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COMPUTATIONAL COMPLEXITY OF DOMINATION INTEGRITY IN GRAPHS

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ABSTRACT. In a graph G, those dominating sets S which give minimum value for $|S| + m(G-S)$, where $m(G-S)$ denotes the maximum order of a component of $G-S$, are called dominating integrity sets of G (briefly called DI-sets of G). This concept combines two important aspects namely domination and integrity in graphs. In this paper, we show that the decision problem domination integrity is NP-complete even when restricted to planar or chordal graphs.

Keywords: Integrity, Domination Integrity

AMS Subject Classification: 05C07, 05C12, 05C35, 05C90.

1. Introduction

One of the important characteristics of a communication network is its ability to function efficiently even when some of its nodes or links are paralyzed. Vulnerability parameters starting from connectivity give measures of the strength of the network in adverse conditions. These parameters aim at finding the nature of the network when a subset of the set of nodes or set of links are removed. Domination gives a measure of the connection that a subset of a nodes has with the complement. The smaller the domination number means that one can effectively communicate to all the nodes outside a small subset of nodes. The study of the effect of removal of a dominating set on a network is interesting in the sense that the damage that the network experiences is more when a core of the network is removed.

In social networks, it means that when a decision making body is removed, the network suffers a heavy set back when the remaining part of the network gets scattered. This gives the motivation to study the vulnerability of a network when dominating sets are removed. For example, in a network with a node having connections with all other nodes which are independent, the removal of this master node will result in complete chaos. On the other hand, when every node has connection with every other node in a network, then the removal of any number of vertices will not affect the cohesiveness of the remaining part. But, such a network is costly. We have to examine various communication / social networks which are economic and at the same time have a high ability to withstand attacks on dominating sets. Domination Integrity is the parameter which will measure the vulnerability of a network when its dominating sets are under attack.

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Domination integrity, a new concept introduced in [16], is defined as $\min\{|S|+m(G-S) : S \text{ is a dominating set of } G\}$. Here $m(G-S)$ is the cardinality of a maximum order component in $G-S$. Clark, Entringer and Fellows (1987)[14], have proved that, for an arbitrary graph G and an arbitrary integer k,the determination of whether the integrity $I(G) \leq k$ is NP-complete, even if G is restricted to planar graphs. Domination Integrity problem is proved to be NP-complete even when G is restricted to planar or chordal graphs.

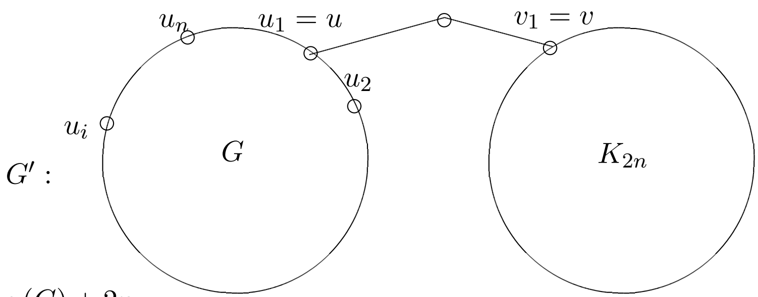
2. Main Results

The decision problem DIG is defined as follows:
Input: A graph $G = (V, E)$ and an integer k.
Question: Is $DI(G) \leq k$?.
Given a graph $G = (V, E)$ and an arbitrary set $S \subseteq V(G)$, it is easy to verify in polynomial time whether S is a dominating set. Also, for any subset S of G, there is a polynomial time algorithm to compute $m(G-S)$. Hence the decision problem DIG is in NP.

Theorem 2.1. *DIG is NP-complete.*

Proof:
Let G be any graph. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$. Let G' be the graph obtained from G as described below:
Let $H = K_{2n}$, where $n = |V(G)|$. Let $V(H) = \{v_1, v_2, \cdots, v_{2n}\}$. Attach a vertex x as a pendent vertex of some vertex u_i, $1 \leq i \leq n$ (say u) to H. Join x and some vertex v_i, $1 \leq i \leq 2n$(say v) of G. The resulting graph is G'.
Claim (i): $\gamma(G') = \gamma(G) + 1$.
Let D be any γ-set of G. If D contains v, then $D \cup \{u\}$ is a γ-set of G'. Hence $\gamma(G') = \gamma(G)+1$. Suppose $v \notin D$. Then also, $D \cup \{u\}$ is a γ-set of G' and hence $\gamma(G') = \gamma(G)+1$. Suppose D_1 is a γ-set of $G - v$. Then $D_1 \cup \{x, u\}$ is a γ-set of G'. Suppose $|D_1| = \gamma(G-v) < \gamma(G)$. But $\gamma(G-v) < \gamma(G) \leq \gamma(G-v)+1$. Therefore, $\gamma(G) = \gamma(G-v)+1$. $|D_1 \cup \{x, u\}| = |D_1| + 2 = |D_1| + 2 = \gamma(G-v) + 2 = \gamma(G) + 1$. Hence the claim(i).
Claim (ii): $DI(G') = \gamma(G) + 2n$.
Let S' be any DI-set of G'. Let $|S' \cap (V(G) \cup \{x\})| = t_1$ and $|S' \cap (V(H))| = t_2$ and $m(G' - S') \geq 2n - 2$.
$DI(G') = |S'| + m(G' - S') \geq t_1 + t_2 + (2n - t_2) = 2n + t_1 \geq 2n + \gamma(G)$, since $t_1 \geq \gamma(G)$. Since $D \cup \{u\}$ is a γ-set of G', where D is a γ-set of G and $m(G' - (D \cup \{u\})) = 2n - 1$, we get that $DI(G') \leq \gamma(G) + 1 + 2n - 1 = \gamma(G) + 2n$. Therefore, $DI(G') = \gamma(G) + 2n$. Hence DIG is NP-complete. ■

Illustration 2.1.



$DI(G') = \gamma(G) + 2n$.

Theorem 2.2. *The decision problem DIG is NP-complete even in the class of chordal graphs.*

Proof:

Proof using Exact Cover by 3-sets.

Instance : A finite set $X = \{x_1, x_2, \cdots, x_{3q}\}$ of cardinality $3q$ for some positive integer q and a set $C = \{C_1, C_2, \cdots, C_m\}$ of 3-element subsets of X.

Question: Does C contain an Exact Cover for X?. That is, a subset $C' \subseteq C$ such that every element of X occurs in exactly one 3-element subset of C'.

Given an instance of exact cover by 3-element subsets, we construct the following graph $G = (V, E)$. For each element $x_i \in X$, we create a vertex $x_i \in V$. For each 3-element subset $C_j \in C$, we create a vertex u_j and make it adjacent with the x_j's in C_j. Add new vertices $w_1, w_2, \cdots, w_m$ and make w_i adjacent with u_i, $1 \leq i \leq m$. At each w_i, construct a complete graph K_{3q+m}, $1 \leq i \leq m$. Make $u_1, u_2, \cdots, w_m$ as a complete graph. Clearly, G is a chordal graph.

Suppose C contains an exact cover C' for X. Then $S = \{w_1, w_2, \cdots, w_m\} \cup \{u_j : C_j \in C'\}$ is a DI-set of G. $|S| = m+q$ and $m(G-S) = 3q+m-1$. $|S|+m(G-S) = m+q+3q+m-1 = 4q+2m-1$.

Any dominating set S_1 of G must contain one element from each complete graph at w_i and hence must contain m elements. Suppose S_1 contains every element u_i, $1 \leq i \leq m$. Then $|S_1| = 2m \geq m+q$, since $m \geq q$.

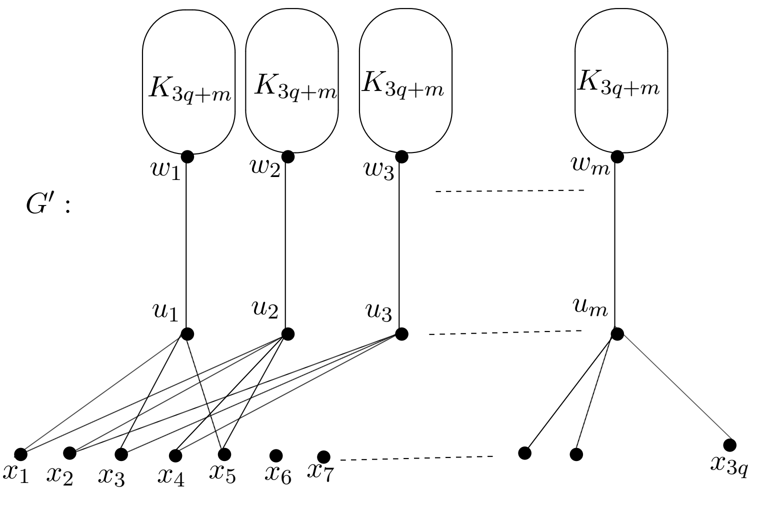
$|S_1| + m(G - S_1) = 2m + 3q + m - 1 = 3m + 3q - 1 > 2m + 4q - 1$, if $m > q$.

Therefore, if $m > q$, then S_1 is not a DI-set of G. Suppose $m = q$. In this case, C is an exact cover of X. Then $S = \{w_1, w_2, \cdots, w_m, u_1, u_2, \cdots, u_m\}$ is the unique dominating set of G and $DI(G) = 2m + 3q + m - 1 = 6m - 1 = 6q - 1$.

In any case, any DI-set of G contains a subset T of $\{u_1, u_2, \cdots, u_m\}$ such that the corresponding subset of C is an exact cover of X.

Conversely, if C' is an exact cover of X and T' is the corresponding subset of $\{u_1, u_2, \cdots, u_m\}$, then $\{w_1, w_2, \cdots, w_m\} \cup T'$ is a DI-set of G. Also, $DI(G) = 4q + 2m - 1$. Hence, DIG is NP-complete in the class of chordal graphs. ■

Illustration 2.2.



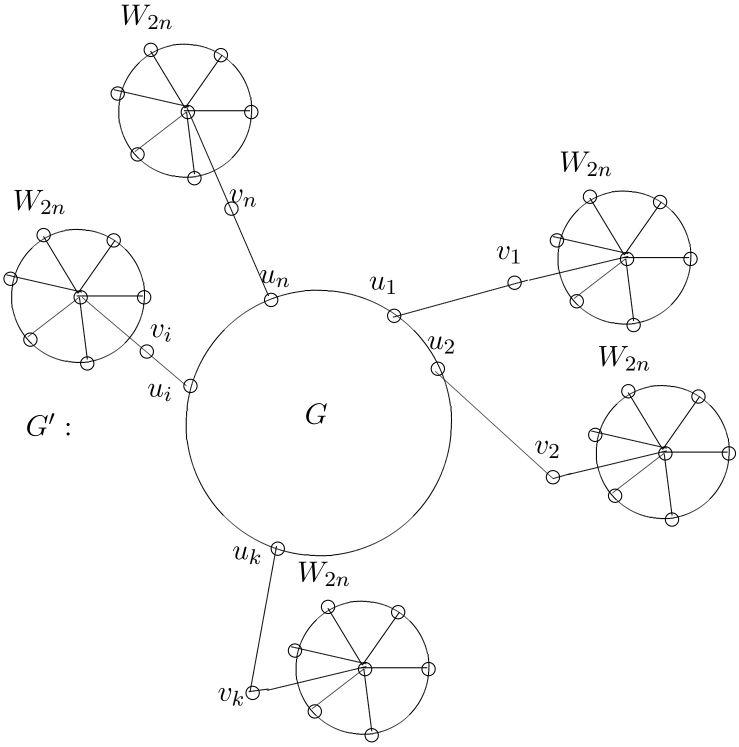
$DI(G') = \gamma(G) + 3n$.

Theorem 2.3. *The decision problem DIG is NP-complete even in the class of planar graphs.*

Proof:

Input: A planar graph $G = (V, E)$ and an integer k.
Question: Is $DI(G) \leq k$?.
Given an instance of DOMINATING SET of a planar graph of maximum vertex degree at most 3, construct a planar graph G' with $2n^2 + 3n$ vertices such that $DI(G') = \gamma(G) + 3n$. Then it is immediate that, DIG is NP-complete for planar graphs.
Let G' be a planar graph with $\Delta(G) \leq 3$. Construct a graph G' from G as follows:
Let $V(G) = \{u_1, u_2, \cdots, u_n\}$. Attach pendent vertices u_i's at each u_i, $1 \leq i \leq n$. For each i, add a Wheel with $2n + 1$ vertices and join u_i' with the center of the Wheel. Then $|V(G')| = 2n^2 + 3n$, $|E(G')| = |E(G)| + 2n + 4n^2$. Clearly, G' is a planar graph. Also, $DI(G') = \gamma(G) + n + 2n = \gamma(G) + 3n$. The domination problem remains NP-complete for planar graphs G with $\Delta(G) \leq 3$ [**?**]. Hence DIG is NP-complete in planar graphs.

Illustration 2.3.



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