to 0}(1/h)\mathrm{Arg}\,(f(t+h)/f(t)) = [\mathrm{Arg}\, f]'(t)$$

in cases when $\mathrm{Arg}\, f(t) = \pi$. Instead, the issue can be improved if we replace $\mathrm{Arg}\, f$, ranging in the interval $(-\pi, \pi]$, by any other branch Θ of $\arg f$, ranging in $(-\pi + \alpha, \pi + \alpha]$ with $\alpha \in \mathbb{R}$, so that $\Theta(t) \neq \pi + \alpha$. But for $\alpha \neq 0$ the principal value of $e^{(1/h)\log(f(t+h)/f(t))}$ differs from

$$e^{(\ln|f(t+h)| - \ln|f(t)|)/h + i(\Theta(t+h) - \Theta(t))/h},$$

refusing the previous development. This discrepancy can be overcome if we use the limit in (1) only for motivation of complex *derivatives, and define $f^*(t)$ directly as $e^{[\ln|f|]'(t)+i\Theta'(t)}$. Clearly, the value of $\Theta'(t)$ is independent on selection of the branch Θ of $\arg f$ among those which range in $(-\pi+\alpha, \pi+\alpha]$ for $\alpha \in \mathbb{R}$ with $\Theta(t) \neq \pi+\alpha$.

Another issue is whether $[\ln|f|]'(t) + i\Theta'(t) = [\log f]'(t)$. Generally speaking, a branch of $\log f$ may not exist. Even if it exists, it may not be a composition of a branch of log and f. In this regard, no problem arises with *derivative because it is based on the *local behavior* of the function f. For definition of $f^*(t)$, selecting a sufficiently small neighborhood $U \subseteq (a,b)$ of t, we can reach the existence of the branches of $\log f$ locally. Additionally, we can get these branches as a composition of the respective branches of log and f, again locally. Moreover, we can use the log-differentiation formula $[\log f]' = f'/f$ independently on selection of a branch of log if f is nowhere-vanishing and f' exists (see, Sarason [19]).

Based on the above discussion, consider a differentiable nowhere-vanishing complex-valued function f of real variable on an open interval (a,b). Select a small neighborhood $U \subseteq (a,b)$ of $t \in (a,b)$ so that $\log f$ on U has branches in the form of composition of branches of log and f. Let $|f| = R$ and let $\arg f = \Theta + 2\pi n$ on U, where Θ is any branch of $\arg f$ on U, and define $f^*(t)$ as

$$f^*(t) = e^{[\ln|f|]'(t)+i\Theta'(t)},$$

or, by the log-differentiation formula,

$$f^*(t) = e^{f'(t)/f(t)}. \tag{2}$$

Defined in this way, $f^*(t)$ exists as a single complex value independently on whether $\operatorname{Arg} f(t) \neq \pi$ or $\operatorname{Arg} f(t) = \pi$.

An important consequence from (2) is that if f takes values on some ray from the origin on the complex plane, i.e.,

$$f(t) = R(t)e^{i(\theta+2\pi n)}, \ \theta = \text{const.},$$

then

$$f^*(t) = e^{R'^{i(\theta+2\pi n)}/R(t)e^{i(\theta+2\pi n)}} = e^{R'(t)/R(t)} = R^*(t).$$

This explains why all order *derivatives of a negative function are positive. It is because $(-\infty, 0)$ is a particular ray. More generally, the *derivative of a complex-valued function f of real variable, taking all the values on a ray from the origin, is a positive function and $f^*(t)$ measures how many times the distance of $f(t)$ from the origin changes at t.

Another unexpectedly important consequence from (2) is that if f takes values on some circle centered at the origin on the complex plane, i.e.,

$$f(t) = re^{i(\Theta(t)+2\pi n)}, \ r = \text{const.},$$

then

$$f^*(t) = e^{ir\Theta'^{i(\Theta(t)+2\pi n)}/re^{i(\Theta(t)+2\pi n)}} = e^{i\Theta'(t)} = e^{i(\Theta'(t)+2\pi m)},$$

demonstrating that f^* takes values on the unit circle centered at the origin on the complex plane and one of the multiple values of $\arg f^*(t)$, that is $\Theta'(t)$, measures the rate of change of all the branches of $\arg f$ at t. In particular, $f^*(t) = -1$ if $\Theta'(t) = \pi$. This is a case when *derivative is a negative number. Also, $f^*(t) = i$ if $\Theta'(t) = \pi/2$, a case when *derivative belongs to the imaginary axis.

More generally, if

$$f(t) = R(t)e^{i(\Theta(t)+2\pi n)},$$

then

$$\begin{aligned} f^*(t) &= e^{(R'(t)+iR(t)\Theta'^{i(\Theta(t)+2\pi n)}/R(t)e^{i(\Theta(t)+2\pi n)}} \\ &= e^{R'(t)/R(t)}e^{i\Theta'(t)} = R^*(t)e^{i(\Theta'(t)+2\pi m)}, \end{aligned}$$

i.e., the modulus and the argument of $f^*(t)$ behave similar to the above mentioned two particular cases.

The most important conclusion from (2) is that in *differentiation the origin on the complex plane acts differently from the other points. More precisely, instead of the complex plane $\mathbb{C}$, the perforated complex plane $\mathbb{C} \setminus \{0\}$ should be considered. This also explains why *derivative can not be applicable to real-valued functions, taking both positive and negative values. This is because instead of the real line $\mathbb{R}$, the perforated real line $\mathbb{R} \setminus \{0\}$ should be considered. Consequently, this removes the real-valued functions with both positive and negative values from the consideration, since a continuous real-valued function with the range in $\mathbb{R}\setminus\{0\}$ is either positive or negative. The functions with both positive and negative values become considerable if they take complex values as well. This is because bypassing the origin when traveling continuously from positive to negative numbers and vice versa is allowed on the complex plane.

All these demonstrate that it is actual to develop *differentiation for complex-valued functions of complex variable. This is done in the following sections.

3. Complex Multiplicative Differentiation

Now assume that f is a nowhere-vanishing differentiable complex-valued function on an open connected set D of the complex plane. Select a sufficiently small neighborhood $U \subseteq D$ of the point $z \in D$ so that the branches of $\log f$ on U exist, they are compositions of the respective branches of log and the restriction of f to U, and the log-differentiation formula is valid for $\log f$ on U. According to (2), define the *derivative of f at $z \in D$ by

$$f^*(z) = e^{f'(z)/f(z)}. \tag{3}$$

By induction, we also obtain the following formula for higher-order multiplicative derivatives:

$$f^{*(n)}(z) = e^{[f'^{(n-1)}(z)}, n = 1, 2, \ldots. \tag{4}$$

To derive the Cauchy–Riemann conditions in *form, let $z = x + iy = re^{i\theta}$ and $f(z) = u(z) + iv(z) = R(z)e^{i(\Theta(z)+2\pi n)}$ for $z \in U$, where Θ is any suitable branch of $\arg f$. Since f is differentiable, all the functions u, v, R and Θ have continuous partial derivatives in x, y, r and θ.

Proposition 3.1. *Under the above conditions and notation,*

$$R^*_x(z) = \left[e^{\Theta}\right]^*_y(z) \quad \textit{and} \quad R^*_y(z) = \left[e^{-\Theta}\right]^*_x(z), \tag{5}$$

*where g^*_x and g^*_y refer to the partial *derivatives in x and in y, respectively, of the positive function g.*

Proof. Indeed, it should be proved that

$$[\ln R]'_x(z) = \Theta'_y(z) \quad \text{and} \quad [\ln R]'_y(z) = -\Theta'_x(z). \tag{6}$$

The proofs of these equalities are similar. Therefore, we prove one of them, say, the second one. At first, let $u(z) \neq 0$. From the Cauchy–Riemann conditions $u'_x = v'_y$ and $u'_y = -v'_x$,

$$\begin{aligned}
[\ln R]'_y &= \left(\ln\sqrt{u^2+v^2}\right)'_y = \frac{1}{\sqrt{u^2+v^2}} \cdot \frac{2(uu'_y + vv'_y)}{2\sqrt{u^2+v^2}} = \frac{uu'_y + vv'_y}{u^2+v^2} \\
&= \frac{-uv'_x + vu'_x}{u^2+v^2} = -\frac{1}{1+v^2/u^2} \cdot \frac{uv'_x - u'_x v}{u^2} \\
&= -\left(\tan^{-1}\frac{v}{u} + \pi k + 2\pi n\right)'_x = -\Theta'_x,
\end{aligned}$$

where k takes one of the values -1, 0 and 1, and n the respective value in $\{0, \pm 1, \pm 2, \ldots\}$. Now assume $u(z) = 0$. Then $v(z) \neq 0$. Consider two subcases. In the first subcase assume that there is a sequence $\{z_n\}$ such that $z_n \to z$ and $u(z_n) \neq 0$. Then the continuity of $[\ln R]'_y$ and Θ'_x implies

$$[\ln R]'_y(z) = \frac{v'_y(z)}{v(z)} = \frac{u'_x(z)}{v(z)} = -\Theta'_x(z).$$

In the second subcase let u be identically zero on some neighborhood of z. Then by Cauchy–Riemann conditions, v takes a nonzero constant value on this neighborhood. Hence, $[\ln R]'_y(z) = 0$. At the same time, on this neighborhood either $\Theta(z) = \pi/2 + 2\pi n$ or $\Theta(z) = -\pi/2 + 2\pi n$, implying $\Theta'_x(z) = 0$. Hence, again $[\ln R]'_y(z) = 0 = -\Theta'_x(z)$. $\square$

Remark 3.1. *The converse of Proposition 3.1 is also valid. For this, assume that the equalities in (6), which are equivalent to the equalities in (5), hold. Then from* $u(z) = R(z)\cos\Theta(z)$ *and* $v(z) = R(z)\sin\Theta(z)$*, we obtain*

$$\begin{aligned}
u'_x(z) &= R'_x(z)\cos\Theta(z) - R(z)\Theta'_x(z)\sin\Theta(z) \\
&= R(z)[\ln R]'_x(z)\cos\Theta(z) - R(z)\Theta'_x(z)\sin\Theta(z) \\
&= R(z)\Theta'_y(z)\cos\Theta(z) + R(z)[\ln R]'_y(z)\sin\Theta(z) \\
&= R(z)\Theta'_y(z)\cos\Theta(z) + R'_y(z)\sin\Theta(z) = v'_y(z).
\end{aligned}$$

In a similar way $u'_y(z) = -v'_x(z)$ *can be proved.*

Thus, the equalities in (5) or in (6) are just another form of the ordinary Cauchy–Riemann conditions and, hence, we call them Cauchy–Riemann *conditions. They can be written in terms of partial derivatives with respect to r and θ as well. Indeed, from

$$\begin{aligned}
\Theta'_\theta &= \Theta'_x x'_\theta + \Theta'_y y'_\theta = -\Theta'_x r\sin\theta + \Theta'_y r\cos\theta \\
&= r([\ln R]'_y y'_r + [\ln R]'_x x'_r) = r[\ln R]'_r
\end{aligned}$$

and

$$\begin{aligned}
[\ln R]'_\theta &= [\ln R]'_x x'_\theta + [\ln R]'_y y'_\theta = -\Theta'_y r\sin\theta - \Theta'_x r\cos\theta \\
&= -r(\Theta'_y y'_r + \Theta'_x x'_r) = -r\Theta'_r,
\end{aligned}$$

we obtain

$$\Theta'_\theta(z) = r[\ln R]'_r(z) \quad \text{and} \quad [\ln R]'_\theta(z) = -r\Theta'_r(z),$$

or in terms of *derivatives

$$R_r^*(z)^r = \left[e^{\Theta}\right]_\theta^*(z) \ \text{ and } \ R_\theta^*(z) = \left[e^{-\Theta}\right]_r^*(z)^r.$$

We think that among different forms of the Cauchy–Riemann *conditions, the ones presented below, which describe modulus and argument of f^*, are most useful.

Theorem 3.1 (Cauchy–Riemann *conditions)**.** *Under the above conditions and notation,*

$$\begin{cases} |f^*(z)| = R_x^*(z) = [e^{\Theta}]_y^*(z), \\ \arg f^*(z) = \Theta_x'(z) + 2\pi n = -[\ln R]_y'(z) + 2\pi n, \ n = 0, \pm 1, \pm 2, \dots. \end{cases} \tag{7}$$

Proof. From

$$f^*(z) = e^{f'(z)/f(z)} = e^{\frac{u_x'(z)+iv_x'(z)}{u(z)+iv(z)}} = e^{\frac{u(z)u_x'(z)+v(z)v_x'(z)}{u(z)^2+v(z)^2}+i\frac{u(z)v_x'(z)-u_x'(z)v(z)}{u(z)^2+v(z)^2}},$$

we have

$$|f^*(z)| = e^{\frac{u(z)u_x'(z)+v(z)v_x'(z)}{u(z)^2+v(z)^2}} = e^{\frac{R(z)R_x'(z)}{R(z)^2}} = e^{\frac{R_x'(z)}{R(z)}} = R_x^*(z)$$

and, by Cauchy–Riemann conditions,

$$\begin{aligned} \arg f^*(z) &= \frac{u(z)v_x'(z) - u_x'(z)v(z)}{u(z)^2+v(z)^2} + 2\pi n = -\frac{u(z)u_y'(z) + v(z)v_y'(z)}{u(z)^2+v(z)^2} + 2\pi n \\ &= -\frac{R(z)R_y'(z)}{R(z)^2} + 2\pi n = -\frac{R_y'(z)}{R(z)} + 2\pi n = -[\ln R]_y'(z) + 2\pi n. \end{aligned}$$

The other two equalities in (7) are from (5)–(6). □

A few examples will be relevant to demonstrate features of complex *differentiation.

Example 3.1. *The function $f(z) = c$, $z \in \mathbb{C}$, where $c = \text{const} \in \mathbb{C} \setminus \{0\}$, is an entire function and its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{0/c} = 1, \ z \in \mathbb{C},$$

*is again an entire function. Respectively, $f^{*(n)}(z) = 1$, $z \in \mathbb{C}$, $n = 1, 2, \dots$. Thus, in complex *calculus the role of 0 (the neutral element of addition) is shifted to 1 (the neutral element of multiplication).*

Example 3.2. *The function $f(z) = e^{cz}$, $z \in \mathbb{C}$, where $c = \text{const} \in \mathbb{C}$, is an entire function and its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{ce^{cz}/e^{cz}} = e^c, \ z \in \mathbb{C},$$

*is again an entire function, taking identically the nonzero value e^c. Respectively, $f^{*(n)}(z) = 1$, $z \in \mathbb{C}$, $n = 2, 3, \dots$. Thus, in complex *calculus $f(z) = e^{cz}$ plays the role of the linear function $g(z) = az$ with $a = e^c$ from Newtonian calculus.*

Example 3.3. *For another entire function $f(z) = e^{ce^z}$, $z \in \mathbb{C}$, with $c = \text{const} \in \mathbb{C}$, that is called a Gompertz function if z takes real values, we have*

$$f^*(z) = e^{f'(z)/f(z)} = e^{ce^z e^{ce^z}/e^{ce^z}} = e^{ce^z}, \ z \in \mathbb{C}.$$

*Hence, f is a solution of the equation $f^* = f$. Respectively, $f^{*(n)}(z) = e^{ce^z}$, $z \in \mathbb{C}$, $n = 1, 2, \dots$. Thus, in complex *calculus $f(z) = e^{ce^z}$ plays the role of the exponential function $g(z) = ce^z$ from Newtonian calculus.*

Example 3.4. *The function* $f(z) = z$, $z \in \mathbb{C}$, *is also entire, but its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{1/z}, \ z \in \mathbb{C} \setminus \{0\},$$

accounts an essential singularity at $z = 0$. *This is because *differentiation is applicable to functions with the range in* $\mathbb{C} \setminus \{0\}$. *Thus, the *derivative of an entire function may not be entire. We also have* $f^{*(n)}(z) = e^{(-1)^{n-1}(n-1)!/z^n}$, $z \in \mathbb{C} \setminus \{0\}$, $n = 1, 2, \ldots$.

Example 3.5. *The holomorphic function* $f(z) = 1/z$, $z \in \mathbb{C} \setminus \{0\}$ *has a simple pole at* $z = 0$, *but its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{-1/z}, \ z \in \mathbb{C} \setminus \{0\},$$

is holomorphic with an essential singularity at $z = 0$. *We also have* $f^{*(n)}(z) = e^{(-1)^n(n-1)!/z^n}$, $z \in \mathbb{C} \setminus \{0\}$, $n = 1, 2, \ldots$.

The **derivative of a multi-valued function* can be defined as derivatives of its branches and it is naturally expected to be multi-valued as in the following case.

Example 3.6. *The function* $f(z) = \log z$, $z \in \mathbb{C} \setminus \{0\}$, *is multi-valued with a branch point at* $z = 0$. *Its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{1/(z \log z)}, \ z \in \mathbb{C} \setminus \{0\},$$

is still multi-valued with a branch point at $z = 0$.

In exceptional cases the *derivative of multi-valued function may be single-valued as in the following case.

Example 3.7. *The function* $f(z) = e^{z \log z}$, $z \in \mathbb{C} \setminus \{0\}$, *is multi-valued with a branch point at* $z = 0$, *but its *derivative*

$$f^*(z) = e^{f'(z)/f(z)} = e^{1+\log z} = ez, \ z \in \mathbb{C} \setminus \{0\},$$

is single-valued and has a removable singularity at $z = 0$, *and its nth order *derivative* $f^{*(n)}(z) = e^{(-1)^n(n-2)!/z^{n-1}}$, $z \in \mathbb{C} \setminus \{0\}$, $n = 2, 3, \ldots$, *accounts an essential singularity at* $z = 0$. *We also see that in complex *calculus the function* $f(z) = e^{z \log z}$, $z \in \mathbb{C} \setminus \{0\}$, *plays the role of the quadratic function* $g(z) = az^2$ *with* $a = e/2$ *from Newtonian calculus.*

We will say that a complex-valued function f of complex variable (single- or multi-valued) is **differentiable* at $z \in \mathbb{C}$ if it is differentiable at z and $f(z) \neq 0$. We will also say that f is **holomorphic* or **analytic* on an open connected set D if $f^*(z)$ exists for every $z \in D$. The above examples demonstrate that for a given *holomorphic function there are a few kinds of significant points on the complex plane which need a special attention:

- **(a):** removable singular points, as in case of holomorphic functions,
- **(b):** poles, as in case of holomorphic functions,
- **(c):** essential singular points, as in case of holomorphic functions,
- **(d):** zeros, at which a function takes zero value,
- **(e):** branch point, if a function is multi-valued.

We can state the following *conservation principle* for the set E_f of all these points for the function f: while *differentiation the points of E_f may change the kind, always remaining in E_f, and no other point falls to E_f. The points of E_f are isolated, their total number is at most countable and E_f has no accumulation point according to the theorems of complex calculus. Note that no need to make an exception for identically zero function since it is not *holomorphic although it is entire. This basic principle is not clearly seen in complex calculus and was not stated before to our knowledge. It become visible in complex *calculus.

The following is an immediate consequence from the above discussion.

Theorem 3.2. *A complex-valued function f of complex variable $z = x+iy$, defined on an open connected set D of the complex plane, is *holomorphic if and only if f is nowhere-vanishing on D, $|f|$ and for every $z \in \mathbb{C}$ one of the local continuous branches of* $\arg f$ *in a small neighborhood of z have continuous partial derivatives in x and in y, and the Cauchy–Riemann *conditions* (7) *hold. Furthermore, a *holomorphic function f has all order *derivatives and its nth order *derivative satisfies* (4).

4. Properties of Complex Multiplicative Derivatives

Some of properties of *derivatives from the real case, listed in Bashirov et al. [3], can immediately be extended to complex case. For example,

- **(a):** $[cf]^*(z) = f^*(z)$, $c = \text{const.} \in \mathbb{C} \setminus \{0\}$;
- **(b):** $[fg]^*(z) = f^*(z)g^*(z)$;
- **(c):** $[f/g]^*(z) = f^*(z)/g^*(z)$,

which can be proved directly by using (3). But some other properties can be extended nontrivially. For example, the equalities

$$[f^g]^*(x) = f^*(x)^{g(x)} f(x)^{g'(x)}, \quad \text{and} \quad [f \circ g]^*(x) = f^*(g(x))^{g'(x)}$$

from the real case fail in the complex case because they include multi-valued functions. They can be stated in the form:

- **(d):** $[e^{g \log f}]^*(z) \subseteq e^{g(z) \log f^*(z)} e^{g'(z) \log f(z)}$ in the sense each branch value of $[e^{g \log f}]^*(z)$ is the product of some branch values of $e^{g(z) \log f^*(z)}$ and $e^{g'(z) \log f(z)}$;
- **(e):** $[f \circ g]^*(z) \in e^{g'^*(g(z))}$ in the sense that $[f \circ g]^*(z)$ equals to some branch value of $e^{g'^*(g(z))}$.

To prove (d), we evaluate

$$\left[e^{g \log f}\right]^*(z) = e^{g'(z) \log f(z)} e^{g(z) f'(z)/f(z)}.$$

On the other hand,

$$e^{g(z) \log f^*(z)} e^{g'(z) \log f(z)} = e^{g(z) \log e^{f'(z)/f(z)}} e^{g'(z) \log f(z)}.$$

Here, note that although the equality $e^{\log w} = w$ holds for all $w \neq 0$, in general we have $\log e^w \neq w$ if we assume any branch of log. The equality $\mathcal{L} e^w = w$ holds if $\mathcal{L}$ is a branch of log, ranging in the strip

$$\{x + iy \in \mathbb{C} : \alpha < y \leq \alpha + 2\pi\},$$

where the imaginary part of w falls into $(\alpha, \alpha + 2\pi]$. Letting $w = f'(z)/f(z)$, we obtain that each branch value of $[e^{g \log f}]^*(z)$ is the product of some branch values of $e^{g(z) \log f^*(z)}$ and $e^{g'(z) \log f(z)}$. This proves (d). In the same way, (e) can be proved.

In particular, if $g(z) \equiv c = \text{const.}$ for $c \in \mathbb{C}$, then

$$\left[e^{c \log f}\right]^*(z) = e^{c f'(z)/f(z)},$$

implying that $[e^{c\log f}]^*$ is single-valued. Hence, (d) reduces to

(f): $[e^{c\log f}]^*(z) \in e^{c\log f^*(z)}$, $c = \text{const.} \in \mathbb{C}$, in the sense that $[e^{c\log f}]^*(z)$ equals to some branch value of $e^{c\log f^*(z)}$.

The properties (d)–(f) are pointwise in sense that for distinct values of z distinct branches of log are required to get the respective branch values of the right hand sides. This makes (d)–(f) less useful than (a)–(c). But for nonnegative integer values of c, both the right and left hand sides in (f) are single valued, reducing (f) to the equality

: (g) $[e^{n\log f}]^*(z) = e^{n\log f^*(z)}$, $n = 0, 1, 2, \ldots$.

The proof of this is just multiple application of (b).

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