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NEW REFINEMENTS AND INTEGRAL INEQUALITIES FOR CONCAVE FUNCTIONS

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Abstract. In this paper, we establish new refinements and integral inequalities including concave functions. The reason why we choose the concave functions in this study is that the methods we use are applicable to these functions. Also some applications are provided.

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1. Introduction

We will start with the following definition that is well-known in the literature:

Definition 1.1. *The function $f : [a, b] \to \mathbb{R}$, is said to be concave, if we have*

$$tf(x) + (1-t) f(y) \leq f(tx + (1-t) y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Geometrically, this means that if P, Q and R are three distinct points under the graph of f with Q between P and R, then Q is on or above chord PR. A huge amount of the researchers interested in this definition and there are several papers based on concavity (or convexity). Many important inequalities are established for the class of concave functions, but one of the most important is so called Hermite-Hadamard's inequality (or Hadamard's inequality).

This double inequality is stated as follows;

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a concave function and let $a, b \in I$, with $a < b$. The following double inequality;

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \ f\left(\frac{a+b}{2}\right) \tag{1}$$

The above inequality is in the reversed direction if f is convex.

Due to the rich geometric interpretation of (1) there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([5]-[33]). Namely, there are numerous inequalities in literature with connected (1) for different kinds of concave (convex) functions.

In [2] and [34], Kırmacı presented the following results for differentiable mappings, which are connected with right hand side of Hadamard's inequality, respectively.

Lemma 1.1. [2] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ $I \subset R$,be a differrentiable mapping on I^*, $a, b \in I^*$ $(I^*$ is interior of $I)$,with $a < b$. If $f' \in L^1[a,b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b) + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)\right]$$

Lemma 1.2. [34] *Let $f : I \subset R \to R$ be twice differentiable function on I^0 with f'' is integrable on $[a,b] \subset I^0$. Then we have*

$$\frac{(b-a)^2}{2}(I_1+I_2) = \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]$$

where $I_1 = \int_0^{\frac{1}{2}} t\left(t-\frac{1}{2}\right)f''(ta+(1-t)b)\,dt$, $I_2 = \int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right)(t-1)f''(ta+(1-t)b)\,dt$ and I^0denotes the interior of I.

In [5], authors established some new integral inequalities connected the left hand side of (1) for concave functions.

The main aim of this paper is to establish some new integral inequalities connected the right hand side of (1) for concave functions. In other sense , This study is the continuation of part of [5].

We consider the following useful inequality:

For all continuous concave functions $f : [a,b] \to R_+$ and all parameters $p > 1$.

$$\left(\frac{1}{b-a}\int_a^b f^{\,p}(x)\,dx\right)^{\frac{1}{p}} \leq \frac{2}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{b-a}\int_a^b f\ (x)\,dx\right) \tag{2}$$

This inequality is well known in the literature as Favard inequality (see [3])

2. MAIN RESULTS

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, $I \subset [0,\infty)$,be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L^1[a,b]$. where $a, b \in I$, $a < b$. If $|f'|^q$ is concave on $[a,b]$, $q > 1$. Then we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq \ &(b-a)\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\left(\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right) \end{aligned} \tag{3}$$

Proof. From Lemma 1, we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ =&\ (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt\right]. \end{aligned}$$

By using Hölder inequality for $q>1$ and $p=\frac{q}{q-1}$, we obtain

$$\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt \le \left(\int_0^{\frac{1}{2}} t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \tag{4}$$

and

$$\int_{\frac{1}{2}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt \le \left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{5}$$

It can be easily checked that

$$\int_0^{\frac{1}{2}} t^{\frac{q}{q-1}}dt = \int_{\frac{1}{2}}^1 (1-t)^{\frac{q}{q-1}}dt = \frac{2^{\frac{1-2q}{q}}(q-1)}{2q-1}.$$

Since $|f'|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality (see [1]) to obtain

$$\begin{aligned} \int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt &= \int_0^{\frac{1}{2}} t^0\left|f'(ta+(1-t)b)\right|^q dt \\ &\le \left(\int_0^{\frac{1}{2}} t^0 dt\right)\left|f'\left(\frac{1}{\int_0^{\frac{1}{2}} t^0 dt}\int_0^{\frac{1}{2}}(ta+(1-t)b)\,dt\right)\right|^q \\ &= \frac{1}{2}\left|f'\left(2\int_0^{\frac{1}{2}}(ta+(1-t)b)\,dt\right)\right|^q = \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q \end{aligned}$$

and analogously

$$\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt \le \frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q.$$

Combining all obtained inequalities we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ \le&\ (b-a)\left[\frac{q-1}{2q-1}\frac{1}{2^{\frac{2q-1}{q}}}\right]^{\frac{q-1}{q}}\frac{1}{2^{\frac{1}{q}}}\left(\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right) \\ =&\ (b-a)\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\left(\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right) \end{aligned}$$

By (4) and (5) We obtain the required inequality (3). □

Remark 2.1. *Since $|f'|^q$ is $-$ concave on $[a,b]$,we can write following simple inequalities*

$$\left|f'(ta+(1-t)b)\right|^q \ge t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q$$

or

$$\left|f'\left(ta+(1-t)\,b\right)\right| \geq \left(t\left|f'\left(a\right)\right|^{q}+(1-t)\left|f'\left(b\right)\right|^{q}\right)^{\frac{1}{q}}.$$

On the other hand, since $q>1$ we can use the power mean inequality (see [1]*):*
$|f'\left(ta+(1-t)\,b\right)| \geq t\,|f'\left(a\right)|+(1-t)\,|f'\left(b\right)|$. *Namely the function $|f'|$ is also concave on $[a,b]$.*
Now using the fact that we can conclude

$$\begin{aligned}
&\frac{\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|}{2}\\
\geq\ &\frac{\frac{3}{4}\,|f'\left(a\right)|+\frac{1}{4}\,|f'\left(b\right)|+\frac{3}{4}\,|f'\left(b\right)|+\frac{1}{4}\,|f'\left(a\right)|}{2}=\frac{|f'\left(a\right)|+|f'\left(b\right)|}{2}
\end{aligned}$$

thus

$$\frac{|f'\left(a\right)|+|f'\left(b\right)|}{2} \leq \frac{\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|}{2}.$$

Furthermore,
$\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\right)\to\frac{1}{8}$ for $q\to\infty$, and $\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\right)\to\frac{1}{2}$ for $q\to1^{+}$, so for $q\in(1,\infty)$ and we obtain

$$\frac{1}{8}<\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\right)<\frac{1}{2}.$$

We can not generally make the decision which estimation is better. We can not write the term $\frac{|f'(a)|+|f'(b)|}{2}$ instead of $\frac{\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{3b+a}{4}\right)\right|}{2}$. But the one given in Theorem 1 becomes better as q increases for $q\in(1,\infty)$. Hence we can write the following Corollary.

Corollary 2.1. *Let $f : I\subset\mathbb{R}\to\mathbb{R}$, $I\subset[0,\infty)$,be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L^{1}[a,b]$. where $a,b\in I$, , $a<b$. If $|f'|^{q}$ is $-$ concave on $[a,b]$. Then we can rewrite inequality (3):*

$$\left|\frac{1}{b-a}\int_a^b f\left(x\right)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{2}\left(\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right)$$

This is a simple consequence of Theorem 2.1

The inequality in Corollary 1 is a variant of (Theorem 2 [5]).

Corollary 2.2. *If f' is linear, we can give the following inequality as variant of (Theorem 2.2,* [4]*) :*

$$\left|\frac{1}{b-a}\int_a^b f\left(x\right)dx-f\left(\frac{a+b}{2}\right)\right|\leq(b-a)\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\frac{1}{2^{2-\frac{2}{q}+\frac{1}{q^2}}}\left(\left|f'\left(a+b\right)\right|\right)$$

Remark 2.2. *In according to Remark 1, since $|f'|^{q}$ is concave on $[a,b]$, we know that the function $|f'|$ is also concave on $[a,b]$. Thus, we can use Favard's inequality (2) in the proof of following Theorem for concave functions.*

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, $I \subset [0, \infty)$,be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L^1[a, b]$. where $a, b \in I$, $a < b$. If $|f'|^q$ is concave on $[a, b]$, $q > 1$, Then we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{2^{\frac{1-2q}{q}}}\int_a^b \left|f'(x)\right| dx \tag{6}$$

Proof. We proceed similarly as in the proof of Theorem 1, but firstly , using inequality of Favard instead of Integral Jensen for integrals including functions in the right hand of side of inequalities (4) and (5), respectively :

$$\begin{aligned}
\left(\int_0^{\frac{1}{2}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} &= \left(\frac{1}{2\left(b-\frac{a+b}{2}\right)}\int_{\frac{a+b}{2}}^b \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \frac{1}{2^{\frac{1}{q}}}\left(\frac{1}{\left(b-\frac{a+b}{2}\right)}\int_{\frac{a+b}{2}}^b \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{2^{\frac{1}{q}}}\frac{2}{(q+1)^{\frac{1}{q}}}\frac{1}{\left(b-\frac{a+b}{2}\right)}\int_{\frac{a+b}{2}}^b \left|f'(x)\right| dx \\
&= \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{(b-a)}\int_{\frac{a+b}{2}}^b \left|f'(x)\right| dx
\end{aligned}$$

and

$$\begin{aligned}
\left(\int_{\frac{1}{2}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} &= \left(\frac{1}{2\left(\frac{a+b}{2}-a\right)}\int_a^{\frac{a+b}{2}} \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \frac{1}{2^{\frac{1}{q}}}\left(\frac{1}{\left(\frac{a+b}{2}-a\right)}\int_a^{\frac{a+b}{2}} \left|f'(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{2^{\frac{1}{q}}}\frac{2}{(q+1)^{\frac{1}{q}}}\frac{1}{\left(\frac{a+b}{2}-a\right)}\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx \\
&= \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx.
\end{aligned}$$

If necessary mathematical operations are performed, we obtain inequality (6). Namely, Combining all obtained inequalities we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\
\le\ & (b-a)\frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} \left|f'(x)\right| dx + \int_{\frac{a+b}{2}}^b \left|f'(x)\right| dx\right) \\
=\ & \frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{2^{\frac{1-2q}{q}}}\int_a^b \left|f'(x)\right| dx
\end{aligned}$$

which completes the proof. □

Corollary 2.3. *Since* $\lim_{q\to 1^+} \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}} = 1$ *and* $\lim_{q\to\infty} \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}} = 4$ *, we can rewrite the inequality (6) with* $|f'(x)| \leq \frac{K}{4}$ *:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq 4\int_a^b \left|f'(x)\right|dx \leq K(b-a)$$

Theorem 2.3. *Let* $f : I \to R,\ I \subset [0,\infty)$ *be twice differentiable function on* I^0 *such that* $f'' \in L[a,b],\ 0 \leq a < \infty$. *If* $|f''|^q$ *is concave function on* $[a,b] \subset I,\ q \geq 1$ *with* $t \in (0,1)$. *Then, we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq\ & \frac{(b-a)^2}{96}\left|f''\left(\frac{1}{4}a+\frac{3}{4}b\right) + f''\left(\frac{3}{4}a+\frac{b}{4}\right)\right| \end{aligned} \tag{7}$$

Proof. From lemma 2 with properties of modulus we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^2}{2}\left(|I_1|+|I_2|\right)$$

Since $|f''|^q$ is concave, then $|f''|$ is also concave function, in this case we can use the Jensen integral inequality for $|I_1|$ and $|I_2|$

$$\begin{aligned} |I_1| &= \int_0^{\frac{1}{2}} t\left(\frac{1}{2}-t\right)\left|f''(ta+(1-t)b)\right|dt \\ &\leq \int_0^{\frac{1}{2}}\left(\frac{t}{2}-t^2\right)dt\left|f''\left(\frac{\int_0^{\frac{1}{2}}\left(\frac{t}{2}-t^2\right)(ta+(1-t)b)\,dt}{\int_0^{\frac{1}{2}}\left(\frac{t}{2}-t^2\right)dt}\right)\right| \\ &= \frac{1}{48}\left|f''\left(\frac{1}{4}a+\frac{3}{4}b\right)\right| \end{aligned} \tag{8}$$

and

$$\begin{aligned} |I_2| &= \int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)(1-t)\left|f''(ta+(1-t)b)\right|dt \\ &\leq \int_{\frac{1}{2}}^1\left(\frac{3t}{2}-t^2-\frac{1}{2}\right)dt\left|f''\left(\frac{\int_{\frac{1}{2}}^1\left(\frac{3t}{2}-t^2-\frac{1}{2}\right)(ta+(1-t)b)\,dt}{\int_{\frac{1}{2}}^1\left(\frac{3t}{2}-t^2-\frac{1}{2}\right)dt}\right)\right| \\ &= \frac{1}{48}\left|f''\left(\frac{3}{4}a+\frac{1}{4}b\right)\right|. \end{aligned} \tag{9}$$

It can be easily checked that

$$\int_0^{\frac{1}{2}} t\left(\frac{1}{2}-t\right)dt = \int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)(1-t)\,dt = \frac{1}{48},$$

$$\left|f''\left(\frac{\int_0^{\frac{1}{2}}\left(\frac{t}{2}-t^2\right)(ta+(1-t)b)\,dt}{\int_0^{\frac{1}{2}}\left(\frac{t}{2}-t^2\right)dt}\right)\right| = \left|f''\left(\frac{1}{4}a+\frac{3}{4}b\right)\right|$$

and

$$\left| f''\left(\frac{\int_{\frac{1}{2}}^{1} \left(\frac{3t}{2} - t^2 - \frac{1}{2} \right) (ta + (1-t)b)\, dt}{\int_{\frac{1}{2}}^{1} \left(\frac{3t}{2} - t^2 - \frac{1}{2} \right) dt} \right) \right| = \left| f''\left(\frac{3}{4}a + \frac{1}{4}b \right) \right|.$$

By (8) and (9) we obtain the required inequality (7). □

Theorem 2.4. *Let $f : I \to R,\ I \subset [0, \infty)$ be twice differentiable function on I^0 such that $f'' \in L[a, b],\ 0 \le a < \infty$. If $|f''|^q$ is concave function on $[a, b] \subset I,\ q > 1$, with $t \in (0, 1)$. Then, we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] \right| \tag{10}$$
$$\le \frac{(b-a)}{2^{-\frac{1}{s}} \left(\frac{2s-1}{s-1} \right)^{\frac{s-1}{s}}} \left(\frac{\sqrt{\pi} 4^{-2s-1} \Gamma(s+1)}{\Gamma\left(s + \frac{3}{2}\right)} \right)^{\frac{1}{s}} \left[\int_a^{\frac{a+b}{2}} |f''(x)|\, dx + \int_{\frac{a+b}{2}}^{b} |f''(x)|\, dx \right]$$
$$= \frac{(b-a)}{2^{-\frac{1}{s}} \left(\frac{2s-1}{s-1} \right)^{\frac{s-1}{s}}} \sqrt[s]{\frac{\sqrt{\pi} 4^{-2s-1} s\Gamma(s)}{\Gamma\left(s + \frac{3}{2}\right)}} \int_a^b |f''(x)|\, dx$$

where $s = \frac{q}{q-1}$ and Γ is Euler Gamma function.

Proof. From Lemma 2, we have

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] \right|$$
$$\le \frac{(b-a)^2}{2}$$
$$\times \left\{ \int_0^{\frac{1}{2}} t\left(t - \frac{1}{2}\right) |f''(ta + (1-t)b)|\, dt + \int_{\frac{1}{2}}^{1} \left(t - \frac{1}{2}\right)(t-1) |f''(ta + (1-t)b)|\, dt \right\}.$$

Using Hölder's inequality for $q > 1$, we obtain

$$\int_0^{\frac{1}{2}} t\left(t - \frac{1}{2}\right) |f''(ta + (1-t)b)|\, dt$$
$$\le \left(\int_0^{\frac{1}{2}} \left(t\left(t - \frac{1}{2}\right) \right)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left(\int_0^{\frac{1}{2}} |f''(ta + (1-t)b)|^q\, dt \right)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^{1} \left(t - \frac{1}{2}\right)(1-t) |f''(ta + (1-t)b)|\, dt$$
$$\le \left(\int_{\frac{1}{2}}^{1} \left(\left(t - \frac{1}{2}\right)(1-t) \right)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left(\int_{\frac{1}{2}}^{1} |f''(ta + (1-t)b)|^q\, dt \right)^{\frac{1}{q}}$$

where we will use the facts that

$$\begin{aligned}\int_0^{\frac{1}{2}}\left(t\left(t-\frac{1}{2}\right)\right)^s dt &= \int_{\frac{1}{2}}^1\left(\left(t-\frac{1}{2}\right)(1-t)\right)^s dt \\ &= \frac{\sqrt{\pi}4^{-2s-1}\Gamma(s+1)}{\Gamma\left(s+\frac{3}{2}\right)} \text{ for } \mathbf{Re}s > -1\end{aligned}$$

For $q>1,\ s=\frac{q}{q-1}$, since $\mathbf{Re}s$ is greater than -1, we can use the equality $s=\frac{q}{q-1}$ in final equality.

On the other hand , using inequality of Favard as in the proof of Theorem 2 for the following inequalities;

$$\int_0^{\frac{1}{2}}\left|f''(ta+(1-t)b)\right|^q dt \le \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{(b-a)}\int_{\frac{a+b}{2}}^b\left|f''(x)\right|dx$$

and

$$\int_{\frac{1}{2}}^1\left|f''(ta+(1-t)b)\right|^q dt \le \frac{1}{2^{\frac{1-2q}{q}}}\frac{1}{(q+1)^{\frac{1}{q}}}\frac{1}{(b-a)}\int_a^{\frac{a+b}{2}}\left|f''(x)\right|dx$$

Combining all obtained inequalities with required procedures we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \\ \le\ & \frac{(b-a)}{2^{-\frac{1}{s}}(q+1)^{\frac{1}{q}}}\left(\frac{\sqrt{\pi}4^{-2s-1}s\Gamma(s)}{\Gamma\left(s+\frac{3}{2}\right)}\right)^{\frac{1}{s}}\left[\int_a^{\frac{a+b}{2}}\left|f''(x)\right|dx+\int_{\frac{a+b}{2}}^b\left|f''(x)\right|dx\right] \\ =\ & \frac{(b-a)}{2^{-\frac{1}{s}}(q+1)^{\frac{1}{q}}}\sqrt[s]{\frac{\sqrt{\pi}4^{-2s-1}s\Gamma(s)}{\Gamma\left(s+\frac{3}{2}\right)}}\int_a^b\left|f''(x)\right|dx.\end{aligned}$$

which gives the inequality (10) □

Corollary 2.4. *Since* $\frac{1}{2^{-\frac{1}{s}}(q+1)^{\frac{1}{q}}}\sqrt[s]{\frac{4^{-2s-1}s\Gamma(s)}{\Gamma\left(s+\frac{3}{2}\right)}}\to\frac{1}{32\,\Gamma\left(\frac{5}{2}\right)}$ *for* $q\to\infty$ *and* $\sqrt{\pi}=\Gamma\left(\frac{1}{2}\right)$ *with* $|f''(x)|\le M$, *we can rewrite inequality (10) as following.*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \\ \le\ & M\frac{(b-a)^2}{32}\frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{2}\right)}=M\frac{(b-a)^2}{32}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\end{aligned}$$

3. Applications to Special Means

We shall consider the means for arbitrary real numbers $\alpha,\beta\ (\alpha\neq\beta)$

$H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\ \alpha,\beta\in R/\{0\}$ (harmonic mean)

$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in R$ (arithmetic mean)

$G(\alpha,\beta)=\sqrt{\alpha\beta},\quad \alpha,\beta\neq 0,\ \alpha,\beta\in R^+$ (geometric mean)

$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\ |\alpha|\neq|\beta|,\alpha\beta\neq 0$ (logarithmic mean)

$L_n(\alpha,\beta)=\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\ ,\quad n\in Z/\{0,1\},\ \alpha,\beta\in R,\alpha\neq\beta$ (generalized log-mean)

Proposition 3.1. *Let* $0 < a < b$, *then for all* $q > 1$, *we can write;*

$$G(a,b)\left[L^{-1}(a,b) - A^{-1}(a,b)\right] \leq \frac{2^{\frac{2q-1}{2}}}{(q+1)^{\frac{1}{q}}}\left(b^2 - a^2\right)$$

Proof. The result follows from Theorem 2 with $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Proposition 3.2. *Let* $a, b \in [0, \infty)$, $a < b$ *and* $n \in Z^{+}$, $n \geq 2$. *Then, we have the following inequality;*

$\log\frac{3}{2} = 0.405\,47$

$$\left|\frac{1}{2}A(a^n, b^n) - A^n(a,b) - L_n^n(a,b)\right| \leq \frac{M(b-a)^2}{32}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}$$

for all $q > 1$.

Proof. The assertion follows from Corollary 4 applied to the 1−concave function $f(x) = -x^n$, $f : [0, \infty) \to R$. □

Proposition 3.3. *Let* $a, b \in [0, \infty)$, $a < b$. *Then, we have the following inequality;*

$$\left|\left(\frac{b^b - a^a}{b-a}\right)L^{-1}(a^a, b^b) - \frac{1}{2}\left[A(\ln a, \ln b) + \ln(A(a,b))\right] - 1\right|$$
$$\leq \frac{(b-a)^2}{96}\left[\left(A^{-1}\left(\frac{a}{2}, \frac{3b}{2}\right)\right)^2 + \left(A^{-1}\left(\frac{3a}{2}, \frac{b}{2}\right)\right)^2\right].$$

Proof. The assertion follows from Theorem 3 applied to the concave function $f(x) = \ln x$, $f : [0, \infty) \to R$. □

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