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HARMONIC MAPPINGS RELATED TO CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER b

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ABSTRACT. Let $CC(b)$ be the class of functions close-to-convex functions of order b, and let S_H be the class of harmonic mappings in the plane. In the present paper we investigate harmonic mappings related to close-to-convex functions of complex order b.

Keywords: Convex and starlike functions of complex order b, Close-to-convex functions of complex order b, Harmonic mappings, Growth and distortion theorems.

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1. INTRODUCTION

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by P the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z)$ is in P if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Moreover, let A be the class of functions in the open unit disc $\mathbb{D}$ that are normalized with $h(0) = h'(0) - 1 = 0$, then a function $h(z) \in A$ is called convex on starlike if it maps $\mathbb{D}$ into a convex or starlike region, respectively. Corresponding classes are denoted by $\mathbb{C}$ and S^*. It is well known that $\mathbb{C} \subset S^*$, that both are subclasses of the univalent functions and have the following analytical representations

$$h(z) \in \mathbb{C} \text{ if and only if } \mathrm{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > 0, \ z \in \mathbb{D}, \tag{2}$$

and

$$h(z) \in S^* \text{ if and only if } \mathrm{Re}\left(z\frac{h'(z)}{h(z)}\right) > 0, \ z \in \mathbb{D}. \tag{3}$$

More on these classes can be found in [3]. Let $h(z)$ be an element of A. If there is a function $s(z)$ in $\mathbb{C}$ and a real β such that

$$\mathrm{Re}\left(\frac{h'(z)}{e^{i\beta} s'(z)}\right) > 0, \ z \in \mathbb{D} \tag{4}$$

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then $h(z)$ is called a close-to-convex function in $\mathbb{D}$, and the class of such functions is denoted by CC [3], and let $h(z) \in A$, $s(z) \in S^*$. If

$$Re(1+\frac{1}{b}(z\frac{h'(z)}{s(z)}-1))>0, z\in\mathbb{D} \tag{5}$$

then $h(z)$ is called the close-to-convex function of complex order b, $b \in \mathbb{C}$ 0, the class of such functions is denoted by $CC(b)$[5].

Further, let $h(z)$, $g(z) \in A$. Then we say that $h(z)$ is subordinate to $g(z)$ and we write $h(z) \prec g(z)$. If there exists a function $\phi(z) \in \Omega$ such that $h(z) = g(\phi(z))$ for all $z \in \mathbb{D}$. Specially if $g(z)$ is univalent in $\mathbb{D}$, then $h(z) \prec g(z)$ if and only if $h(0) = g(0)$, $h(\mathbb{D}) \subset g(\mathbb{D})$, implies $h(\mathbb{D}_r) \subset g(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z||z| < r, 0 < r < 1\}$ (Subordination and Lindelof Principle [1]).

In the terms of subordination we have

$$P=\Big\{p(z)=1+\sum_{n=1}^{\infty}p_nz^n|p(z) \text{ regular in } \mathbb{D}, p(z)\prec\frac{1+z}{1-z}\Big\}, \tag{6}$$

$$S^*=\Big\{h(z)\in A|z\frac{h'(z)}{h(z)}\prec\frac{1+z}{1-z}\Big\}, \tag{7}$$

$$C=\Big\{h(z)\in A|\left(1+z\frac{h''(z)}{h'(z)}\right)\prec\frac{1+z}{1-z}\Big\}, \tag{8}$$

and

$$CC=\Big\{h(z),s(z)\in A|\frac{h'(z)}{e^{i\beta}s'(z)}\prec\frac{1+z}{1-z}, s(z)\in C\Big\}. \tag{9}$$

$$CC(b)=\Big\{h(z),s(z)\in A|Re(1+\frac{1}{b}(z\frac{h'(z)}{s(z)}-1))>0, (1+\frac{1}{b}(z\frac{h'(z)}{s(z)}-1))\prec\frac{1+z}{1-z}, \\ s(z)\in S^*, b\in C-\{0\}\Big\}.$$

Using Alexander Theorem the class $CC(b)$ can be written in the following form

$$CC(b)=\Big\{h(z),s(z)\in A|Re(1+\frac{1}{b}(z\frac{h'(z)}{s'(z)}-1))>0, (1+\frac{1}{b}(z\frac{h'(z)}{s'(z)}-1))\prec\frac{1+z}{1-z}, \\ s(z)\in C, b\in C-\{0\}\Big\}.$$

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions
$h(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbb{D}$.
where $a_n, b_n \in C$, $n = 0, 1, 2, 3, ...$ as usual we call $h(z)$ analytic part of f and $g(z)$ co-analytic part of f an elegant and complete account of the theory harmonic mapping in given Duren's monograph [2]. Lewy [2] proved in 1936 that the harmonic mapping f locally univalent in $\mathbb{D}$ if and only if its jacobien $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-preserving if $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ or sense-reserving if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc $\mathbb{D}$, and the second dilatation $w(z) = (\frac{g'(z)}{h'(z)})$

has the property $|w(z)| < 1$ in $\mathbb{D}$.
The class of all sense-preserving harmonic mapppings of the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by S_H. Thus S_H contains the standard class S of univalent functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by S_H^0. Thus it is clear that $S \subset S_H^0 \subset S_H$. [2]. Now we consider the following class of harmonic mappings

$$S_{HCC}(b) = \left\{ f = h(z) + \overline{g(z)} | \frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + (2b-1)z}{1-z}, b \in C - \{0\}, h(z) \in C \right\}, \tag{10}$$

The aim of this paper we need the following well known lemma and theorems.

Lemma 1.1. *([4]) Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$. Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.1. *([3]) Let $h(z)$ be an element of C, then*

$$\frac{r}{1+r} \leq |h(z)| \leq \frac{r}{1-r}$$

and

$$\frac{r}{(1+r)^2} \leq |h'(z)| \leq \frac{r}{(1-r)^2}$$

for all $|z| = r < 1$.

Theorem 1.2. *([3]) If $h(z) \in C$ then*

$$Re z\frac{h'(z)}{h(z)} > \frac{1}{2} \Rightarrow z\frac{h'(z)}{h(z)} \prec \frac{1}{1-z}.$$

2. Main Results

Theorem 2.1. *Let $f = h(z) + \overline{g(z)}$ be an elemet of $S_{HCC(b)}$, then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + (2b-1)z}{1-z}, \quad z \in \mathbb{D}$$

Proof. Since $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then we have

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + (2b-1)z}{1-z} \Rightarrow Re[1 + \frac{1}{b}(\frac{1}{b_1}\frac{g'(z)}{h'(z)} - 1)] > 0 \Rightarrow$$

$$\left| \frac{g'(z)}{h'(z)} - \frac{b_1(1 + (2b-1)r^2)}{1-r^2} \right| \leq \frac{2|b_1||b|r}{1-r^2},$$

this shows that the values of $(\frac{g'(z)}{h'(z)})$ for $|z| < 1$ are inside the disc with the centre

$$C(r) = \frac{b_1(1 + (2b-1)r^2)}{1-r^2}$$

and the radius

$$\rho(r) = \frac{2|b_1||b|r}{1-r^2},$$

at the some time we can write (using Theorem 1.2)

$$Re(z\frac{h'(z)}{h(z)}) > \frac{1}{2} \Rightarrow z\frac{h'(z)}{h(z)} \prec \frac{1}{1-z} \Rightarrow \frac{h(z)}{zh'(z)} = (1 + \phi(z)), \phi(z) \in \Omega$$

Now we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+(2b-1)\phi(z)}{1-\phi(z)},$$

taking the derivative from this equality we obtain.

$$\frac{g'(z)}{h'(z)} = b_1 \left(\frac{1+(2b-1)\phi(z)+2bz\phi'(z)}{1-\phi(z)}\right) \tag{11}$$

Now, it is easy to realize that the subordination

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+(2b-1)z}{1-z}$$

(from the definition of $S_{HCC(b)}$) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.
Indeed assume the contrary that there exists $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. Then by I. S. Jack lemma (Lemma 1.1) $z_1\phi'(z_1) = k\phi(z_1)$, $k \geq 1$, such z_1 we have

$$w(z_1) = \frac{g'(z)}{h'(z)} = \frac{1+(1+k)(2b-1)\phi(z_1)}{1-\phi(z_1)} = w(\phi(z_1)) \notin w(\mathbb{D}).$$

But this is a contradiction to the condition of the definition of $S_{HCC(b)}$ and so assumption is wrong, i.e, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

Corollary 2.1. *Let $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then*

$$\frac{|b_1|\left[1-2|b|r+|1-2b|r^2\right]}{1-r^2} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1|\left[1+2|b|r+|1-2b|r^2\right]}{1-r^2} \tag{12}$$

$$\frac{|b_1|\left[1-2|b|r+|1-2b|r^2\right]}{1-r^2} \leq \left|\frac{g(z)}{h(z)}\right| \leq \frac{|b_1|\left[1+2|b|r+|1-2b|r^2\right]}{1-r^2} \tag{13}$$

Proof. Since $(\frac{g'(z)}{h'(z)})$ and $(\frac{g(z)}{h(z)})$ are subordinate to $(b_1\frac{1+(2b-1)z}{1-z})$, then using subordination and Lindelf Principle we get (12) and (13).

□

Corollary 2.2. *Let $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then*

$$|b_1| F(|b|, -r) \leq |g'(z)| \leq |b_1| F(|b|, r) \tag{14}$$

$$|b_1| r.G(|b|, -r) \leq |g(z)| \leq |b_1 r.| G(|b|, r) \tag{15}$$

where

$$F(|b|, r) = \frac{1+2|b|r+|1-2b|r^2}{(1+r)(1-r)^3}$$

$$G(|b|, r) = \frac{1+2|b|r+|1-2b|r^2}{(1+r)(1-r)^2}$$

Proof. Using Corollary 2.1 and Theorem 1.1 we obtain (14) and (15).

□

Lemma 2.1. *If $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then*

$$\frac{|b_1| - r}{1+|b_1|r} \leq |w(z)| \leq \frac{|b_1| + r}{1+|b_1|r} \tag{16}$$

$$\frac{(1-r^2)(1-|b_1|)^2}{(1+|b_1|r)^2} \leq (1-|w(z)|^2) \leq \frac{(1-r^2)(1-|b_1|)^2}{(1-|b_1|r)^2} \tag{17}$$

$$\frac{(1-r)(1+|b_1|)}{1-|b_1|r} \leq (1+|w(z)|) \leq \frac{(1+r)(1+|b_1|)}{1+|b_1|r} \tag{18}$$

$$\frac{(1-r)(1-|b_1|)}{1+|b_1|r} \leq (1-|w(z)|) \leq \frac{(1+r)(1-|b_1|)}{1-|b_1|r} \tag{19}$$

Proof. Since $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, it follows that

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2 z + ...}{1 + 2a_2 z + ...} \Rightarrow w(0) = b_1, |w(z)| < 1.$$

So, the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} = \frac{w(z) - b_1}{1 - \overline{b_1}w(z)}, (z \in \mathbb{D})$$

satisfies the conditions of Schwarz lemma. Therefore, we have

$$w(z) = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} ifandonlyif w(z) \prec \frac{b_1 + z}{1 + \overline{b_1}z}, (z \in \mathbb{D})$$

On the other hand, the linear transformation $(\frac{b_1+z}{1+\overline{b_1}z})$ maps $|z| = r$ onto the disc with the centre

$$C(r) = (\frac{(1-r^2)Reb_1}{1-r^2}, \frac{(1-r^2)Imb_1}{1-r^2})$$

and the radius

$$\rho(r) = \frac{(1-|b_1|^2)r}{1-r^2}.$$

Then we have (16) which gives (17), (18) and (19). □

Corollary 2.3. *Let $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then*

$$\frac{(1-r)(1-|b_1|^2)}{(1+r)^3(1+|b_1|r)^2} \leq J_f \leq \frac{(1+r)(1-|b_1|^2)}{(1-r)^3(1+|b_1|r)^2} \tag{20}$$

$$\leq |f| \leq \tag{21}$$

Proof. Since

$$J_f = \left|h'(z)\right|^2 (1 - |w(z)|^2), \tag{22}$$

and

$$\left|h'(z)\right| (1 - |w(z)|)dr \leq |df| \leq \left|h'(z)\right| (1 + |w(z)|)dr \tag{23}$$

Using Theorem 1.1 and Lemma 2.1 in the inequalities (22) and (23), then we obtain (20) and (21).

□

Theorem 2.2. *Let $f = h(z) + \overline{g(z)} \in S_{HCC(b)}$, then*

$$\sum_{k=2}^{n} k^2 |b_k - b_1 a_k|^2 \leq \left|1 - b_1^2\right|^2 + \sum_{k=2}^{n+1} k^2 |a_k - b_1 b_k|^2 \tag{24}$$

Proof. Using Lemma 2.1 we can write,

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1}z} \Rightarrow \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} \Rightarrow$$

$$g'(z)(1 + \overline{b_1}\phi(z)) = h'(z)(b_1 + \phi(z)) \Rightarrow g'(z) + \overline{b_1}g'(z)\phi(z) = b_1 h'(z) + h'(z)\phi(z)$$

$$\Rightarrow (g'(z) - b_1 h'(z)) = (h'(z) - \overline{b_1}g'(z))\phi(z) \Rightarrow$$

$$(\sum_{n=1}^{\infty} b_n z^n)' - b_1(z + \sum_{n=2}^{\infty} a_n z^n)' = [(z + \sum_{n=2}^{\infty} a_n z^n)' - b_1(z + \sum_{n=1}^{\infty} a_n z^n)']\phi(z) \Rightarrow$$

$$\sum_{k=2}^{n} k(b_k - b_1 a_k) z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1} = [(1 - b_1^2) + \sum_{k=2}^{n} k(a_k - b_1 b_k) z^{k-1}]\phi(z) \quad (25)$$

Since the last equality has the form

$$f_1(z) = f_2(z)\phi(z)$$

with $|\phi(z)| < 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \left| f_1(re^{i\theta}) \right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left| f_2(re^{i\theta}) \right| \quad (26)$$

for each r, $(0 < r < 1)$. Expressing (26) in terms of coefficients in (24) we obtain the inequality

$$\sum_{k=2}^{n} k \left| b_k - b_1 a_k \right|^2 r^{2k} + \sum_{k=n+1}^{\infty} \left| d_k \right|^2 r^{2k} \leq [\left| 1 - b_1^2 \right|^2 + \sum_{k=2}^{n+1} k^2 \left| a_k - b_1 b_k \right|^2] r^{2k} \quad (27)$$

By letting $r \to 1^-$ in (27) we obtain the desired result. The proof of this method is due to Clunie [1].

□

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