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TIGHT JUST EXCELLENT GRAPHS

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Abstract. A graph G is χ-excellent if for every vertex v, there exists a chromatic partition π such that $\{v\} \in \pi$.A graph G is just χ-excellent if every vertex appears as a singleton in exactly one χ-partition. In this paper, a special type of just χ-excellence namely tight just χ-excellence is defined and studied.

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1. Definition and Properties of tight just χ-excellent graphs

Definition 1.1. *G is χ-excellent if for every vertex v, there exists a chromatic partition π such that $\{v\} \in \pi$.*

Example 1.1. *:*

1. K_n is χ-excellent.
2. C_{2n} is not χ-excellent but C_{2n+1} $(n \geq 1)$ is χ-excellent.
3. W_{2n} $(n \geq 2)$ is χ-excellent.

Definition 1.2. *A graph G is just χ-excellent if every vertex appears as a singleton in exactly one χ-partition.*

Example 1.2.

1. K_n is just χ-excellent.
2. C_{2n+1} is just χ-excellent

Definition 1.3. ***Harary graphs*** $H_{n,m}$ with n vertices and $m < n$ are defined as follows:
Case(i):
n is even and $m = 2r$. Then $H_{n,2r}$ has n vertices $0, 1, 2, \cdots, n-1$ and i, j are joined if $i - r \leq j \leq i + r$, where the addition is taken with respect to modulo n.
Case(ii):

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m is odd and n is even. Let $m = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to the vertex $i + \frac{n}{2}$, for $0 \leq i \leq \frac{n}{2}$.

Case(iii):

m, n are odd. Let $m = 2r + 1$. Then $H_{n,2r+1}$ is constructed by drawing $H_{n,2r}$ and then adding edges joining vertex 0 to the vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and vertex i to $i + \frac{n+1}{2}$, for $1 \leq i \leq \frac{n-1}{2}$.

Definition 1.4. ***Kneser Graph*** Let k, n be two positive integers, such that $2 \leq k \leq n$. Let M be a set with n elements. The Kneser graph $K(n, k)$ is defined as the graph with vertex set V as the set of all subsets of n of cardinality k. Two vertices of $K(n, k)$ are adjacent if and only if the corresponding sets are disjoint. This concept was introduced by Kneser in 1978. When $n = 2k + 1$, the Kneser graph is also called odd by Mulder.
The domination number of $K(n, 2)$ is 3 for every n.

Definition 1.5. *A just χ-excellent graph of order n having exactly n χ-partitions is called a tight just χ-excellent graph.*

Example 1.3.

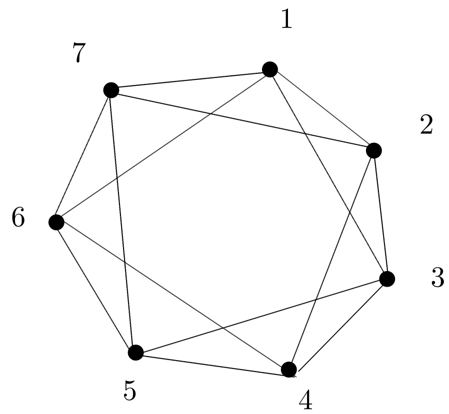

Fig 1: $H_{4,7}$

The only χ-partitions are:
$\pi_1 = \{\{1\}, \{2, 5\}, \{3, 6\}, \{4, 7\}\}$; $\pi_2 = \{\{2\}, \{1, 5\}, \{3, 6\}, \{4, 7\}\}$
$\pi_3 = \{\{3\}, \{1, 5\}, \{2, 6\}, \{4, 7\}\}$; $\pi_4 = \{\{4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$
$\pi_5 = \{\{5\}, \{1, 4\}, \{2, 6\}, \{3, 7\}\}$; $\pi_6 = \{\{6\}, \{1, 4\}, \{2, 5\}, \{3, 7\}\}$
$\pi_7 = \{\{7\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}$
Examples of graphs which are χ- just excellent but not tight just χ-excellent: $H_{4,10}, H_{5,10}, H_{7,13}, H_{9,13}$.

Corollary 1.1. *If G is a just χ-excellent graph, then either it is tight or it contains a χ-partition in which no singleton appears(That is it contains at least n+1 χ-partitions).*

Example 1.4.

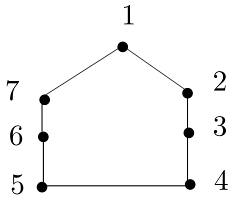

Fig 2: C_7

The χ-partitions of are :
$\{\{1\}, \{2, 4, 6\}, \{3, 5, 7\}\}$; $\{\{2\}, \{3, 5, 7\}, \{1, 4, 6\}\}$; $\{\{3\}, \{1, 4, 6\}, \{2, 5, 7\}\}$

$\{\{4\},\{2,5,7\},\{1,3,6\}\}$; $\{\{5\},\{1,3,6\},\{2,4,7\}\}$; $\{\{6\},\{2,4,7\},\{1,3,5\}\}$
$\{\{7\},\{1,3,5\},\{2,4,6\}\}$
Some other χ-partitions are:
$\{\{1,3\},\{2,4,6\},\{5,7\}\}$; $\{\{1,3\},\{2,5,7\},\{4,6\}\}$; $\{\{1,4\},\{3,5,7\},\{2,6\}\}$
$\{\{1,4\},\{2,5,7\},\{3,6\}\}$; $\{\{1,5\},\{2,4,6\},\{3,7\}\}$; $\{\{1,5\},\{2,4,7\},\{3,6\}\}$
$\{\{1,6\},\{3,5,7\},\{2,4\}\}$; $\{\{1,6\},\{2,4,7\},\{3,5\}\}$; $\{\{2,4\},\{1,3,6\},\{5,7\}\}$
$\{\{2,5\},\{1,3,6\},\{4,7\}\}$; $\{\{2,5\},\{1,4,6\},\{3,7\}\}$; $\{\{2,6\},\{1,3,5\},\{4,7\}\}$
$\{\{2,7\},\{1,3,5\},\{4,6\}\}$; $\{\{2,7\},\{1,4,6\},\{3,5\}\}$

Total number of chromatic partitions = 21. Of these14 Chromatic partitions do not involve singletons. It is an example of a non-tight just χ-excellent graph. In general C_{2n+1} is a non-tight just χ-excellent graph.

Remark 1.1. *If G is just χ-excellent and not tight, then any chromatic partition with a singleton class contains at least one class with more than two elements.*

Proposition 1.1. *Let G be a just χ-excellent graph. Then G is a tight χ-excellent graph if and only if $n = 2\chi - 1$.*

Proof. Let G be a just χ-excellent graph with $n = 2\chi - 1$. Since G is just χ-excellent, given any vertex u, there exists a chromatic partition with $\{u\}$ as an element of the partition. The remaining $\chi - 1$ partitions must have atleast two elements each since in a just χ-excellent graph no chromatic partition can contain two singletons. Therefore the minimum number of elements in any partitions are $2(\chi - 1) + 1 = 2\chi - 1 = n$. But the total number of elements are n. Therefore every chromatic partition containing a singleton must contain only two elements sets as other elements of the partition. If a chromatic partition does not contain a singleton then the total number of elements in the partition are at least $2\chi > n$ a contradiction. Therefore the graph is tight just χ-excellent.
The converse is obvious. □

Remark 1.2. *C_5 is χ-excellent and number of χ-partitions is 5.*

Proposition 1.2. *If G is a tight just χ-excellent graph, then $\chi - 1 \leq deg(u) \leq 2\chi - 4 = |V(G)| - 3$ for any $u \in V(G)$.*

Proof. Since G is a tight just χ-excellent graph, $|V(G)| = 2\chi - 1$. Clearly u is not a full degree vertex. Therefore $deg\ u \leq n - 2 = 2\chi - 3$. Suppose $deg u = n - 2$. Then u is not adjacent to exactly one vertex of G say v. Let $\pi = \{\{v\}, V_2, \cdots, V_\chi\}$ be a χ-partition of G containing $\{v\}$. Then $u \in V_i$ for some i, $2 \leq i \leq \chi$. But u is adjacent to every vertex other than v. Therefore $|V_i| = \{u\}$, a contradiction, since in a just χ-excellent graph any χ-partition can contain at most one singleton class. Hence $deg(u) \leq n - 3 \leq 2\chi - 4$. □

Proposition 1.3. *Given a positive integer k, there exists a tight just χ-excellent regular hamiltonian graph G such that $\chi(G) = k+1$, $|V(G)| = 2k+1$ and every vertex that appears as a singleton in a chromatic partition is adjacent to every element of $(k-2)$ doubletons in that partition and adjacent with exactly one element in the remaining two doubleton classes.*

Proof. Consider the graph $H_{2k-2,2k+1}$. $\beta_0(H_{2k-2,2k+1}) = 2$. (For: Suppose S is an independent set with 3 vertices say $\{u_1, u_2, u_3\}$. But u_1 is not adjacent with only two vertices say v, w where $d(u_1, v) = k$ and $d(u_1, w) = k + 1$. Therefore $u_2 = v$ and $u_3 = w$. But $d(v, w) = 1$ and hence u_2 and u_3 are adjacent, a contradiction. Clearly $\{u_1, v\}$ is independent). Therefore $\frac{n}{\beta_0} \leq \chi$ gives $\frac{2k+1}{2} \leq \chi$. Therefore $\chi \geq k + 1$.

Let $\pi = \{\{1\}, \{2, k+2\}, \{3, k+3\}, \cdots, \{k+1, 2k+1\}\}$. Then π is a proper colour partition of cardinality $k+1$ and hence $\chi = k+1$ and π is a χ-partition in which 1 is adjacent with $2, 3, 4, \cdots, k, 2k+1, 2k, \cdots, k+3$. Therefore 1 is adjacent with exactly 1 element namely 2 and $2k+1$ in the remaining two doubleton classes $\{2, k+2\}, \{k+1, 2k+1\}$. □

Observation 1.1. *The graph $G = H_{2k-2,2k+1}$ is $2k-2$ regular, $\beta_0 = 2$ and $\chi \geq k+\frac{1}{2}$. The graph admits a $k+1$-colour partition. Therefore $\chi(H_{2k-2,2k+1}) = k+1$. Degree of every vertex $= 2k-2 = 2\chi-4$. $|V(G)| = 2k+1 = 2\chi-1$. This graph is tight just χ-excellent and the degree of every vertex is $2\chi-4$. As illustrations, the graphs $H_{6,9}, H_{8,11}, H_{10,13}$ are drawn and the chromatic partitions are exhibited.*

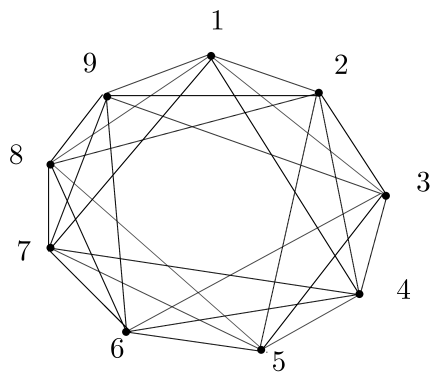

Fig 3: $H_{6,9}$

The chromatic partitions are:

$\pi_1 = \{\{1\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{5, 9\}\}$
$\pi_2 = \{\{2\}, \{1, 6\}, \{3, 7\}, \{4, 8\}, \{5, 9\}\}$
$\pi_3 = \{\{3\}, \{1, 6\}, \{2, 7\}, \{4, 8\}, \{5, 9\}\}$
$\pi_4 = \{\{4\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{5, 9\}\}$
$\pi_5 = \{\{5\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}\}$
$\pi_6 = \{\{6\}, \{1, 5\}, \{2, 7\}, \{3, 8\}, \{4, 9\}\}$
$\pi_7 = \{\{7\}, \{1, 5\}, \{2, 6\}, \{3, 8\}, \{4, 9\}\}$
$\pi_8 = \{\{8\}, \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 9\}\}$
$\pi_9 = \{\{9\}, \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$

In $H_{8,11}$,

The chromatic partitions are:

$\pi_1 = \{\{1\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}, \{6, 11\}\}$
$\pi_2 = \{\{2\}, \{1, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}, \{6, 11\}\}$
$\pi_3 = \{\{3\}, \{1, 7\}, \{2, 8\}, \{4, 9\}, \{5, 10\}, \{6, 11\}\}$
$\pi_4 = \{\{4\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{5, 10\}, \{6, 11\}\}$
$\pi_5 = \{\{5\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{6, 11\}\}$
$\pi_6 = \{\{6\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}\}$
$\pi_7 = \{\{7\}, \{1, 6\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}\}$
$\pi_8 = \{\{8\}, \{1, 6\}, \{2, 7\}, \{3, 9\}, \{4, 10\}, \{5, 11\}\}$
$\pi_9 = \{\{9\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 10\}, \{5, 11\}\}$
$\pi_{10} = \{\{10\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 11\}\}$
$\pi_{11} = \{\{11\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}\}$

Remark 1.3. *There exist hamiltonian graphs which are tight just χ-excellent and for every k, $\chi - 1 \leq k \leq 2\chi - 4$, there exists a vertex u with degree k.*

Observation 1.2. *Two families of tight just χ-excellent graphs are given below. Both are obtained from Harary graphs by removing suitable edges. Construct $H_{2n-2,2n+1}$. Remove the edges with one end at vertex i $(1 \le i \le 2n+1)$ and the other end at the vertices shown against each i. The resulting graph is tight just χ-excellent with $\chi = n+1$. Every positive integral value in the range $[\chi-1, 2\chi-4]$ (that is n to $2n-2$) is realized as degree of the vertices.*

Case 1: n is odd. Consider $H_{16,19}$ with specified edges removed.

Vertex	degree	Non-adjacent vertices	Other end of the Edges
1	$n+1$	$n+1, n+2$	$n+3, n+4, \cdots, 2n-1$
2	$n+1$	$n+2, n+3$	$n+4, ..., 2n$
3	$n+2$	$n+3, n+4$	$n+5, ..., 2n$
4	$n+2$	$n+4, n+5$	$n+6, ..., 2n+1$
5	$n+3$	$n+5, n+6$	$n+7, ..., 2n+1$
6	$n+4$	$n+6, n+7$	$n+8, ..., 2n+1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$n-1$	$2n-3$	$2n-1, 2n$	$2n+1$
n	$2n-2$	$2n, 2n+1$	$\cdots$
$n+1$	$2n-3$	$2n+1, 2n+2$	$2n$
$n+2$	$2n-3$	$2n+2, 2n+3$	$2n+1$
$n+3$	$2n-4$	$2n+3, 2n+4$	$2n+1, 1$
$n+4$	$2n-4$	$2n+4, 2n+5$	$1, 2$
$n+5$	$2n-5$	$2n+5, 2n+6$	$1, 2, 3$
$n+6$	$2n-6$	$2n+6, 2n+7$	$1, \cdots, 4$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2n-1$	$n+1$	$n-2, n-1$	$1, 2, \cdots, n-3$
$2n$	n	$n-1, n$	$n+1, 2, \cdots, n-2$
$2n+1$	n	$n, n+1$	$n+2, n+3, 4, ..., n-1$

Illustration 1.1. *: For $H_{16,19}$ with specified edges removed, $\chi = 10$. Every positive integral value in $[9, 16]$ is realized as the degree of the vertices.*

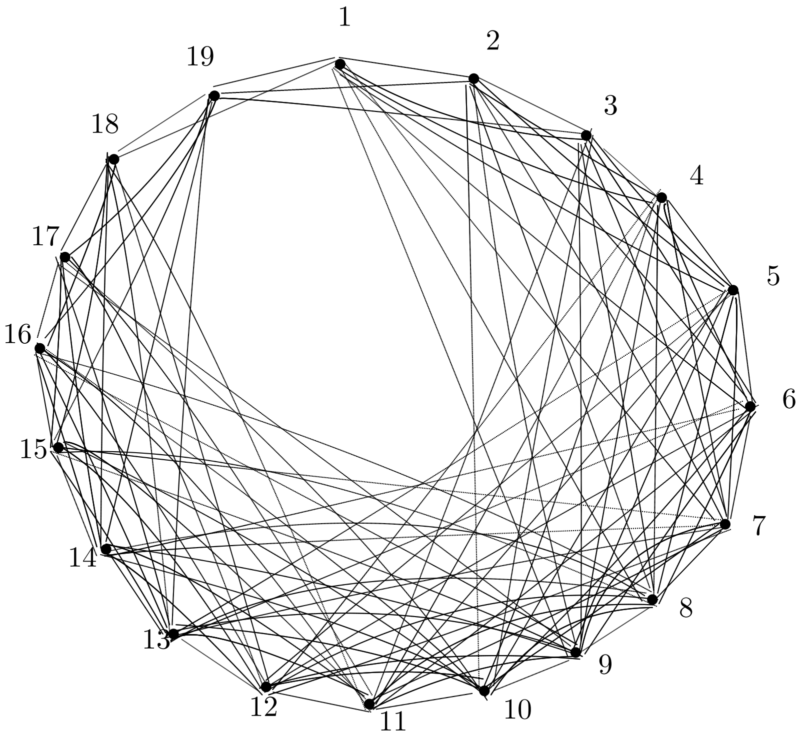

Fig 5: $H_{16,19}$

Vertex	Degree	Non-adjacent vertices	Other end of the edges
1	10	10, 11	12, 13, 14, 15, 16, 17
2	10	11, 12	13, 14, 15, 16, 17, 18
3	11	12, 13	14, 15, 16, 17, 18
4	11	13, 14	15, 16, 17, 18, 19
5	12	14, 15	16, 17, 18, 19
6	13	15, 16	17, 18, 19
7	14	16, 17	18, 19
8	15	17, 18	19
9	16	18, 19	− − − − − − − − − −
10	15	19, 1	18
11	15	1, 2	19
12	14	2, 3	19, 1
13	14	3, 4	1, 2
14	13	4, 5	1, 2, 3
15	12	5, 6	1, 2, 3, 4
16	11	6, 7	1, 2, 3, 4, 5
17	10	7, 8	1, 2, 3, 4, 5, 6
18	9	8, 9	2, 3, 4, 5, 6, 7, 10
19	9	9, 10	4, 5, 6, 7, 8, 11, 12

Case 2: n is even.

Vertex	degree	Non-adjacent vertices	Other end of the Edges
1	$n+2$	$n+1, n+2$	$n+3, n+4, ..., 2n-2$
2	$n+2$	$n+2, n+3$	$n+4, ..., 2n-1$
3	$n+2$	$n+3, n+4$	$n+5, ..., 2n$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{n+2}{2}$	$n+2$	$\frac{3n+2}{2}, \frac{3n+2}{2}+1$	$\frac{3n+2}{2}+2, \cdots, 2n+1,$ $1, \cdots, \frac{n-6}{2}$
$\frac{n+2}{2}+1$	$n+3$	$\frac{3n+2}{2}+1, \frac{3n+2}{2}+2$	$\frac{3n+2}{2}+3, \cdots, 2n+1,$ $1, \cdots, \frac{n-6}{2}$
$\frac{n+2}{2}+2$	$n+4$	$\frac{3n+2}{2}+2, \frac{3n+2}{2}+3$	$\frac{3n+2}{2}+4, \cdots, 2n+1,$ $1, 2, \cdots, \frac{n-8}{2}$
$. \cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{3n+2}{2}-4$	$2n-2$	$\frac{5n+2}{2}-4, \frac{5n+2}{2}-3$	$\cdots$
$\frac{3n+2}{2}-3$	$n+2$	$\frac{3n+2}{2}+n-3, \frac{3n+2}{2}+n-2$	$2n, 2n-1, \cdots, 2n-(n-5)$
$\frac{3n+2}{2}-2$	$n+2$	$\frac{3n+2}{2}+n-2, \frac{3n+2}{2}+n-1$	$2n+1, 2n, \cdots, 2n+1-(n-5)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2n-2$	$n+2$	$3n-2, 3n-1$	$\frac{n+2}{2}-2, \frac{n+2}{2}-3, \cdots, 1, 2n+1,$ $2n, \cdots, 2n+1-(\frac{n-8}{2})$
$2n-1$	$n+1$	$3n-1, 3n$	$2n+3, 2n+4, \cdots, 3n-2, 2n-5$
$2n$	n	$3n, 3n+1$	$2n+4, \cdots, 3n-1, 2n-5, 2n-4$
$2n+1$	n	$3n+1, 3n+2$	$2n+5, \cdots, 3n, 2n-4, 2n-3$

Illustration 1.2. *For $H_{10,13}$ with specified edges removed, $\chi = 7$. Every positive integral value in $[6, 10]$ is realized as the degree of the vertices.*

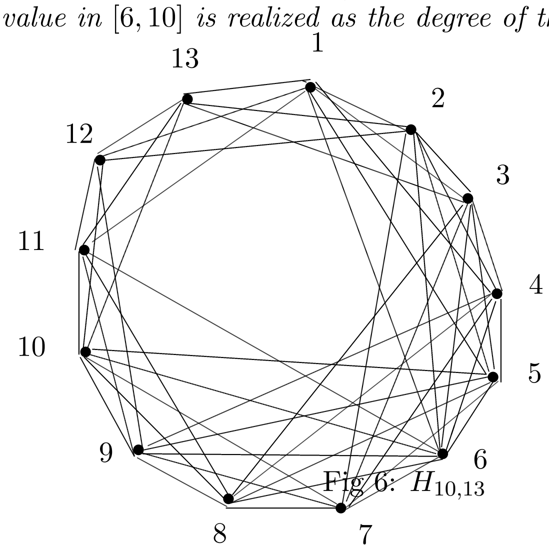

Fig 6: $H_{10,13}$

Vertex	Degree	Non-adjacent vertices	Other end of the edges
1	8	7,8	9,10
2	8	8,9	10,11
3	8	9,10	11,12
4	8	10,11	12,13
5	9	11,12	13
6	10	12,13	——-
7	8	13,1	11,12
8	8	1,2	12,13
9	8	2,3	13
10	8	3,4	1,2
11	7	4,5	2,3,7
12	6	5,6	3,4,7,8
13	6	6,7	4,5,8,9

Proposition 1.4. *The Kneser graph $K(n, 2)$ is not χ-excellent for $n \geq 3$.*

Proof. $\chi(K(n, 2)) = n - 2$. $\chi(K(n, 2) - \{u\}) = \chi(K(n, 2))$ for any $u \in V(K(n, 2))$. Therefore $K(n, 2)$ is not χ-critical and hence not χ-excellent. □

Proposition 1.5. *The Kneser graph $K(n, k)$ $(k \leq \lfloor \frac{n}{2} \rfloor)$ is not χ-excellent for $n \geq 3$.*

Proof. Let $u = \{1, 2, ..., k\}$. Then $\chi(G - u) = \chi(G) = n - 2k + 2$. Therefore G is not χ-excellent. □

Observation 1.3. *C_{2n+1} is just χ-excellent. It is not tight just χ-excellent if $n \geq 1$. Further there exists a chromatic partition in which every vertex of the cycle is colourful if and only if $2n + 1 \equiv 0 (mod\ 3)$.*

Proof. Consider C_{3n} where n is odd. The chromatic number is 3. The partition $\pi = \{\{u_1, u_4, \cdots, u_{3n-2}\},$
$\{u_2, u_5, \cdots, u_{3n-1}\}, \{u_3, u_6, \cdots, u_{3n}\}\}$ is a chromatic partition in which every vertex is colourful. Consider C_{3n+1} where n is even. A chromatic partition giving $3n - 1$ colourful

vertices is $\{\{u_1, u_4, u_7, u_{10}, \cdots, u_{3n-2}\},$
$\{u_2, u_5, \cdots, u_{3n-1}, u_{3n+1}\}, \{u_3, u_6, \cdots, u_{3n}\}\}$.
Here u_1 and u_{3n+1} are not colourful and all other vertices are colourful.
Let $\pi = \{V_1, V_2, V_3\}$ be a chromatic partition of C_{3n+1} (n even). In any V_i, if $u_i \in V_i$ then u_{i-2} and u_{i+2} can not be in V_i. Therefore $V_1 = \{u_1, u_4, \cdots\}$, $V_2 = \{u_2, u_5, \cdots\}, V_3 = \{u_3, u_6, \cdots\}$. Since total number of vertices is $3n + 1$, there exists at least one V_i such that $|V_i| \geq n + 1$. Suppose $|V_1| \geq n + 1$. If $|V_1| = n + 1$, then the $(n + 1)^{th}$ term in V_1 is u_{3n+1} which is adjacent to u_1 in V_1, a contradiction. A similar contradiction arises if $|V_1| > n + 1$. Therefore $|V_1| \leq n$. Similarly $|V_2| \leq n$ and $|V_3| \leq n$, a contradiction since $|V| = 3n + 1$.
Further if $V_1 = \{u_1, u_4, \cdots, u_{3n-2}\}$, $V_2 = \{u_2, u_5, \cdots, u_{3n-1}\}$, and
$V_3 = \{u_3, u_6, \cdots, u_{3n}\}$, then u_{3n+1} cannot be accomodated in V_1 and V_3, since they contain the adjacent vertices u_1 and u_{3n} respectively. Therefore u_{3n+1} has to be included in V_2. In this case u_{3n} and u_1 will not be colourful. Hence the number of colourful vertices is at most $3n - 1$. Since we have already shown that there exists a chromatic partition containing $3n - 1$ colourful vertices, we get that the maximum number of colourful vertices in any chromatic partition of C_{3n+1} (n even) is $3n - 1$.
Similar proof can be given for C_{3n+2} where n is odd to show that the maximum number of colourful vertices in any chromatic partition is $3n$. Hence the observation. □

Remark 1.4. *In a tight just χ-excellent graph of order n, the maximum number of colourful vertices in any chromatic partition is $n - 2$. For:*
In any tight just χ-excellent graph of order n every chromatic partition contains exactly one singleton class and the maximum degree of a vertex is $2\chi - 4$ where $n = 2\chi - 1$. The number of colourful vertices in any chromatic partition is equal to $1 + deg(v)$ where $\{v\}$ appears as a colour class in that partition. Therefore maximum number of colourful vertices in any chromatic partition is equal to $1 + 2\chi - 4 = 2\chi - 3 = n - 2$. Thus there is a vertex of degree $n - 3$ in that graph.

Remark 1.5. *$H_{2r,2r+3}$ is tight just χ-excellent, in which every colour partition has $n-2$ $(= 2r + 1)$ colourful vertices.*

Remark 1.6. *There are tight just excellent graphs of order n in which the maximum number of colourful vertices is less than $n - 2$.*

Example 1.5.

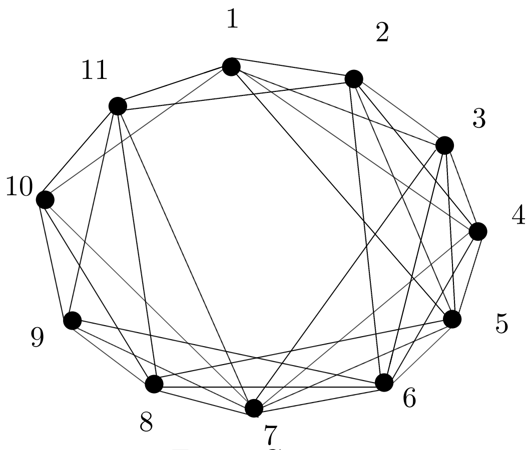

Fig 7: G

The χ-partitions of G are:
$\pi_1 = \{\{1\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}, \{6,11\}\}$
$\pi_2 = \{\{2\}, \{1,7\}, \{3,8\}, \{4,9\}, \{5,10\}, \{6,11\}\}$
$\pi_3 = \{\{3\}, \{1,7\}, \{2,8\}, \{4,9\}, \{5,10\}, \{6,11\}\}$
$\pi_4 = \{\{4\}, \{1,7\}, \{2,8\}, \{3,9\}, \{5,10\}, \{6,11\}\}$
$\pi_5 = \{\{5\}, \{1,7\}, \{2,8\}, \{3,9\}, \{4,10\}, \{6,11\}\}$
$\pi_6 = \{\{6\}, \{1,7\}, \{2,8\}, \{3,9\}, \{4,10\}, \{5,11\}\}$
$\pi_7 = \{\{7\}, \{1,7\}, \{2,8\}, \{3,9\}, \{4,10\}, \{5,11\}\}$
$\pi_8 = \{\{8\}, \{1,6\}, \{2,7\}, \{3,9\}, \{4,10\}, \{5,11\}\}$
$\pi_9 = \{\{6\}, \{1,6\}, \{2,7\}, \{3,8\}, \{4,10\}, \{5,11\}\}$
$\pi_{10} = \{\{10\}, \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,11\}\}$
$\pi_{11} = \{\{11\}, \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}\}$
The colourful vertices with respect to
π_1 are: $1,2,3,4,5,10,11$; π_2 are: $1,2,3,4,5,6,11$;
π_3 are: $1,2,3,5,6,7$; π_4 are: $1,2,3,4,5,6,7$;
π_5 are: $1,2,3,4,5,6,7,8$; π_6 are: $2,4,5,6,7,8,9$;
π_7 are: $3,4,5,6,7,9$; π_8 are: $5,6,7,8,9,10,11$;
π_9 are: $6,7,8,9,10,11$; π_{10} are: $1,7,8,9,10,11$;
π_{11} are: $1,2,7,8,9,10,11$.
Hence maximum number of colourful vertices is 8 and this is realized in the partition π_5. Here $n = 11$ and $8 < 9 = n - 2$

Proposition 1.6. *Consider C_{3n}. There is no chromatic partition containing exactly $(n-1)$ colourful vertices.*

Proof. Let $V(C_{3n}) = \{u_1, u_2, \cdots, u_{3n}\}$. Suppose there exists a chromatic partition say π, containing exactly $n-1$ colourful vertices. Let $\pi = \{V_1, V_2, V_3\}$. Since exactly one vertex say u_i is not colourful, u_{i-1} and u_{i+1} belong to the same colour class of π say V_1. Every element of V_2 and V_3 is colourful. Let $V_2 = \{u_{i1}, u_{i2}, \cdots, u_{ir}\}$ and $V_3 = \{u_{j1}, u_{j2}, \cdots, u_{js}\}$, where ($i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$). Further in V_2 and V_3 i_k *and* i_{k+1} must have difference at least 3 and so also j_k *and* j_{k+1}. The maximum cardinality of V_2 satisfying the above property is n. The same condition holds in V_3. Moreover no V_i can have cardinality more than n since $\beta_0(C_{3n}) = n$. If $|V_1|$ or $|V_2|$ or $|V_3|$ is less than n, then one or two of the remaining elements of the partition will have more than n elements a contradiction. Therefore $|V_1| = n = |V_2| = |V_3|$. Since V_2 and V_3 satisfy the property that the difference between any two suffixes is 3, V_1 also satisfies the same condition, a contradiction. Therefore exactly $n-1$ colourful vertices in a chromatic partition is not possible. □

Observation 1.4. *Every tight just χ-excellent graph is of odd order. But a just χ-excellent graph may be of even order. For example, $H_{5,10}$ is 5-regular and is just χ-excellent with $\chi = 4$, $n = 2\chi + 2$.*

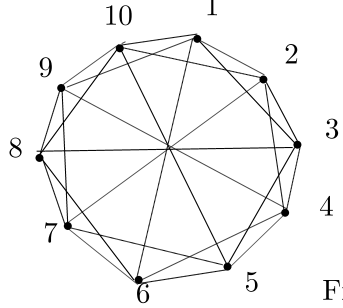

Fig 8: $H_{5,10}$

The partitions are:
$\pi_1 = \{\{1\}, \{2, 5, 8\}, \{3, 6, 9\}, \{4, 7, 10\}\}$; $\pi_2 = \{\{2\}, \{1, 5, 8\}, \{3, 6, 9\}, \{4, 7, 10\}\}$
$\pi_3 = \{\{3\}, \{1, 5, 8\}, \{2, 6, 9\}, \{4, 7, 10\}\}$; $\pi_4 = \{\{4\}, \{1, 5, 8\}, \{2, 6, 9\}, \{3, 7, 10\}\}$
$\pi_5 = \{\{5\}, \{1, 4, 8\}, \{2, 6, 9\}, \{3, 7, 10\}\}$; $\pi_6 = \{\{6\}, \{1, 4, 8\}, \{2, 5, 9\}, \{3, 7, 10\}\}$
$\pi_7 = \{\{7\}, \{1, 4, 8\}, \{2, 5, 9\}, \{3, 6, 10\}\}$; $\pi_8 = \{\{8\}, \{1, 4, 7\}, \{2, 5, 9\}, \{3, 6, 10\}\}$
$\pi_9 = \{\{9\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 10\}\}$; $\pi_{10} = \{\{10\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}$

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