TWMS J. App. Eng. Math. V.8, N.1, 2018, pp. 144-154

SOLUTIONS FOR A DISCRETE BOUNDARY VALUE PROBLEM INVOLVING KIRCHHOFF TYPE EQUATION VIA VARIATIONAL METHODS

ZEHRA YÜCEDAG[1]

Abstract. In this paper, Mountain Pass theorem is applied together with Ekeland variational principle, and we show the existence of nontrivial solutions for a discrete boundary value problem of $p(k)$-Kirchhoff-type in a finite dimensional Hilbert space.

Keywords: Kirchhoff type equation, discrete boundary value problem, Varitional methods, Ekeland variational principle, Mountain Pass theorem.

AMS Subject Classification: 47A75, 35B38, 35P30, 34L05, 35A15, 39A10, 39A70.

1. Introduction

1. Introduction

We are concerned with the following problem

$$\begin{cases} -M\left(A(k-1,\Delta u\left(k-1\right))\right)\ \Delta\left(a(k-1,\triangle u\left(k-1\right))\right)=\lambda f\left(k,u\left(k\right)\right), \\ u\left(0\right)=u\left(T+1\right)=0, \qquad k\in\mathbb{Z}\left[1,T\right], \end{cases} \qquad (\mathbf{P})$$

where $T\geq 2$ is a positive integer, $\mathbb{Z}\left[a,b\right]$ denotes the discrete interval $\{a,a+1,...,b\}$ with a and b are integers such that $0<a<b$; $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator; $f:\mathbb{Z}\left[1,T\right]\times\mathbb{R}\to(0,+\infty)$ is a continuous function; λ is a positive constant. Moreover, we assume that the function $a(k,\xi):\mathbb{Z}\left[1,T\right]\times\mathbb{R}\to\mathbb{R}$ is continuous derivative with respect to ξ of the mapping $A:\mathbb{Z}\left[1,T\right]\times\mathbb{R}\to\mathbb{R}$, $A=A(k,\xi)$, i.e. $a(k,\xi)=\nabla_{\xi}A(k,\xi)$; $p:\mathbb{Z}\left[0,T\right]\to[2,\infty)$ satisfies

$$p^{-}=\min_{k\in\mathbb{Z}[0,T]}p\left(k\right)\leq p^{+}=\max_{k\in\mathbb{Z}[0,T]}p\left(k\right)<\infty.$$

We suppose that f, M,a and A satisfy the following conditions:

$(\mathbf{M}_1)$ $M:(0,+\infty)\to(0,+\infty)$ is continuous function such that

$$\left(1-\eta\right)s^{\alpha-1}\leq M(s)\leq\left(1+\eta\right)s^{\alpha-1},$$

for all $s>0$, $0\leq\eta<1$ and $\alpha\geq 1$.

$(\mathbf{f}_0)$ There exists a function $q(k):\mathbb{Z}\left[1,T\right]\to[2,\infty)$ such that

$$\left|f(k,t)\right|\leq c_0\left(1+\left|t\right|^{q(k)-1}\right),$$

for all $(k,t)\in\mathbb{Z}\left[1,T\right]\times\mathbb{R}$, where c_0 is pozitif constant.

[1] Dicle University, Faculty of Science, Department of Mathematics, 21280 Diyarbakir, Turkey.
e-mail: zyucedag@dicle.edu.tr; ORCID: http://orcid.org/0000-0003-1950-0163.

§ Manuscript received: June 9, 2017; accepted: August 24, 2017.
TWMS Journal of Applied and Engineering Mathematics, Vol.8, No.1

(**A1**) There exists a constant c_0 such that

$$|a(k,\xi)| \leq c_0(1+|\xi|^{p(k)-1}),$$

for all $(k,\xi) \in \mathbb{Z}[0,T] \times \mathbb{R}$

(**A2**) The following inequalitiy holds true

$$|\xi|^{p(k)} \leq a(k,\xi) \cdot \xi \leq p(k)\, A(k,\xi),$$

for all $(k,\xi) \in \mathbb{Z}[0,T] \times \mathbb{R}$.

(**A3**) The following inequality holds true

$$(a(k,\xi) - a(k,\eta)) \cdot (\xi - \eta) > 0,$$

for all $k \in \mathbb{Z}[0,T]$ and $\xi, \eta \in \mathbb{R}$ such that $\xi \neq \eta$.

(**A4**) $A(k,0) = 0$, for all $k \in \mathbb{Z}[0,T]$.

Problem (P) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [20]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

Discrete boundary value problems have been extensively studied by applying variational methods in the last few years because they can be used to models various phenomena arising from the study of elastic mechanics [30], electrorheological fluids [26] and image restoration [9]. We refer to the recent results of involving the discrete $p-$Laplacian operator and $p(k)-$ Laplacian operator [6, 7, 8, **?**, 19, 22, 29]. The discrete problems of type (**P**) involving anisotropic exponents have first been discussed by [24, 21]. Moreover, Koné and Guiro studied a more general operator in [21, 13]. In [24], by using critical point theory, the authors showed the existence of a continuous spectrum of eigenvalues for a discrete anisotropic problem. In [21], using minimization method, Koné and Ouaro obtained the existence and uniqueness of weak solutions for anisotropic discrete boundary value problems. Then, the authors studied the existence and multiplicity of positive solutions for a discret anisotropic equation by variational methods and a critical point theory in [11, 28]. Moreover, many interesting results are obtained see for examples, in [2, 3, 4, 5, 12, 15, 16, 17, 18, 25]

In this paper, applying Mountain-Pass theorem together with Ekeland variational principle of Ambrosetti-Rabinovitz's (see [24]), we obtain the existence of at least one nontrivial weak solution of an anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, we get some existence results for (**P**).

2.Preliminaries

Let us define the function space

$$W = \{u : \mathbb{Z}[0, 1+T] \to \mathbb{R};\ \text{such that}\ \ u(0) = u(T+1) = 0\}.$$

Then, W is a T-dimensional Hilbert space [1] with the inner product

$$\langle u,v\rangle=\sum_{k=1}^{T+1}\Delta u(k-1)\Delta v(k-1),\ \forall u,v\in W.$$

Then, the associated norm is defined by

$$\|u\|=\left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2\right)^{1/2}.$$

On the other hand, it is useful to introduce other equivalent norms on W, namely

$$|u|_m=\left(\sum_{k=1}^{T}|u(k)|^m\right)^{1/m},\ \forall u\in W \text{ and } m\geq 2.$$

It can be verified [7, 24] that

$$T^{2-m/(2m)}\,|u|_2\leq |u|_m\leq T^{1/m}\,|u|_2\,,\ \forall u\in W \text{ and } m\geq 2. \tag{2.1}$$

Lemma 2.1[24]
(*i*) *For every* $u\in W$ *with* $\|u\|>1$,we have

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}\geq T^{\left(2-p^-\right)/2}\,\|u\|^{p^-}-(T+1)\,.$$

(*ii*) *For every* $u\in W$ *with* $\|u\|\leq 1$,we have

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}\geq T^{\left(p^+-2\right)/2}\,\|u\|^{p^+}\,.$$

(*iii*) *For any* $m\geq 2$ *there exist a positive constant* c_m *such that*

$$\sum_{k=1}^{T}|u(k)|^m\leq c_m\sum_{k=1}^{T+1}|\Delta u(k-1)|^m\,,\ \forall u\in W.$$

Furthermore, from (2.1) and Lemma 2.1(*iii*), it reads

$$|u|_m^m\leq T\,|u|_2^m\leq c_mT\left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2\right)^{\frac{m}{2}}=c_mT\,\|u\|^m\,. \tag{2.2}$$

Definition 2.2 *Let* X *be a real Banach spaces and let* I_λ *be a functional such that* $I_\lambda\in C\,(X,\mathbb{R})$. *We say that* I_λ *satisfies Palais-Smale condition* ((PS) *condition for short) if any sequence* $\{u_n\}$ *in* X *such that* $\{I_\lambda\,(u_n)\}$ *is bounded and* $I'_\lambda\,(u_n)\to 0$ *as* $n\to\infty$, *has a convergent subsequence.*

Theorem 2.3 (Mountain-Pass lemma)(see [27]**)** *Let* X *be a Banach space and* $I\in C^1\,(X,\mathbb{R})$ *satisfy the Palais-Smale condition. Assume that* $I\,(0)=0$, *and*
(*i*) *There exist two positive real numbers* γ *and* r *such that* $I\,(u)\geq r>0$, *for all* $u\in X$ *with* $\|u\|=\gamma$.
(*ii*) *There exists* $u\in X$ *with* $\|u\|>\gamma$ *such that and* $I_\lambda\,(u)<0$.
Then, I *has a critical value* $\beta\geq\alpha$. *Moreover,* β *can be characterized as*

$$0 < \beta := \inf_{\varphi \in G} \max_{t \in [0,1]} I\left(\varphi\left(t\right)\right),$$

where $G = \{\varphi \in C\left([0,1], E\right) : \varphi\left(0\right) = 0,\ \varphi\left(1\right) = u\}$.

3. The Main results

The energy functional corresponding to problem $(\mathbf{P})$ is defined as $I_\lambda : W \to \mathbb{R}$,

$$I_\lambda\left(u\right) = \widehat{M}\left(\sum_{k=1}^{T+1} A\left(k-1, \Delta u\left(k-1\right)\right)\right) - \lambda \sum_{k=1}^{T} F\left(k, u\left(k\right)\right),$$

where $k \in \mathbb{Z}\left[1, T\right]$, $\widehat{M}\left(t\right) = \int_0^t M\left(s\right) ds$ and $F\left(k, t\right) = \int_0^t f\left(k, \xi\right) d\xi$.

A critical point to I_λ is a point $u \in W$ such that

$$\begin{aligned} & M\left(\sum_{k=1}^{T+1} A\left(k-1, \Delta u\left(k-1\right)\right)\right) \sum_{k=1}^{T+1} a\left(k-1, \Delta u\left(k-1\right)\right) \triangle \upsilon\left(k-1\right) \\ = \ & \lambda \sum_{k=1}^{T} f\left(k, u\left(k\right)\right) \upsilon\left(k\right),\ \forall k \in \mathbb{Z}\left[1, T\right] \end{aligned}$$

which in turn is a weak solution to $(\mathbf{P})$ for any $\upsilon \in W$. Since we work in a finite dimensional space, we see that any weak solution of $(\mathbf{P})$ is in fact a strong, i.e. classical solution. Hence, in order to solve $(\mathbf{P})$, we need to find critical points of I_λ .

Lemma 3.1
(i) The functional I_λ is well defined on W.
(ii) The functional I_λ is *of class $C^1(W, \mathbb{R})$ and*

$$\begin{aligned} \left\langle I_\lambda'(u), \upsilon \right\rangle &= M\left(\sum_{k=1}^{T+1} A\left(k-1, \Delta u\left(k-1\right)\right)\right) \sum_{k=1}^{T+1} a\left(k-1, \Delta u\left(k-1\right)\right) \triangle \upsilon\left(k-1\right) \\ & \quad -\lambda \sum_{k=1}^{T} f\left(k, u\left(k\right)\right) \upsilon\left(k\right), \forall k \in \mathbb{Z}\left[1, T\right], \end{aligned}$$

for all $\upsilon, \upsilon \in W$.

Since the proof of Lemma 3.2 is very similar to that of the proof of Lemma 3.4 in [13], we omit it.

Lemma 3.2 [23] *A varifies the following condition*

$$A\left(k, t\xi\right) \leq A\left(k, \xi\right) t^{p(k)},\ \text{for all } \left(k, t\right) \in \mathbb{Z}\left[0, T\right] \times \mathbb{R}, t \geq 1.$$

Theorem 3.3 *Assume that (M_1) and $(\mathbf{f}_0)$ hold with $q^+ < \alpha p^-$. Then, there exists $\lambda_{**} > 0$ such that for any $\lambda \in (0, \lambda_{**})$ the problem $(\mathbf{P})$ has at least one nontrivial solution.*

Proof. Let $\|u\| > 1$.Using the condition $(\mathbf{M}_1)$, we have

$$
\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\sum_{k=1}^{T+1} A\left(k-1, \Delta u\left(k-1\right)\right)\right) - \lambda \sum_{k=1}^{T} F\left(k, u\left(k\right)\right) \qquad (3.1) \\
&\geq (1-\eta) \int\limits_{0}^{\sum\limits_{k=1}^{T+1} A(k-1,\Delta u(k-1))} s^{\alpha-1} ds - \lambda c_0 \sum_{k=1}^{T} \frac{\left|u\left(k\right)\right|^{q(k)}}{q\left(k\right)}.
\end{aligned}
$$

If we use the inequalities $|u(k)|^{q(k)} \leq |u(k)|^{q^-} + |u(k)|^{q^+}$ for $\forall k \in \mathbb{Z}\left[1, T\right], u \in W$, Lemma 2.1(iii) and relation (2.2), we obtain

$$
\begin{aligned}
|u(k)|^{q^-} + |u(k)|^{q^+} &\leq c_{q^+} \sum_{k=1}^{T+1} \left|\Delta u\left(k-1\right)\right|^{q^+} + c_{q^-} \sum_{k=1}^{T+1} \left|\Delta u\left(k-1\right)\right|^{q^-} \qquad (3.2) \\
&\leq c_{q^+} T \left\|u\right\|^{q^+} + c_{q^-} T \left\|u\right\|^{q^-}.
\end{aligned}
$$

For $u \in W$ with $\|u\| > 1$, the above inequalities combined with (3.1) ,Lemma 2.1(i) , $(\mathbf{f}_0)$ and relation $(\mathbf{A2})$, imply that

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{1-\eta}{\alpha} \left(\sum_{k=1}^{T+1} A\left(k-1, \Delta u\left(k-1\right)\right)\right)^{\alpha} - \frac{\lambda c_{q^+}}{q^-} T \left\|u\right\|^{q^+} \\
&\geq \frac{1-\eta}{\alpha \left(p^+\right)^{\alpha}} T^{\left(\frac{2-p^-}{2}\right)^{\alpha}} \left\|u\right\|^{\alpha p^-} - \frac{\lambda c_{q^+}}{q^-} T \left\|u\right\|^{q^+}.
\end{aligned}
$$

Thus, if we choose as

$$
\lambda_{**} = \frac{(1-\eta) q^- T^{\left(\frac{2-p^-}{2}\right)^{\alpha}}}{\alpha \left(p^+\right)^{\alpha} c_{q^+} T}
$$

then for any $\lambda \in (0, \lambda_{**})$ and $q^+ < \alpha p^-$, I_λ is coercive.

A continuous coercive functional on a finite dimensional space is bounded from below. On the other hand, it is obvious that it is also weakly lower semicontinuous on the finite dimensional Hilbert space W. Therefore, $(\mathbf{P})$ has at least one nontrivial solution in W.

Example 3.1 As examples of a and A functions satisfying assumptions $(\mathbf{A2})$-$(\mathbf{A4})$, we can give the following.

If we take

$$
a(k, \xi) = |\xi|^{p(k)-2} \xi, \text{ for all } (k, \xi) \in \mathbb{Z}\left[1, T\right] \times \mathbb{R}
$$

we have

$$
A(k, \xi) = \frac{1}{p\left(k\right)} \left|\xi\right|^{p(k)}.
$$

On the other hand, if we let

$$
a(k, \xi) = \left(1 + |\xi|^2\right)^{(p(k)-2)/2} \xi, \text{ for all } (k, \xi) \in \mathbb{Z}\left[1, T\right] \times \mathbb{R}
$$

then we have

$$
A(k, \xi) = \frac{1}{p\left(k\right)} \left[\left(1 + |\xi|^2\right)^{p(k)/2} - 1\right].
$$

Example 3.2 As an example of application of the Theorem 3.3, we consider the following: Let

$f(k,t) = |t|^{k+2}$, $p(k-1) = e^{k-1}+5, q(k) = k+3, M(t) = t, T = 2, \alpha = 2$, $c_{q^+} = 1$ and $\eta = 1/2$,

then we have

$$F(k,t) = \frac{1}{k+3}|t|^{k+3}, \ p^- = 6, \ p^+ = e+5, \ q^- = 4, \ q^+ = 5 \text{ and } \lambda_{**} = 0,134291561.$$

Thus, all the assumptions requested in Theorem 3.3 are provided.

Theorem 3.4 *Assume that* (M_1),$(\mathbf{f}_0)$ *hold. Suppose, additionally, that the following conditions hold:*

$(\mathbf{f}_1)$: $f(k,t) = o(|t|^{\alpha p^+ - 1})$, $t \to 0$,*with* $\alpha p^+ < q^-$ *for* $\forall k \in \mathbb{Z}[1,T]$*;*

$(\mathbf{AR})$: *Ambrosetti-Rabinovitz's condition holds, i.e. There exists constants* $\exists t_* > 0$, $0 \leq \eta < 1$

$$\theta > \frac{1+\eta}{1-\eta}\alpha p^+$$

such that

$$0 < \theta F(k,t) \leq f(k,t)t, \ \ |t| \geq t_*, \ \forall k \in \mathbb{Z}[1,T].$$

Then, the problem $(\mathbf{P})$ *has at least a nontrivial solution for any* $\lambda > 0$.

Lemma 3.5 *Assume that the assumptions of Theorem 3.4 hold. Then,*

(i) *There exist two positive real numbers* γ *and* r *such that* $I_\lambda(u) \geq r > 0$, $u \in W$ *with* $\|u\| = \gamma$.

(ii) *There exists* $u \in W$ *such that* $\|u\| > \gamma$ *and* $I_\lambda(u) < 0$.

Proof. (i) Let $\|u\| < 1$. Then, using condition $(\mathbf{M}_1)$, $(\mathbf{A2})$, $(\mathbf{A4})$ and relation (2.2), we have

$$\begin{aligned} I_\lambda(u) &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1))\right) - \lambda\sum_{k=1}^{T} F(k, u(k)) \\ &\geq \frac{(1-\eta)}{\alpha(p^+)^\alpha} T^{(\frac{2-p^-}{2})^\alpha} \|u\|^{\alpha p^+} - \lambda\sum_{k=1}^{T} F(k, u(k)). \end{aligned}$$

From $(\mathbf{f}_0)$ and $(\mathbf{f}_1)$, we can write the following inequality

$$F(k,t) \leq \varepsilon |t|^{\alpha p^+} + c_1 |t|^{q(k)},$$

where c_1 is pozitif constant and $t \in \mathbb{R}$. Let $\varepsilon > 0$ be small enough such that $\lambda\varepsilon c_{\alpha p^+} T \leq \frac{1-\eta}{2\alpha(p^+)^\alpha} T^{(\frac{2-p^-}{2})^\alpha}$. Thus, considering also inequality (2.2), (3.2) and Lemma 2.1(ii) we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{(1-\eta)}{\alpha(p^+)^\alpha} T^{(\frac{2-p^-}{2})^\alpha} \|u\|^{\alpha p^+} - \lambda\varepsilon \sum_{k=1}^{T} |u(k)|^{\alpha p^+} - \lambda c_1 \sum_{k=1}^{T} |u(k)|^{q(k)} \\ &\geq \frac{(1-\eta)}{\alpha(p^+)^\alpha} T^{(\frac{2-p^-}{2})^\alpha} \|u\|^{\alpha p^+} - \lambda\varepsilon c_{\alpha p^+} T \|u\|^{\alpha p^+} - \lambda c_1 \left(c_{q^+} T \|u\|^{q^+} + c_{q^-} T \|u\|^{q^-}\right) \\ &\geq \frac{1-\eta}{2\alpha(p^+)^\alpha} T^{(\frac{2-p^-}{2})^\alpha} \|u\|^{\alpha p^+} - c_2 c_{q^-} T \|u\|^{q^-}. \end{aligned}$$

Since$\|u\| < 1$ and $\alpha p^+ < q^-$, there exist two positive real numbers γ and r such that $I_\lambda(u) \geq r > 0$, $u \in W$ with $\|u\| = \gamma \in (0,1)$.

(ii) From $(\mathbf{AR})$ and for each $t \geq 1$, we can write

$$F(k, tu) \geq t^\theta F(k,u), \ \forall k \in \mathbb{Z}[1,T].$$

Thus, for $\psi \in W, \psi \neq 0$ and $t > 1$, we have

$$\begin{aligned} I_\lambda(t\psi) &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta t\varphi(k-1))\right) - \lambda \sum_{k=1}^{T} F(k, t\varphi(k)) \\ &\leq \frac{1+\eta}{\alpha(p^-)^{\frac{1+\eta}{1-\eta}\alpha p^+}} t^{\frac{1+\eta}{1-\eta}\alpha p^+} \left(\sum_{k=1}^{T+1} A(k-1, \Delta\varphi(k-1))\right)^\alpha - \lambda c_3 t^\theta \sum_{k=1}^{T} F(k, \varphi(k)). \end{aligned}$$

Since $\theta > \frac{1+\eta}{1-\eta}\alpha p^+$, it can be obtanied that $I_\lambda(t\psi) \to -\infty$ as $t \to +\infty$.

Lemma 3.6 *Assume that the conditions* $(\mathbf{M}_1)$, $(\mathbf{f}_0)$ *and* $(\mathbf{AR})$ *hold with* $\alpha p^+ < q^-$*.Then, for any* $\lambda > 0$ *the functional* I_λ *satisfies Palais-Smale condition.*

Proof. First, we deduce the existence of a sequence $\{u_n\} \subset W$ such that

$$|I_\lambda(u_n)| \leq c \ \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty.$$

We prove that $\{u_n\}$ is bounded in W. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may assume that for n large enough, we have $\|u_n\| > 1$. Moreover, using the conditions $(\mathbf{M}_1)$, $(\mathbf{AR})$, $(\mathbf{f}_0)$, and relations $(\mathbf{A2})$, $(\mathbf{A4})$, we can write

$$\begin{aligned} c + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\theta}\left\langle I'_\lambda(u_n), u_n\right\rangle \\ &\geq (1-\eta) \int_0^{\sum_{k=1}^{T+1} A(k-1,\Delta u_n(k-1))} s^{\alpha-1} ds - \frac{(1+\eta)p^+}{\theta}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1))\right)^\alpha \\ &\quad + \lambda\left(\frac{1}{\theta}\sum_{k=1}^{T} f(k, u_n(k)) u_n(k) - \sum_{k=1}^{T} F(k, u_n(k))\right) \\ &\geq \left(\frac{1-\eta}{\alpha} - \frac{(1+\eta)p^+}{\theta}\right)\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1))\right)^\alpha, \end{aligned}$$

for n large enough. From $(\mathbf{A2})$, $(\mathbf{AR})$ and Lemma 2.1 (i), we obtain

$$c + \|u_n\| \geq \frac{1}{(p^+)^\alpha}\left(\frac{1-\eta}{\alpha} - \frac{(1+\eta)p^+}{\theta}\right) T^{\left(2-p^-\right)/2} \|u\|^{p^-} - K(\alpha, T) T^\alpha.$$

Dividing the last inequality above by $\|u\|^{p^-}$, and passing to the limit as $n \to \infty$,we infer that $\{\|u_n\|\}$ is bounded in W. This information combined with the fact that W is a finite dimensional Hilbert space imply that there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in W$ such that $\{u_n\}$ converges to u_0 in W. Thus, I_λ satisfies (PS) condition.

Proof of Theorem 3.4 From Lemma 3.5, Lemma 3.6 and the fact that $I_\lambda(0) = 0$, I_λ satisfies the conditions of Theorem 2.3. Therefore, the problem $(\mathbf{P})$ has at least one nontrivial solution.

Example 3.3 We consider the function

$$f(k,t) = |t|^{q(k)-2}\, t \quad \text{for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R},$$

From the above definition of f, we get $F(k,t) = \frac{1}{q(k)} |t|^{q(k)}$ .If we take

$$p(k-1) = k+1,\ q(k) = 2(k+1), M(t) = 2t, T = 2, \alpha = 1 \text{ and } \eta = 0,1316309013,$$
$$f(1,t) = t^3 \text{and } f(2,t) = t^5,$$

we obtain

$$p^- = 2,\ p^+ = 3,\ q^- = 4,\ q^+ = 6,\ F(1,t) = t^4/4 \text{ and } F(2,t) = t^6/6.$$

Hence, all the assumptions requested in Theorem 3.4 hold.

Theorem 3.7 *Assume that* (M_1) *and* $(\mathbf{f}_0)$ *hold with* $q^- < \alpha p^-$. *Then, there exists* $\lambda_* > 0$ *such that for any* $\lambda \in (0, \lambda_*)$ *the problem* $(\mathbf{P})$ *has at least one solution.*

Lemma 3.8 *Assume that the assumptions of Theorem 3.7 hold. Then, there exist* η, $a > 0$ *and* $\lambda_* > 0$ *such that for any* $\lambda \in (0, \lambda_*)$, *we have*

$$I_\lambda(u) \geq a > 0,\ \forall u \in W \quad \textit{with} \quad \|u\| = \eta.$$

Proof. Let $\|u\| < 1$. Using conditions $(M_1), (\mathbf{f}_0)$ and relations $(\mathbf{A2})$, $(\mathbf{A4})$, (2.2), we obtain that for $u \in W$ with $\|u\| = \eta$ the following inequalities hold true

$$\begin{aligned} I_\lambda(u) &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1))\right) - \lambda \sum_{k=1}^{T} F(k, u(k)) \\ &\geq \frac{(1-\eta)}{\alpha (p^+)^\alpha} T^{\frac{(p^+-2)\alpha}{2}} \|u\|^{\alpha p^+} - \lambda \frac{c_2 c_{q^-}}{q^-} T \|u\|^{q^-} \qquad (3.3) \\ &= \left(c_3 \rho^{\alpha p^+ - q^-} - \lambda c_4\right) \eta^{q^-}, \end{aligned}$$

where c_3 and c_4 are positive constants. If we use (3.3) and $q^- < \alpha p^- \leq \alpha p^+$, and choose λ_* as

$$\lambda_* = \frac{c_3 \eta^{\alpha p^+ - q^-}}{2c_4},$$

then for any $\lambda \in (0, \lambda_*)$ and $\forall u \in W$ with $\|u\| = \eta$ there exists $a = \frac{c_3 \eta^{\alpha p^+}}{2}$ such that $I_\lambda(u) \geq a > 0$.

Lemma 3.9 *Assume that* (M_1) *and* $(\mathbf{f}_0)$ *hold with* $q^- < \alpha p^-$. *Then, there exists* $\varphi \in W$ *such that* $\varphi \geq 0, \varphi \neq 0$ *and* $I_\lambda(t\varphi) < 0$, *for* $t > 0$ *small enough.*

Proof. For any fixed $\varphi \in W, \varphi \neq 0$ and each $t \in (0,1)$, using conditions $(M_1), (\mathbf{f}_0)$ and Lemma 2.1(ii), we have

$$\begin{aligned} I_\lambda(t\varphi) &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta t\varphi(k-1))\right) - \lambda \sum_{k=1}^{T} F(k, t\varphi(k)) \\ &\leq \frac{1+\eta}{\alpha} t^{\alpha p^-} \left(\sum_{k=1}^{T+1} A(k-1, \Delta\varphi(k-1))\right)^\alpha - \lambda \frac{c_1 t^{q^+}}{q^+} \sum_{k=1}^{T} \frac{|\varphi(k)|^{q(k)}}{m(k)} \\ &\leq \frac{1+\eta}{\alpha} t^{\alpha p^-} \left(\sum_{k=1}^{T+1} A(k-1, \Delta\varphi(k-1))\right)^\alpha - \lambda \frac{c_1 t^{q^-}}{q^-} \sum_{k=1}^{T} |\varphi(k)|^{q(k)}. \end{aligned}$$

Thus,

$$I_\lambda(t\varphi) < 0,$$

for all $t < \delta^{\frac{1}{\alpha p^- - q^-}}$ with

$$0 < \delta < \min\left\{1, \lambda\alpha c_1 \sum_{k=1}^{T} |\varphi(k)|^{m(k)} \Big/ \left((1+\eta) q^- \left(\sum_{k=1}^{T+1} A(k-1, \Delta\varphi(k-1))\right)^{\alpha}\right)\right\}.$$

The proof of of Lemma 3.9 is complete.

Proof of Theorem 3.7 By Lemma 3.8, we infer that there exists a ball centered at the origin $B \subset W$, such that

$$\inf_{\partial B_\rho(0)} I_\lambda > 0.$$

Moreover, from Lemma 3.9, there exists $\varphi \in W$ such that $I_\lambda(t\varphi) < 0$, for all $t > 0$ small enough. Thus, by taking into account (3.3), we obtain the following

$$-\infty < \underline{c} := \inf_{B_\rho(0)} I_\lambda < 0.$$

Let choose $\varepsilon > 0$. Then, it follows

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda - \inf_{B_\rho(0)} I_\lambda.$$

Applying Ekeland's variational principle [10] to the functional $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} I_\lambda + \varepsilon \text{ and } I_\lambda(u_\varepsilon) < I_\lambda(u) + \varepsilon \|u - u_\varepsilon\|, \ u \neq u_\varepsilon.$$

By the fact that

$$I_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} I_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda,$$

we can infer that $u_\varepsilon \in B_\rho(0)$ .Now, we define $\phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $\phi_\lambda(u) = I_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of ϕ, and thus

$$\frac{\phi_\lambda(u_\varepsilon + tv) - \phi_\lambda(u_\varepsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in B_1(0)$ .By the above relation, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0.$$

Letting $t \to 0$, we have that $\left\langle I'_\lambda(u_\varepsilon), v\right\rangle + \varepsilon \|v\| > 0$, and hence, we infer that $\left\| I'_\lambda(u_\varepsilon)\right\| \leq \varepsilon$. The information obtained so far shows that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$I_\lambda(u_n) \to c = \inf_{B_\rho(0)} I_\lambda < 0 \text{ and } \ I'_\lambda(u_n) \to 0. \tag{3.4}$$

Since the sequence $\{u_n\}$ is bounded in W, there exists $u \in W$ such that, up to a subsequence, $\{u_n\}$ conververges to u in W. So, we conclude that I_λ has at least one nontrivial critical point, i.e., the problem $(\mathbf{P})$ has a nontrivial solution.

Acknowledgements. The author would like to thank anonymous referees for their valuable suggestions.

References

[1] Agarwal, R. P., Perera, K. and O'Regan, D., (2005), Multiple positive solutions of singular discrete $p-$Laplacian problems via variational methods, Adv. Diff. Equ. 2, pp. 93–99.

[2] Moghadam, M.Khaleghi and Avci, M., (2017), Existence results to a nonlinear $p(k)$-Laplacian difference equation, Journal of Difference Equations and Appl.https://doi.org/10.1080/10236198.2017.1354991.

[3] Avci, M., (2017), On a nonlocal Neumann problem in Orlicz-Sobolev spaces, Journal of Nonlinear Functional Analysis, Vol. 2017 (2017), Article ID 42, 1-11.

[4] Avci, M., (2016), Existence results for anisotropic discrete boundary value problems, Electron. J. Differ. Equ. 148, pp. 1-11.

[5] Avci, M. and Pankov, A., (2015), Nontrivial solutions of discrete nonlinear equations with variable exponent, J.Math.Anal.Appl. 431, pp. 22-33.

[6] Cabada, A., Lannizzotto A. and Tersian, S., (2009), Multiple solutions for discrete boundary value problems, J. Math. Anal. Appl. 356, pp. 418-428.

[7] Cai, X. and Yu, J., (2006), Existence theorems for second-order discrete boundary value problems, J. Math. Anal. Appl. 320, pp. 649–661.

[8] Candito, P. and Giovannelli, N., (2008), Multiple solutions for a discrete boundary value problem involving the $p-$Laplacian, Comput. Math. Appl. 56, pp. 959-964.

[9] Chen, Y., Levine S. and Rao, M., (2006), Variable exponent linear growth functionals in image processing, SIAM J. Appl. Math. 66 (4), pp. 1383-1406.

[10] Ekeland, I., (1974), On the variational principle. J Math Anal Appl. 47, pp. 324-353.
discrete boundary value problem for anisotropic equation, J. Math. Anal. Appl. 386, pp. 956-965.

[11] Galewski, M. and WieteskaI, R., (2013), Existence and multiplicity of positive solutions for discrete anisotropic equations, Turk. J. Math. doi:10.3906/mat-1303-6.

[12] Galewski, M., Heidarkhani, S. and Salari, A., Multiplicity results for discrete anisotropic equations, Discrete Contin. Dyn. Syst. Ser. B, to appear.

[13] Guiro, A., Nyanquini, I. and Ouaro, S., (2011), On the solvability of discrete nonlinear Neumann problems involving the $p(x)-$Laplacian, Adv. Diff. Equ. 32.

[14] Halsey, T. C., (1992), Electrorheological fluids, Science 258, pp. 761-766.

[15] Heidarkhani, S., Afrouzi, G.A., Moradi, S. and Caristi, G., Existence of multiple solutions for a perturbed discrete anisotropic equation, J. Differ. Equ. Appl., doi: 10.1080/10236198.2017.1337108.

[16] Heidarkhani, S., Afrouzi, G.A., Henderson, J., Moradi,S. and G. Caristi, Variational approaches to $p-$Laplacian discrete problems of Kirchhoff-type, J. Differ. Equ. Appl., https://doi.org/10.1080/10236198.2017.1306061.

[17] Heidarkhani, S., Caristi, G. and Salari, A., Perturbed Kirchhoff-type p-Laplacian discrete problems, Collect. Math., doi: 10.1007/s13348-016-0180-4.

[18] Heidarkhani, S., Afrouzi, G.A., Caristi, G., Henderson, J. and Moradi, S., (2016)A variational approach to difference equations, J. Differ. Equ. Appl. 22, pp. 1761-1776.

[19] Jebelean, P. and Şerban, C., (2011), Ground state periodic solutions for difference equations with discrete $p-$Laplacian, Appl. Math. Comput. 217, pp. 9820-9827.

[20] Kirchhoff, G., (1883), Mechanik, Teubner, Leipzig.

[21] Koné, B. and Ouaro, S., (2010), Weak solutions for anisotropic discrete boundary value problems, J. Diff. Equ. Appl. 17 (10), pp. 1-11.

[22] Mashiyev, R. A., Yucedag, Z. and Ogras, S., (2011), Existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)$-Laplacian operator, E. J. Qualitative Theory of Diff. Equ. 67, pp. 1-10.

[23] Mashiyev, R. A., Cekic, B., Avci, M. and Yucedag, Z., (2012), Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition, Complex Variables and Elliptic Equations, 57, No. 5, pp. 579–595.

[24] Mihăilescu, M., Rădulescu, V. and Tersian, S., (2009), Eigenvalue problems for anisotropic discrete boundary value problems, J. Differ. Equ. Appl. 15 (2009), pp. 557–56.

[25] Moghadam, M. K., Heidarkhaniand, S., Henderson, J., (2014), Infinitely many solutions for perturbed difference equations, J. Differ. Equ. Appl., vol. 20, no. 7, pp. 1055-1068.

[26] Růžička, M., (2000), Electrorheological fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin.

[27] Willem, M., (1996), Minimax Theorems, Birkhauser, Basel.

[28] Yücedag, Z., (2014), Existence of solutions for anisotropic discrete boundary value problem of Kirchhoff type, Int. J. Differ. Equ. Appl. 13(1), pp. 1-15.
[29] Zhang, X. and Tang, X., (2012), Existence of solutions for a nonlinear discrete system involving the $p-$Laplacian. Appl. Math. Praha 57, No. 1, pp. 11-30.
[30] Zhikov, V. V., (1987), Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR. Izv. 9, pp. 33–66.

Zehra Yucedag is an assistant professor of mathematics at Dicle University. His research interests are Variable Exponent Lebesgue and Sobolev Spaces; Differential equations with variable exponents; Calculus of variations. He has been doing research in the mentioned areas for over ten years.