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# A NEW GENERALIZATION OF OSTROWSKI TYPE INEQUALITIES ON ARBITRARY TIME SCALE

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Abstract. In this paper, a new generalization of Ostrowski type inequalities for twice differentiable mappings on time scales and some other interesting inequalities as special cases are given.
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## 1. Introduction

In 1938, Ostrowski [17] proved the following interesting integral inequality.

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in $(a,b)$ and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in $(a,b)$. Then, for any $x \in [a,b]$,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left( \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty$$

*where $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.*

In [4], Bohner and Matthews obtained Ostrowski inequality by using the Montgomery identity on time scales as follow.

**Theorem 1.2.** *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(\sigma(s))\triangle s \right| \leq \frac{M}{b-a}\left(h_2(t,a) + h_2(t,b)\right) \tag{1}$$

*where $h_2(.,.)$ is defined by Definition 2.8 below and $M = \sup_{a<t<b} \left|f^{\triangle}(t)\right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1) cannot be replaced by a smaller one.*

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B. G. Pachpatte [18] established the Ostrowski type inequalities for twice differentiable mappings as follows.

**Theorem 1.3.** *Let $f,g:[a,b]\to\mathbb{R}$ be twice differentiable mappings on $(a,b)$ and $f'',g'':(a,b)\to\mathbb{R}$ are bounded i.e. $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$, $\|g''\|_\infty=\sup_{t\in(a,b)}|g''(t)|<\infty$. Then*

$$\begin{aligned}&\left|2\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)-\left[f(t)-\left(t-\frac{a+b}{2}\right)f'(t)\right]\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\right.\\&\left.-\left[g(t)-\left(t-\frac{a+b}{2}\right)g'(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\right|\\&\leq\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b|g(s)|\,ds\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b|f(s)|\,ds\right)\right]\\&\times\left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right].\end{aligned}\tag{2}$$

**Theorem 1.4.** *Let $f,g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and twice differentiable on $(a,b)$, whose second derivatives $f'',g'':(a,b)\to\mathbb{R}$ are bounded on $(a,b)$ i.e. $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$, $\|g''\|_\infty=\sup_{t\in(a,b)}|g''(t)|<\infty$. Then*

$$\begin{aligned}&\left|f(t)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)+g(t)\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\right.\\&-2\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)-\left[\left(t-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\\&\left.-\left[\left(t-\frac{a+b}{2}\right)\frac{g(b)-g(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\right|\\\leq&\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b g(s)ds\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\right]\\&\times\frac{1}{2}\left[\left(\frac{\left(t-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right](b-a)^2.\end{aligned}\tag{3}$$

In 1988, S. Hilger [8] introduced the time scales theory to unify continuous and discrete analysis. For some Ostrowski, Grüss and Čebyšev type inequalities on time scales, see the papers [10, 11, 12, 13, 14, 15, 16, 19, 20, 21] where further references are provided.

In the present paper, a new generalization of Ostrowski type inequalities for twice differentiable mappings on time scales which provides a generalization of the inequalities (2) and (3) is studied. Also some other interesting inequalities as special cases are given.

## 2. General Definitions

For a general introduction to the time scales theory, the reader is refered to Hilger's Ph.D. thesis [8], the books [2, 3, 9], and the survey [1].

**Definition 2.1.** *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real numbers.*

We assume throughout in this paper that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$ and also the interval $[a,b]$ means the set $\{t \in \mathbb{T}\colon\ a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, the following concept of jump operators is needed.

**Definition 2.2.** *For each* $t \in \mathbb{T}$*, the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by* $\sigma(t) = \inf\{s \in \mathbb{T}{:}s > t\}$ *and the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

**Definition 2.3.** *If* $\sigma(t) > t$ *then* $t$ *is right-scattered, if* $\rho(t) < t$ *then* $t$ *is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If* $\sigma(t) = t$*, then* $t$ *is called right-dense, anf if* $\rho(t) = t$ *then* $t$ *is called left-dense. Points that are both right-dense and left-dense are called dense.*

**Definition 2.4.** *The mapping* $\mu : \mathbb{T} \to [0,\infty)$ *defined by* $\mu(t) = \sigma(t) - t$ *is called the graininess function. The set* $\mathbb{T}^k$ *is defined as follows: if* $\mathbb{T}$ *has a left-scattered maximum* $m$*, then* $\mathbb{T}^k = \mathbb{T} - \{m\}$*; otherwise,* $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

**Definition 2.5.** *Let* $f : \mathbb{T} \to \mathbb{R}$ *and fix* $t \in \mathbb{T}^k$*. Then* $f^{\Delta}(t)$ *is defined to be the number (provided it exists) with the property that for any given* $\epsilon > 0$*, there exists a neighborhood* $U$ *of* $t$ *such that*

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)\left[\sigma(t) - s\right]\right| \leq \epsilon \left|\sigma(t) - s\right|, \qquad \forall s \in U.$$

*We call* $f^{\Delta}(t)$ *the delta derivative of* $f(t)$ *at* $t$.

In the case $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = \frac{df(t)}{dt}$. In the case $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, that is, is the usual foward diference operator.

**Theorem 2.1.** *If* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$*. Then the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t$ *and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

**Definition 2.6.** *The function* $f : \mathbb{T} \to \mathbb{R}$ *is said to be rd-continuous(denote* $f \in C_{rd}(\mathbb{T},\mathbb{R})$*) on* $\mathbb{T}$ *provided it is continuous at all right-dense points* $t \in \mathbb{T}$ *and its left-sided limits exist at all left-dense points* $t \in \mathbb{T}$.

It follows from [2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

**Definition 2.7.** *Let* $F : \mathbb{T} \to \mathbb{R}$ *be a function. Then* $F : \mathbb{T} \to \mathbb{R}$ *is called the antiderivative of* $f$ *on* $\mathbb{T}$ *if it satisfies* $F^{\Delta}(t) = f(t)$ *for any* $t \in \mathbb{T}^k$*. In this case the Cauchy integral*

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

**Theorem 2.2.** *Let* $f, g$ *be rd-continuous,* $a, b, c \in \mathbb{T}$ *and* $\alpha, \beta \in \mathbb{R}$*. Then*

*(1)* $\int_a^b \left[\alpha f(t) + \beta g(t)\right]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t,$

*(2)* $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$

*(3)* $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$

*(4)* $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t,$

**Theorem 2.3.** *If $f$ is $\Delta-$integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\,\Delta t.$$

**Definition 2.8.** *Let $h_k, g_k : \mathbb{T}^2 \to \mathbb{R}$ , $k \in \mathbb{N}_0$ be defined by $h_0(t,s) := g_0(t,s) := 1$, for all $s,t \in \mathbb{T}$ and then recursively by $g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau$, $h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau$, for all $s,t \in \mathbb{T}$.*

## 3. Main Results

The following generalization of the Ostrowski type inequalities containing three functions on time scales holds:

**Theorem 3.1.** *Let $a,b,s,t \in \mathbb{T}$, $a < b$. Suppose that $f,g,h \in C^2_{rd}(\mathbb{T},\mathbb{R})$ are such that*

$$\left\|f^{\triangle\triangle}\right\|_{\infty} := \sup_{t\in(a,b)} \left|f^{\triangle\triangle}(t)\right| < \infty, \left\|g^{\triangle\triangle}\right\|_{\infty} := \sup_{t\in(a,b)} \left|g^{\triangle\triangle}(t)\right| < \infty, \ \left\|h^{\triangle\triangle}\right\|_{\infty} := \sup_{t\in(a,b)} \left|h^{\triangle\triangle}(t)\right| < \infty.$$

*Then, for all* $t \in [a,b]$,

$$\begin{aligned}
&\Bigg| \left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&-\frac{1}{3}\left[f(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&-\frac{1}{3}\left[g(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&-\frac{1}{3}\left[h(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\Bigg| \\
\leq\ & \frac{1}{3}\Bigg[\left\|f^{\triangle\triangle}\right\|_\infty\left(\frac{1}{b-a}\int_a^b |g(s)|\Delta s\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\Delta s\right) \\
&+\left\|g^{\triangle\triangle}\right\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\Delta s\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\Delta s\right) \\
&+\left\|h^{\triangle\triangle}\right\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\Delta s\right)\left(\frac{1}{b-a}\int_a^b |g(s)|\Delta s\right)\Bigg] \\
&\times\left[h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\right]. \qquad (4)
\end{aligned}$$

*Proof.* For all $t \in [a,b]$, in pp.7 from [5], the following identities were given

$$\frac{1}{b-a}\int_a^b f(s)\Delta s = \left[f(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\triangle}(t)\right]+\frac{1}{b-a}\int_a^b u(\sigma(s))\,f^{\triangle\triangle}(s)\Delta s, \quad (5)$$

similarly,

$$\frac{1}{b-a}\int_a^b g(s)\Delta s = \left[g(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\triangle}(t)\right]+\frac{1}{b-a}\int_a^b u(\sigma(s))\,g^{\triangle\triangle}(s)\Delta s \quad (6)$$

and

$$\frac{1}{b-a}\int_a^b h(s)\Delta s = \left[h(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\triangle}(t)\right]+\frac{1}{b-a}\int_a^b u(\sigma(s))\,h^{\triangle\triangle}(s)\Delta s \quad (7)$$

where

$$u(\sigma(s)) = \begin{cases} g_2(s,a), & s\in[a,t), \\ h_2(b,s), & s\in[t,b\,]. \end{cases}$$

Multiplying (5), (6) and (7) by $\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$, $\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$ and $\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)$, respectively, adding the resulting identities and dividing by three, for all $t\in[a,b]$,

$$\begin{aligned}
&\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\\
=\ &\frac{1}{3}\left[f(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\\
&+\frac{1}{3}\left[g(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\\
&+\frac{1}{3}\left[h(t)-\left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\triangle}(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\\
&+\frac{1}{3}\left(\frac{1}{b-a}\int_a^b u(\sigma(s))f^{\triangle\triangle}(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\\
&+\frac{1}{3}\left(\frac{1}{b-a}\int_a^b u(\sigma(s))g^{\triangle\triangle}(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\\
&+\frac{1}{3}\left(\frac{1}{b-a}\int_a^b u(\sigma(s))h^{\triangle\triangle}(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right). \quad (8)
\end{aligned}$$

Additionally, in pp.7 from [5], the following identity was given

$$\int_a^b |u(\sigma(s))|\Delta s = g_3(b,a)+(t-b)g_2(b,a)+(b-a)h_2(b,t). \quad (9)$$

From (8), taking absolute values and using (9), (4) can be easily obtained. □

**Corollary 3.1.** *Let $\mathbb{T}=\mathbb{R}$ in Theorem 3.1. Then*

$$\begin{aligned}
&\Bigg|\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)\\
&-\frac{1}{3}\left[f(t)-\left(t-\frac{a+b}{2}\right)f'(t)\right]\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)\\
&-\frac{1}{3}\left[g(t)-\left(t-\frac{a+b}{2}\right)g'(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)\\
&-\frac{1}{3}\left[h(t)-\left(t-\frac{a+b}{2}\right)h'(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\Bigg|\\
\leq\ &\frac{1}{3}\Bigg[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,ds\right)\\
&+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,ds\right)\\
&+\|h''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |g(s)|\,ds\right)\Bigg]\\
&\times\left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]
\end{aligned}\tag{10}$$

*where* $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)| < \infty, \|g''\|_\infty := \sup_{t\in(a,b)}|g''(t)| < \infty$ *and* $\|h''\|_\infty := \sup_{t\in(a,b)}|h''(t)| < \infty$.

**Remark 3.1.** *For $h(t)=1$ in the Corollary 3.1, then*

$$\begin{aligned}
&\Bigg|2\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\\
&-\left[f(t)-\left(t-\frac{a+b}{2}\right)f'(t)\right]\left(\frac{1}{b-a}\int_a^b g(s)ds\right)
\end{aligned}$$

$$
\begin{aligned}
& -\left[g(t)-\left(t-\frac{a+b}{2}\right)g'(t)\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\Bigg| \\
\leq\ & \left[\left\|f''\right\|_\infty\left(\frac{1}{b-a}\int_a^b |g(s)|\,ds\right)+\left\|g''\right\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\,ds\right)\right] \\
& \times\left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right] \qquad (11)
\end{aligned}
$$

*where* $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)| < \infty$ *and* $\|g''\|_\infty := \sup_{t\in(a,b)}|g''(t)| < \infty$. *This inequality can be found in* [18] *as Theorem 1 with inequality (2.1).*

**Remark 3.2.** *If we take* $g(t)=1$ *in the (11), then, for all* $t\in[a,b]$,

$$
\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds-\left(t-\frac{a+b}{2}\right)f'(t)\right| \leq \|f''\|_\infty\left[\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right] \qquad (12)
$$

*where* $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)| < \infty$. *This inequality is Ostrowski type inequality given by P. Cerone et al.* [6] *as Thorem 2.1.*

**Corollary 3.2.** *Let* $\mathbb{T}=\mathbb{Z}$ *in Theorem 3.1. Then*

$$
\begin{aligned}
& \Bigg|\left(\frac{1}{b-a}\sum_{s=a}^{b-1}f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}h(s)\right) \\
& -\frac{1}{3}\left[f(t)-\left(t-\frac{a+b-1}{2}\right)\triangle f(t)\right]\left(\frac{1}{b-a}\sum_{s=a}^{b-1}g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}h(s)\right) \\
& -\frac{1}{3}\left[g(t)-\left(t-\frac{a+b-1}{2}\right)\triangle g(t)\right]\left(\frac{1}{b-a}\sum_{s=a}^{b-1}f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}h(s)\right) \\
& -\frac{1}{3}\left[h(t)-\left(t-\frac{a+b-1}{2}\right)\triangle h(t)\right]\left(\frac{1}{b-a}\sum_{s=a}^{b-1}f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}g(s)\right)\Bigg| \\
\leq\ & \frac{1}{3}\Bigg[\left\|\triangle^2 f\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|g(s)|\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|h(s)|\right) \\
& +\left\|\triangle^2 g\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|f(s)|\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|h(s)|\right) \\
& +\left\|\triangle^2 h\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|f(s)|\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1}|g(s)|\right)\Bigg] \\
& \times\frac{1}{2}\left[\left(t+1-\frac{a+b}{2}\right)^2+\frac{(b-a+2)(b-a-2)}{12}\right] \qquad (13)
\end{aligned}
$$

*where* $\left\|\triangle^2 f\right\|_\infty := \sup_{a\leq t\leq b-1}\left|\triangle^2 f(t)\right| < \infty$, $\left\|\triangle^2 g\right\|_\infty := \sup_{a\leq t\leq b-1}\left|\triangle^2 g(t)\right| < \infty$ *and* $\left\|\triangle^2 h\right\|_\infty := \sup_{a\leq t\leq b-1}\left|\triangle^2 h(t)\right| < \infty$.

**Theorem 3.2.** *Let $a, b, t, x \in \mathbb{T}$, $a < b$. Suppose that $f, g, h \in C_{rd}^2(\mathbb{T},\mathbb{R})$ are such that*

$$\left\|f^{\triangle\triangle}\right\|_\infty := \sup_{t\in(a,b)} \left|f^{\triangle\triangle}(t)\right| < \infty, \quad \left\|g^{\triangle\triangle}\right\|_\infty := \sup_{t\in(a,b)} \left|g^{\triangle\triangle}(t)\right| < \infty, \quad \left\|h^{\triangle\triangle}\right\|_\infty := \sup_{t\in(a,b)} \left|h^{\triangle\triangle}(t)\right| < \infty.$$

*Then, for all $t \in [a, b]$,*

$$\begin{aligned}
&\left| f(t) \left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)\right. \\
&+g(t) \left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&+h(t) \left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right) \\
&-3\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&-\left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{f(b)-f(a)}{b-a}\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&-\left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{g(b)-g(a)}{b-a}\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
&\left.-\left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{h(b)-h(a)}{b-a}\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\right| \\
\leq\ &\left[\left\|f^{\triangle\triangle}\right\|_\infty \left(\frac{1}{b-a}\int_a^b |g(s)|\,\Delta s\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,\Delta s\right)\right. \\
&+\left\|g^{\triangle\triangle}\right\|_\infty \left(\frac{1}{b-a}\int_a^b |f(s)|\,\Delta s\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,\Delta s\right) \\
&\left.+\left\|h^{\triangle\triangle}\right\|_\infty \left(\frac{1}{b-a}\int_a^b |f(s)|\,\Delta s\right)\left(\frac{1}{b-a}\int_a^b |g(s)|\,\Delta s\right)\right] \\
&\times\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |p(t,\tau)|\,|p(\tau,s)|\,\Delta s\Delta\tau\right) \qquad (14)
\end{aligned}$$

*where*

$$p(t,s) = \begin{cases} \sigma(s) - a, & s \in [a,t), \\ \sigma(s) - b, & s \in [t,b\,]. \end{cases}$$

*Proof.* For all $t \in [a,b]$, in page pp.14 from [5], the following identities were given

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\Delta s + \left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{f(b)-f(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,f^{\triangle\triangle}(s)\Delta s\Delta\tau, \tag{15}$$

similarly,

$$g(t) = \frac{1}{b-a}\int_a^b g(s)\Delta s + \left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{g(b)-g(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,g^{\triangle\triangle}(s)\Delta s\Delta\tau \tag{16}$$

and

$$h(t) = \frac{1}{b-a}\int_a^b h(s)\Delta s + \left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{h(b)-h(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,h^{\triangle\triangle}(s)\Delta s\Delta\tau \tag{17}$$

where

$$p(t,s) = \begin{cases} \sigma(s) - a, & s \in [a,t), \\ \sigma(s) - b, & s \in [t,b\,]. \end{cases}$$

Multiplying (15), (16) and (17) by $\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$, $\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$ and $\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)$, respectively and adding the resulting identities, for all $t \in [a,b]$,

$$\begin{aligned}
& f(t)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
& + g(t)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
& + h(t)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right) \\
= {} & 3\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right) \\
& + \left(\left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{f(b)-f(a)}{b-a}\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)
\end{aligned}$$

$$+\left(\left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{g(b)-g(a)}{b-a}\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$$

$$+\left(\left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{h(b)-h(a)}{b-a}\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)$$

$$+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,f^{\triangle\triangle}(s)\Delta s\Delta\tau\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$$

$$+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,g^{\triangle\triangle}(s)\Delta s\Delta\tau\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b h(s)\Delta s\right)$$

$$+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)\,p(\tau,s)\,h^{\triangle\triangle}(s)\Delta s\Delta\tau\right)\left(\frac{1}{b-a}\int_a^b f(s)\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right) \quad (18)$$

From (18), taking absolute values, the inequality (14) is proved. □

**Corollary 3.3.** *In case of* $\mathbb{T}=\mathbb{R}$ *in Theorem 3.2, then*

$$\Bigg| f(t)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)$$

$$+g(t)\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)$$

$$+h(t)\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)$$

$$-3\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)$$

$$-\left[\left(t-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)$$

$$-\left[\left(t-\frac{a+b}{2}\right)\frac{g(b)-g(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b h(s)ds\right)$$

$$-\left[\left(t-\frac{a+b}{2}\right)\frac{h(b)-h(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)\Bigg|$$

$$\leq \Bigg[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,ds\right)$$

$$+\left\|g''\right\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |h(s)|\,ds\right)$$

$$+\left\|h''\right\|_\infty\left(\frac{1}{b-a}\int_a^b |f(s)|\,ds\right)\left(\frac{1}{b-a}\int_a^b |g(s)|\,ds\right)\Bigg]$$

$$\times\frac{1}{2}\left[\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right](b-a)^2 \tag{19}$$

*where* $\left\|f''\right\|_\infty := \sup_{t\in(a,b)}\left|f''(t)\right| < \infty, \left\|g''\right\|_\infty := \sup_{t\in(a,b)}\left|g''(t)\right| < \infty$ *and* $\left\|h''\right\|_\infty := \sup_{t\in(a,b)}\left|h''(t)\right| < \infty$ *is obtained.*

**Remark 3.3.** *For* $h(t)=1$ *, (19) gives*

$$\Bigg|f(t)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)+g(t)\left(\frac{1}{b-a}\int_a^b f(s)ds\right)$$

$$-2\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\left(\frac{1}{b-a}\int_a^b g(s)ds\right)$$

$$-\left[\left(t-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b g(s)ds\right)$$

$$-\left[\left(t-\frac{a+b}{2}\right)\frac{g(b)-g(a)}{b-a}\right]\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\Bigg|$$

$$\leq \left[\left\|f''\right\|_\infty\left(\frac{1}{b-a}\int_a^b g(s)ds\right)+\left\|g''\right\|_\infty\left(\frac{1}{b-a}\int_a^b f(s)ds\right)\right]$$

$$\times\frac{1}{2}\left[\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right](b-a)^2 \tag{20}$$

*where* $\left\|f''\right\|_\infty := \sup_{t\in(a,b)}\left|f''(t)\right| < \infty$ *and* $\left\|g''\right\|_\infty := \sup_{t\in(a,b)}\left|g''(t)\right| < \infty$. *This inequality can be found in* [18] *as Theorem 2 with inequality (2.3).*

**Remark 3.4.** *For* $g(t)=1$ *in the (20), then*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds-\left(t-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right|$$

$$\leq \frac{1}{2}\left[\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right](b-a)^2\left\|f''\right\|_\infty \tag{21}$$

*where* $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. *This inequality is Ostrowski type inequality given by S.S. Dragomir et al.* [7] *as Theorem 2.1.*

**Corollary 3.4.** *In case of* $\mathbb{T}=\mathbb{Z}$ *in Theorem 3.2,*

$$\begin{aligned}
&\Bigg| f(t)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&+g(t)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&+h(t)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right) \\
&-3\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&-\left(t-\frac{a+b-1}{2}\right)\frac{f(b)-f(a)}{b-a}\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&-\left(t-\frac{a+b-1}{2}\right)\frac{g(b)-g(a)}{b-a}\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&-\left(t-\frac{a+b-1}{2}\right)\frac{h(b)-h(a)}{b-a}\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\Bigg| \\
\leq\ & \Bigg[\left\|\triangle^2 f\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&+\left\|\triangle^2 g\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} h(s)\right) \\
&+\left\|\triangle^2 h\right\|_\infty\left(\frac{1}{b-a}\sum_{s=a}^{b-1} f(s)\right)\left(\frac{1}{b-a}\sum_{s=a}^{b-1} g(s)\right)\Bigg] \\
&\times\left(\frac{1}{(b-a)^2}\sum_{\tau=a}^{b-1}\sum_{s=a}^{b-1}|p(t,\tau)|\,|p(\tau,s)|\right) \qquad (22)
\end{aligned}$$

*where* $\left\|\triangle^2 f\right\|_\infty := \sup_{a\le t\le b-1}\left|\triangle^2 f(t)\right| < \infty$, $\left\|\triangle^2 g\right\|_\infty := \sup_{a\le t\le b-1}\left|\triangle^2 g(t)\right| < \infty$, $\left\|\triangle^2 h\right\|_\infty := \sup_{a\le t\le b-1}\left|\triangle^2 h(t)\right| < \infty$ *and*

$$p(t,s)=\begin{cases} s+1-a, & s\in[a,t-1), \\ s+1-b, & s\in[t,b-1]. \end{cases}$$

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