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RECTIFYING DEVELOPABLE SURFACE OF TIMELIKE BIHARMONIC CURVE IN THE LORENTZIAN HEISENBERG GROUP HEIS^3

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ABSTRACT. In this paper, we study in particular developable surfaces, a special type of ruled surface in the Lorentzian Heisenberg group Heis^3. We find out explicit parametric equations of rectifying developable of timelike biharmonic curve in the Lorentzian Heisenberg group Heis^3. Additionally, we illustrate three figures of our main theorem.

Keywords: Heisenberg group, biharmonic curve, helix, developable surface.

AMS Subject Classification: 58E20

1. Introduction

Use of developable surfaces has a long history [4,6,11]. Real developable surfaces have natural applications in many areas of engineering and manufacturing. For instance, an aircraft designer uses them to design the airplane wings, and a tinsmith uses them to connect two tubes of different shapes with planar segments of metal sheets.

A developable surface is a surface that can be (locally) unrolled onto a flat plane without tearing or stretching it. If a developable surface lies in three-dimensional Euclidean space, and is complete, then it is necessarily ruled, but the converse is not always true. For instance, the cylinder and cone are developable, but the general hyperboloid of one sheet is not. More generally, any developable surface in three-dimensions is part of a complete ruled surface, and so itself must be locally ruled. There are surfaces embedded in four dimensions which are however not ruled.

Let (N,h) and (M,g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathcal{T}(\phi) \right|^2 dv_h,$$

where the section $\mathcal{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ [1,2,3,5].

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \operatorname{tr}R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the bitension field of ϕ, [7-14]. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

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In this paper, we study in particular developable surfaces, a special type of ruled surface in the Lorentzian Heisenberg group Heis^3. We find out explicit parametric equations of rectifying developable of biharmonic curve in the Lorentzian Heisenberg group Heis^3.

2. The Lorentzian Heisenberg Group Heis^3

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial z},\quad \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2,\mathbf{e}_3]=2\mathbf{e}_1,\quad [\mathbf{e}_3,\mathbf{e}_1]=0,\quad [\mathbf{e}_2,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\quad g(\mathbf{e}_3,\mathbf{e}_3)=-1.$$

Proposition 2.1. *(see [15]) For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla=\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2\\ \mathbf{e}_3 & 0 & \mathbf{e}_1\\ \mathbf{e}_2 & -\mathbf{e}_1 & 0\end{pmatrix},$$

where the (i,j)-element in the table above equals $\nabla_{e_i}e_j$ for our basis

$$\{\mathbf{e}_k, k=1,2,3\}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

3. Timelike Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3

Let $\gamma: I\longrightarrow Heis^3$ be a timelike curve in the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N}&=\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},\end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.2}$$

Lemma 3.1. *(see [16]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve in the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \tau &= \text{constant}, \\ N_1B_1 &= 0, \\ \kappa^2 - \tau^2 &= -1 + 4B_1^2. \end{aligned} \tag{3.3}$$

Theorem 3.2. *(see [16]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then the tangent vector of γ is*

$$\mathbf{T}(s) = \sinh \mathcal{M}\mathbf{e}_1 + \cosh \mathcal{M} \sinh (\mathcal{N}s + \mathcal{C})\,\mathbf{e}_2 + \cosh \mathcal{M} \cosh (\mathcal{N}s + \mathcal{C})\,\mathbf{e}_3, \tag{3.4}$$

where $\mathcal{N} = \frac{\kappa - \sinh 2\mathcal{M}}{\cosh \mathcal{M}}$.

4. Rectifying Developable of Timelike Biharmonic Curve in the Lorentzian Heisenberg Group Heis^3

Ruled surfaces are swept out by the motion of a straight line in a space. More formally, the image of the map $\Omega_{(\gamma,\delta)} : I \times \mathbb{R} \to Heis^3$ defined by

$$\Omega_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s), \quad s \in I, \quad u \in \mathbb{R}$$

is called a ruled surface in Heis^3 where $\gamma : I \to Heis^3$, $\delta : I \to Heis^3 \setminus \{0\}$ are smooth mappings and I is an open interval.

We call γ the base curve and δ the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings.

Definition 4.1. *A smooth surface $\Omega_{(\gamma,\delta)}$ is called a developable surface in $Heis^3$ if its Gaussian curvature K vanishes everywhere on the surface.*

We define a modified Darboux vector as

$$\tilde{\mathbf{D}}(s) = -\left(\frac{\tau}{\kappa}\right)\gamma'(s) - \mathbf{B}(s). \tag{4.1}$$

Definition 4.2. *The rectifying developable of γ in $Heis^3$ is*

$$\Omega_{(\gamma,\tilde{\mathbf{D}})}(s,u) = \gamma(s) + u\tilde{\mathbf{D}}(s). \tag{4.2}$$

The rectifying developable of the curve γ is the envelope of the family of rectifying planes at $\gamma(s)$, where the rectifying plane at $\gamma(s)$ is defined to be the plane generated by the tangent vector $\gamma'(s)$ and the binormal vector $\mathbf{B}(s)$.

Theorem 4.3. *$\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve in Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then, the equation of rectifying developable of $\gamma$is given by*

$$\begin{aligned}
&\Omega_{(\gamma,\tilde{\mathbf{D}})}(s,u) = \qquad (4.3)\\
&[(\sinh\mathcal{M}+\frac{1}{\mathcal{N}}\cosh^2\mathcal{M})s-\frac{1}{4\mathcal{N}^2}\cosh^2\mathcal{M}\sinh 2(\mathcal{N}s+\mathcal{C})-\frac{\mathcal{A}_1}{\mathcal{N}}\cosh(\mathcal{N}s+\mathcal{C})\\
&+\mathcal{A}_3+(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_1)(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_2)\\
&+u[-\frac{\tau}{\kappa}\sinh\mathcal{M}-\frac{1}{\kappa}\cosh^2\mathcal{M}(\mathcal{N}+2\sinh\mathcal{M})]]\mathbf{e}_1+[\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_2\\
&+u\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa}+\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}]]\mathbf{e}_2\\
&+[\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_1\\
&+u\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa}+\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M}\sinh\mathcal{M}]]\mathbf{e}_3,
\end{aligned}$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are constants of integration, $\mathcal{M}$ is constant angle and $\mathcal{N}=\frac{\kappa-\sinh 2\mathcal{M}}{\cosh\mathcal{M}}$.

Proof. Using (2.1) in (3.4), we obtain

$$\begin{aligned}
\mathbf{T} &= (\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C}),\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C}),\sinh\mathcal{M} \qquad (4.4)\\
&-\frac{1}{\mathcal{N}}\cosh^2\mathcal{M}\sinh^2(\mathcal{N}s+\mathcal{C})-\mathcal{A}_1\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})),
\end{aligned}$$

where $\mathcal{A}_1$ is constant of integration.

On the other hand, using Frenet formulas (3.1) and (4.4), we have

$$\mathbf{N}=\frac{1}{\kappa}\cosh\mathcal{M}(\mathcal{N}+2\sinh\mathcal{M})[\cosh(\mathcal{N}s+\mathcal{C})\mathbf{e}_2+\sinh(\mathcal{N}s+\mathcal{C})\mathbf{e}_3].$$

Also, using (4.4) and Lemma 3.1, we obtain

$$\begin{aligned}
\mathbf{B} &= \frac{1}{\kappa}\cosh\mathcal{M}(\mathcal{N}+2\sinh\mathcal{M})[\cosh\mathcal{M}\mathbf{e}_1 \qquad (4.5)\\
&-\sinh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})\mathbf{e}_2-\sinh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})\mathbf{e}_3].
\end{aligned}$$

Therefore, the modified Darboux vector of γ is

$$\begin{aligned}
\tilde{\mathbf{D}}(s) &= [-\frac{\tau}{\kappa}\sinh\mathcal{M}-\frac{1}{\kappa}\cosh^2\mathcal{M}(\mathcal{N}+2\sinh\mathcal{M})]\mathbf{e}_1 \qquad (4.6)\\
&+\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa}+\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}]\mathbf{e}_2\\
&+\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa}+\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}]\mathbf{e}_3.
\end{aligned}$$

On the other hand, using (4.4) the position vector of γ is

$$\begin{aligned}\gamma(s) = & [(\sinh\mathcal{M}+\frac{1}{\mathcal{N}}\cosh^2\mathcal{M})s-\frac{1}{4\mathcal{N}^2}\cosh^2\mathcal{M}\sinh 2(\mathcal{N}s+\mathcal{C}) \\ & -\frac{\mathcal{A}_1}{\mathcal{N}}\cosh^2(\mathcal{N}s+\mathcal{C})+\mathcal{A}_3+(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C}) \\ & +\mathcal{A}_1)(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_2)]\mathbf{e}_1 \\ & +[\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_2]\mathbf{e}_2 \\ & +[\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_1]\mathbf{e}_3,\end{aligned} \tag{4.7}$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ is constant of integration.

So, substituting (4.6) and (4.7) in (4.2), we have (4.3), which completes the proof.

Theorem 4.4. *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve in Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then, the parametric equations for rectifying developable of γ are*

$$\begin{aligned}x_{(\gamma,\tilde{\mathbf{D}})}(s,u) = & [\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})+u\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa} \\ & +\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}]]+\mathcal{A}_1, \\ y_{(\gamma,\tilde{\mathbf{D}})}(s,u) = & [\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+u\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa} \\ & +\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}]]+\mathcal{A}_2, \\ z_{(\gamma,\tilde{\mathbf{D}})}(s,u) = & [(\sinh\mathcal{M}+\frac{1}{\mathcal{N}}\cosh^2\mathcal{M})s-\frac{1}{4\mathcal{N}^2}\cosh^2\mathcal{M}\sinh 2(\mathcal{N}s+\mathcal{C}) \\ & -\frac{\mathcal{A}_1}{\mathcal{N}}\cosh^2(\mathcal{N}s+\mathcal{C})+\mathcal{A}_3+(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C}) \\ & +\mathcal{A}_1)(\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C})+\mathcal{A}_2)+u[-\frac{\tau}{\kappa}\sinh\mathcal{M} \\ & -\frac{1}{\kappa}\cosh^2\mathcal{M}(\mathcal{N}+2\sinh\mathcal{M})]][1-[\frac{1}{\mathcal{N}}\cosh\mathcal{M}\cosh(\mathcal{N}s+\mathcal{C}) \\ & +u\cosh\mathcal{M}\sinh(\mathcal{N}s+\mathcal{C})[-\frac{\tau}{\kappa}+\frac{1}{\kappa}(\mathcal{N}+2\sinh\mathcal{M})\sinh\mathcal{M}+\mathcal{A}_2]]],\end{aligned} \tag{4.8}$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are constants of integration, $\mathcal{M}$ is constant angle and $\mathcal{N}=\frac{\kappa-\sinh 2\mathcal{M}}{\cosh\mathcal{M}}$.

Proof. Substituting equation (2.1) to (4.3), we have (4.8). Thus, the proof is completed.

Hence, the obtained parametric equations are illustrated in Fig. 1, 2, 3:

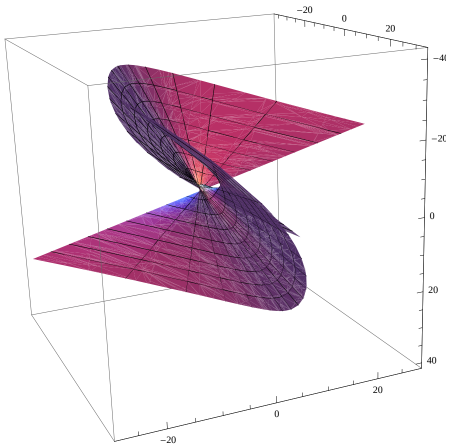

Figure 1.

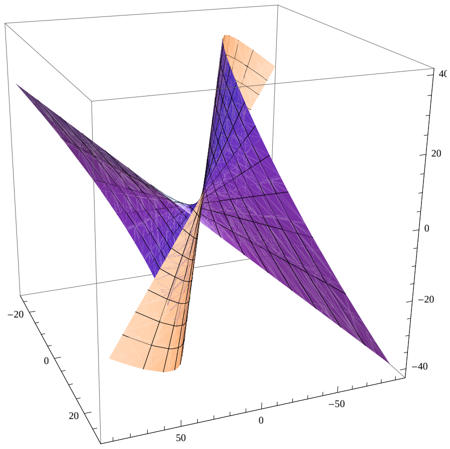

Figure 2.

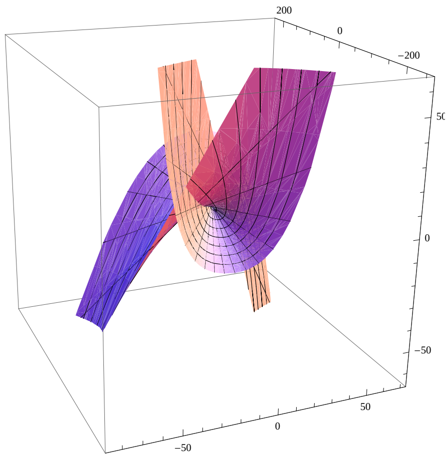

Figure 3.

Corollary 4.4. *$\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve in Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then, the modified Darboux vector of γ in terms of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is given by*

$$\begin{aligned}\tilde{\mathbf{D}}(s) &= [-\frac{\tau}{\kappa}\sinh\mathcal{M} - \frac{1}{\kappa}\cosh^2\mathcal{M}(\mathcal{N} + 2\sinh\mathcal{M})]\mathbf{e}_1 \\ &\quad + \cosh\mathcal{M}\sinh(\mathcal{N}s + \mathcal{C})\,[-\frac{\tau}{\kappa} + \frac{1}{\kappa}(\mathcal{N} + 2\sinh\mathcal{M})\sinh\mathcal{M}]\mathbf{e}_2 \\ &\quad + \cosh\mathcal{M}\cosh(\mathcal{N}s + \mathcal{C})\,[-\frac{\tau}{\kappa} + \frac{1}{\kappa}(\mathcal{N} + 2\sinh\mathcal{M})\sinh\mathcal{M}]\mathbf{e}_3.\end{aligned}$$

where $\mathcal{M}$ *is constant angle and* $\mathcal{N} = \frac{\kappa - \sinh 2\mathcal{M}}{\cosh \mathcal{M}}$.

Proof. From (2.1) and (4.5), we obtain above system, which completes the proof.

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