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TRIVIALLY EXTENDABLE GRAPHS

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Abstract. Let G be a simple graph. Let k be a positive integer. G is said to be k-extendable if every independent set of cardinality k is contained in a maximum independent set of G. G is said to be trivially extendable if G is not k-extendable for $1 \leq k \leq (\beta_0(G) - 1)$. A well covered graph is one in which every maximal independent set is maximum. Study of k-extendable graphs has been made in [7,8,9]. In this paper a study of trivially extendable graphs is made. Characterization of graphs with $\beta_0(G) = (n-3)$ and which is trivially extendable has been done. Similarly graphs with $\beta_0(G) = (n-2)$ is also studied for trivial extensibility.
Keywords: Berge graph, Extensibility in graphs, Trivially extendable graphs

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1. Introduction

Extendable graphs are those for which all independent sets of some cardinality are contained in maximum independent sets. The well covered graph is k-extendable for every k. That is any independent set of cardinality k, $1 \leq k \leq \beta_0(G)$ is contained in maximum independent sets. There are graphs which are just the opposite of well covered graphs. That is G is not k-extendable for any $k < \beta_0(G)$, these graphs are called trivially extendable graphs. In this paper, a study of trivially extendable graphs are made. Characterization of graphs which are trivially extendable with specific values of $\beta_0(G)$ are done.

2. Trivially Extendable graphs

Definition 2.1. *Let $G = (V, E)$ be a simple graph. Let k be a positive integer. G is said to be k-extendable if every independent set of cardinality k in G is contained in a maximum independent set of G.*

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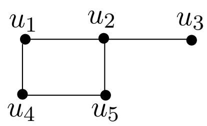
Definition 2.2. *G is said to be trivially extendable if G is k-extendable only for $k = \beta_0(G)$.*

Example 2.1. *Let G :*



$\beta_0(G) = 3$. $\{u_1, u_3, u_5\}$ is the unique β_0-set of G. Clearly $\{u_4\}$, $\{u_2, u_4\}$ are not contained in β_0-set of G. So G is trivially extendable graph.

Theorem 2.1. *Let G be a graph in which every vertex supports at least two pendent vertices. Then G is trivially extendable.*

Remark 2.1. *There are graphs which are trivially extendable but do not contain pendent vertices, for example the Wheel W_7.*

Theorem 2.2. *Let G be a graph with $\beta_0(G) = 2$. G is trivially extendable if and only if G has a full degree vertex (or) $|V(G)| = 2$.*

Proof: Suppose $\beta_0(G) = 2$. Suppose $|V(G)| = 2$. Then $G = \overline{K_2}$ and G is trivially extendable. Suppose $|V(G)| \geq 3$ and G is trivially extendable. Then there exists a vertex $u \in V(G)$ such that u is adjacent to every vertex of G. Therefore u is a full degree vertex of G. Conversely, suppose G has a full degree vertex say u. Then $\{u\}$ is not extendable. Therefore G is trivially extendable.

Theorem 2.3. *Let G be a graph with $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $V - S = \{u_{n-1}, u_n\}$.*
(i).If u_{n-1} and u_n are not adjacent, then G is trivially extendable if and only if both u_{n-1} and u_n are of degree less than 3 and either one of u_{n-1}, u_n is of degree 2 with $N(u_n) - N(u_{n-1}) \neq \phi$, if $deg(u_{n-1}) = 2$ ((or) $N(u_{n-1}) - N(u_n) \neq \phi$, if $deg(u_n) = 2$) (or) both are of degree 2.
(ii).If u_{n-1} and u_n are adjacent, then G is trivially extendable if and only if u_{n-1} and u_n are adjacent to exactly 2 vertices from S.

Proof: (i).Let $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $V - S = \{u_{n-1}, u_n\}$. u_{n-1} and u_n are not adjacent. Then G is bipartite.
Case(1): u_{n-1} has degree 2 and $N(u_n) - N(u_{n-1}) \neq \phi$. Without loss of generality, let u_{n-1} be adjacent to u_1 and u_2. Let $1 \leq k \leq (n-3)$. Let $S_1 = \{u_{n-1}, u_3, \ldots, u_{k+1}\}$. S_1 is independent and $|S_1| = k$. Suppose S_1 is contained in a maximum independent set say T. Then $u_1, u_2 \notin T$. Since $(V - S_1) = \{u_n, u_1, u_2, u_{k+2}, \ldots, u_{n-2}\}$, $|V - S_1| = (n-2) - (k+1) + 2 + 1 = n - k$. Let u_n be adjacent with some $u_j, j \geq 3$.
Subcase(1): T contains u_{n-2}.
Then $u_j \notin T$. Already $u_1, u_2 \notin T$. Therefore $|T| \leq (n-3)$, a contradiction.
Subcase(2): T does not contain u_{n-2}.
Then $u_1, u_2, u_{n-2} \notin T$. Therefore $|T| \leq (n-3)$, a contradiction. Hence S is not contained in any maximum independent set of G. But S is independent and cardinality of k,
$1 \leq k \leq (n-3)$. Therefore G is trivially extendable. Similar proof if u_n has degree 2 and $N(u_{n-1}) - N(u_n) \neq \phi$.
Case(2): u_{n-1} (or) u_n have degree are greater than (or) equal to 3.
Suppose u_{n-1} have degree≥ 3. Let T be an independent set of cardinality k,
$1 \leq k \leq (n-t)$, where $|N(u_{n-1})| = t$. Then T is not extendable. Let S_2 be an independent set of cardinality $(n-t)+1$. Let u_{n-1} (or) $u_n \in S_2$. S_2 can contain at most $(n-2)-t$ vertices

from $\{u_2, u_3, \ldots, u_{n-2}\}$. Therefore $|S_2| \leq (n-2) - t + 2 = n - t$. But $|S_2| = (n-t) + 1$, a contradiction. Therefore there exists no independent set of cardinality $(n-t)+1$ containing u_{n-1} (or) u_n, which implies any independent set of cardinality $(n-t)+1$ is contained in the maximum independent set $\{u_1, u_2, \ldots, u_{n-2}\}$. Hence G is k-extendable for all $k \geq (n-t)+1$. Therefore G is k-extendable exactly for $(n-t)+1 \leq k \leq (n-2)$, which means G is not trivially extendable.

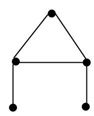
Conversely, suppose G is trivially extendable with $\beta_0(G) = (n-2)$. Then there exists an independent set say T of cardinality k, $1 \leq k \leq (n-3)$ which is not contained in any β_0-set of G. Since any set of $(n-3)$ elements in S is independent, T contains either u_{n-1} (or) u_n. If u_{n-1} and u_n have degree 1, then every $(n-1)$ subset of G is independent, a contradiction. Therefore $deg(u_{n-1})$ (or) $deg(u_n)$ is greater than or equal to 2. suppose $deg(u_{n-1})$ and $deg(u_n)$ are greater than or equal to 3. If $deg(u_{n-1})$ (or) $deg(u_n)$ are greater than or equal to 4, then any $(n-3)$-independent set is contained in S and hence extendable, a contradiction. Therefore $deg(u_{n-1})$ and $deg(u_n)$ are less than or equal to 3. If $deg(u_{n-1}) = deg(u_n) = 3$ and $N(u_n) = N(u_{n-1})$ then there exists a $(n-3)$-independent set containing u_{n-1} and u_n which is not contained in β_0-set of G. If $deg(u_{n-1}) = deg(u_n) = 3$ and $N(u_n) \neq N(u_{n-1})$ then any $(n-3)$-independent set cannot contain u_{n-1} (or) u_n. Therefore G is $(n-3)$-extendable, a contradiction. Therefore $deg(u_{n-1})$ (or) $deg(u_n)$ is less than 3. Therefore $deg(u_{n-1})$ (or) $deg(u_n) = 2$. If $N(u_n) = N(u_{n-1})$, then G is 1-extendable, a contradiction. Therefore $N(u_n) - N(u_{n-1}) \neq \phi$ (or) $N(u_{n-1}) - N(u_n) \neq \phi$. That is if $deg(u_{n-1}) = 2$ then $N(u_n) - N(u_{n-1}) \neq \phi$ and if $deg(u_n) = 2$ then $N(u_{n-1}) - N(u_n) \neq \phi$.

(ii) is analogous to that of (i).

Remark 2.2. *Let G be a graph for which $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $(V-S) = \{u_{n-1}, u_n\}$. Suppose u_{n-1} and u_n are independent. Then G is trivially extendable if and only if G is either $P_3 \cup K_2 \cup (n-5)K_1$ (or) $P_5 \cup (n-5)K_1$ (or) $2P_3 \cup (n-6)K_1$.*

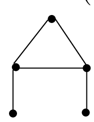
Remark 2.3. *Let G be a graph and let $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $(V-S) = \{u_{n-1}, u_n\}$. Suppose u_{n-1} and u_n are independent. If G is connected then G is trivially extendable if and only if $G = P_5$.*

Remark 2.4. *Let G be a graph and let $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $(V-S) = \{u_{n-1}, u_n\}$. Suppose u_{n-1} and u_n are not independent. Then G is trivially extendable if and only if G is either $(K_4 - e) \cup (n-4)K_1$ (or) $D_{2,2} \cup (n-6)K_1$ (or) $G \cup (n-5)K_1$, where G is*



Remark 2.5.

Let G be a connected graph with $\beta_0(G) = (n-2)$. Let $S = \{u_1, u_2, \ldots, u_{n-2}\}$ be a β_0-set of G. Let $V - S = \{u_{n-1}, u_n\}$. Suppose u_{n-1} and u_n are not independent. Then G is trivially extendable if and only if G is either $(K_4 - e)$ (or) $D_{2,2}$ (or) G is



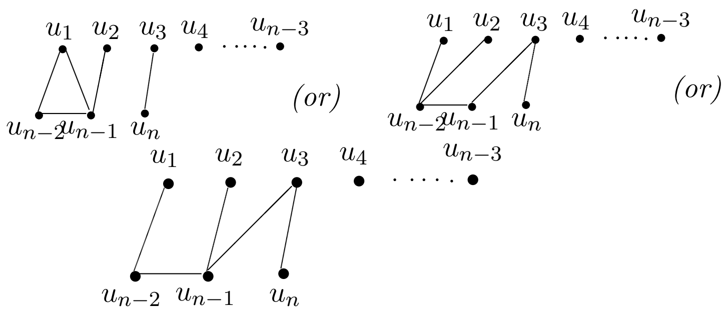
Remark 2.6.
Let G be a graph. If $\beta_0(G) = (n-1)$, then G is k-extendable for all k, $2 \leq k \leq (n-1)$.

The following theorem gives a characterization of graphs which are trivially extendable with $\beta_0(G) = (n-3)$. The proof is lengthy and hence omitted.

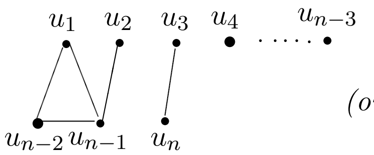
Theorem 2.4. *Let G be a simple graph of order $n \geq 7$ with $\beta_0(G) = (n-3)$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be a β_0-set of G, $V - S = \{u_{n-2}, u_{n-1}, u_n\}$.*
(i). If u_{n-2}, u_{n-1}, u_n are independent then G is trivially extendable if and only if $|N(u_{n-2}) \cup N(u_{n-1}) \cup N(u_n)| = 4$.
(ii).(a). Let u_{n-2} and u_{n-1} be adjacent and u_n be not adjacent with u_{n-2} and u_{n-1} then G is trivially extendable if and only if $|(N(u_{n-1}) \cup N(u_n)) \cap S|$ (or) $|(N(u_{n-2}) \cup N(u_n)) \cap S|$ is equal to 3.
(b). Let the remaining u_{n-2}, u_{n-1} and u_n form P_3 with u_n adjacent with u_{n-2} and u_{n-1}. Then G is trivially extendable if and only if either $|N(u_n) \cap S| = 2$ (or) $|(N(u_{n-2}) \cup N(u_{n-1})) \cap S| = 3$.
(c). Let the remaining u_{n-2}, u_{n-1} and u_n form K_3. Then G is trivially extendable if and only if at least $|N(u_n) \cap S|$, $|N(u_{n-1}) \cap S|$, $|N(u_{n-2}) \cap S|$ is equal to 2.

Remark 2.7. *Let G be a simple graph of order $n \geq 7$ with $\beta_0(G) = (n-3)$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be a β_0-set of G, $V - S$ be independent. G is trivially extendable if and only if G is $P_3 \cup 2K_2 \cup (n-7)K_1$.*

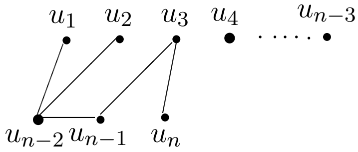
Remark 2.8. *Let G be a simple graph of order $n \geq 6$ with $\beta_0(G) = (n-3)$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be a β_0-set of G, $V - S = \{u_{n-2}, u_{n-1}, u_n\}$ Let $\langle V - S \rangle = K_2 \cup K_1$. Then G is trivially extendable if and only if G is $K_3 \cup P_3 \cup (n-6)K_1$ (or) $P_6 \cup (n-6)K_1$ (or) G is any one of the following graph*



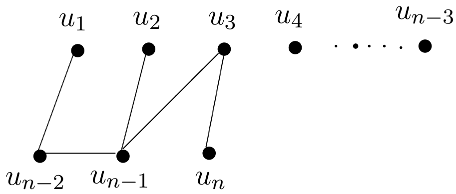
Proof: *Since $\beta_0(G) = (n-3)$. Each of u_{n-2}, u_{n-1}, u_n has at least one neighbour in S. G is trivially extendable if and only if $|(N(u_{n-1}) \cup N(u_n)) \cap S| = 3$ (or) $|(N(u_{n-2}) \cup N(u_n)) \cap S| = 3$. Therefore either u_{n-1} and u_{n-2} have exactly one neighbour each in S and u_n has 2 neighbour in S (or) u_{n-1} and u_n have one neighbour each in S and u_{n-2} has 2 neighbours in S (or) u_{n-2} and u_n have one neighbour each in S and u_{n-1} has 2 neighbours in S. Suppose u_{n-1} and u_{n-2} have distinct neighbours in S. Then u_n has one of the neighbours coincident with the neighbour of u_{n-1} (or) the neighbour of u_{n-2}. In this case $G = P_6 \cup (n-6)K_1$. Suppose u_{n-1} and u_{n-2} have the same neighbour in S. Then u_n has 2 neighbours in S which are distinct from the neighbour of u_{n-1}. In this case $G = K_3 \cup P_3 \cup (n-6)K_1$. Suppose u_{n-1} has 2 neighbours in S. Then either u_n has a neighbour distinct from the 2 neighbour of u_{n-1} in S (or) u_n has a neighbour in S which is also a neighbour of u_{n-1}. In the former case, either the neighbour of u_{n-2} is coincident with a neighbour of u_{n-1} (or) coincident with the neighbour of u_n in S. Therefore G is one of the following graph*



(or)

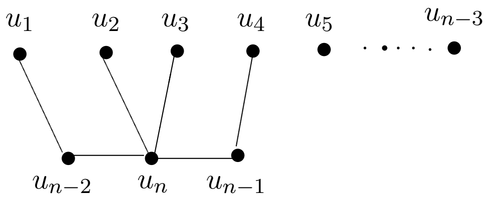


In the latter case, u_{n-2} has a neighbour distinct from the neighbour of u_{n-1}. Therefore G is

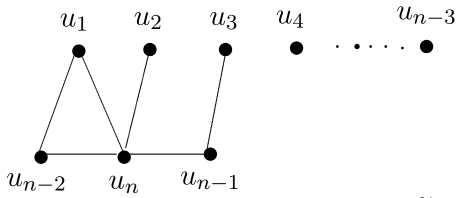


Remark 2.9. *Let G be a simple graph of order $n \geq 6$ with $\beta_0(G) = (n-3)$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be a β_0-set of G, $V - S = \{u_{n-2}, u_{n-1}, u_n\}$. Let u_{n-2}, u_n, u_{n-1} form a P_3 with u_n being adjacent to both u_{n-1} and u_{n-2}. Then G is trivially extendable if and only if G is one of the following graphs*

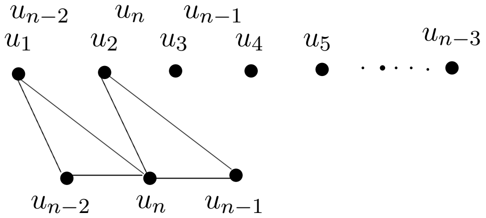
(i)



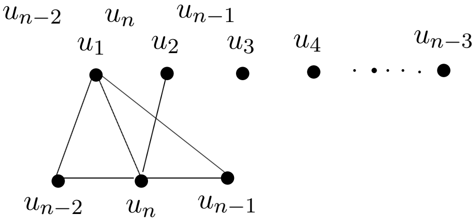
(ii)



(iii),



(iv),



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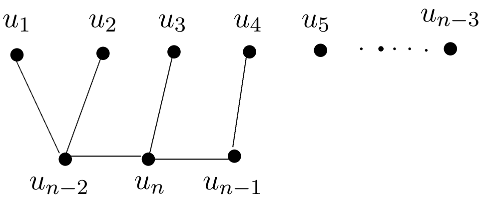

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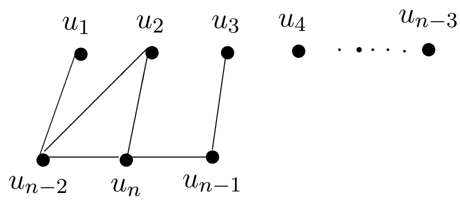

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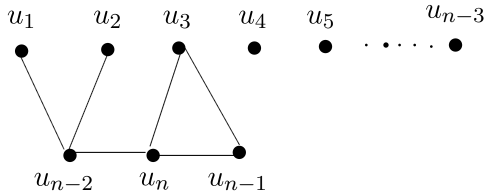
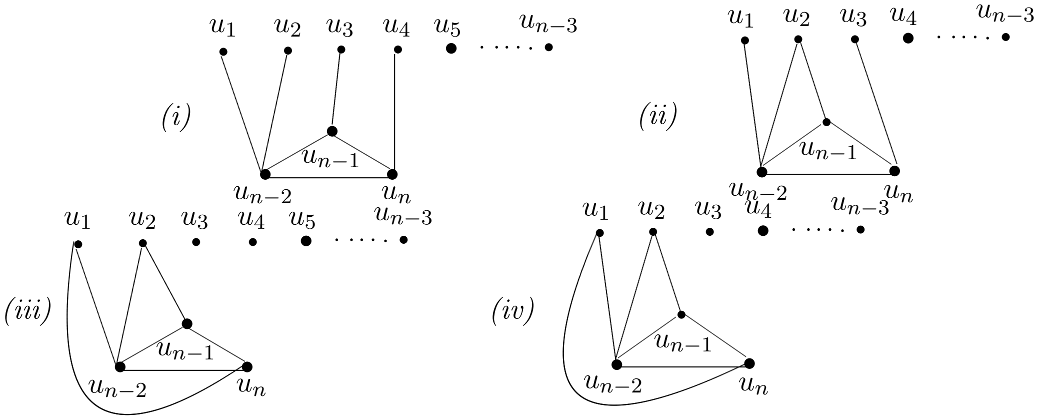


Proof follows from the fact that G is trivially extendable if and only if $|N(u_n) \cap S| = 2$ (or) $|(N(u_{n-1}) \cup N(u_{n-2})) \cap S| = 3$ and u_{n-2}, u_{n-1}, u_n has at least one neighbour in S.

Remark 2.10. *Let G be a simple graph of order $n \geq 6$ with $\beta_0(G) = (n-3)$. Let $S = \{u_1, u_2, \ldots, u_{n-3}\}$ be a β_0-set of G, $V - S = \{u_{n-2}, u_{n-1}, u_n\}$. Let u_{n-2}, u_n, u_{n-1} form a K_3. Then G is trivially extendable if and only if G is one of the following graphs*



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