TWMS J. App. Eng. Math. V.9, N.3, 2019, pp. 687-692

ON STAR CHROMATIC NUMBER OF PRISM GRAPH FAMILIES

VERNOLD VIVIN J.[1], KOWSALYA V.[2], VIMAL KUMAR S.[3], §

Abstract. In this paper, we find the star chromatic number χ_s for the central graph of prism graph $C(Y_n)$, line graph of prism graph $L(Y_n)$, middle graph of prism graph $M(Y_n)$ and the total graph of prism graph $T(Y_n)$ for all $n \geq 3$.

Keywords: Star coloring, prism graph, central graph, line graph, middle graph and total graph.

AMS Subject Classification:05C15, 05C75

1. Introduction

The star coloring in graphs is a part of graph coloring which has many applications. Its application includes computing Jacobian or Hessian matrix and many other. A *star coloring* [1, 3, 6] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two colors have connected components that are star graphs.

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. The star chromatic number $\chi_s(G)$ of G is the minimum number of colors needed to star color G. Guillaume Fertin et al.[6] proved the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids $(d \geq 3)$, d-dimensional tori $(d \geq 2)$, graphs with bounded treewidth and cubic graphs.

Albertson et al.[1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, when G is a graph that is both planar and bipartite. The problem of finding star colorings is NP-hard and remain so even for bipartite graphs [7, 8].

[1] Department of Mathematics, University College of Engineering Nagercoil, (Anna University Constituent College), Konam, Nagercoil-629 004, Tamil Nadu, India.
e-mail: vernoldvivin@yahoo.in; ORCID: https://orcid.org/0000-0002-3027-2010.

[2] Assistant Professor, Department of Mathematics, Sri Ramakrishna College of Arts and Science, Coimbatore-641 006, Tamil Nadu, India.
e-mail: kowsalya@srcas.ac.in; ORCID: https://orcid.org/0000-0003-0395-5542.

[3] Department of Mathematics, RVS Technical Campus-Coimbatore, Coimbatore-641 402, Tamil Nadu, India.
e-mail: svimalkumar16@gmail.com; ORCID: https://orcid.org/0000-0002-4556-0808.

§ Manuscript received: January 8, 2017; accepted: February 14, 2017.

2. Preliminaries

The *prism* graph is a graph corresponding to the skeleton of an n-prism, they are both planar and polyhedral. An n-prism graph has $2n$ vertices and $3n$ edges, and is equivalent to the generalized Petersen graph $P_{n,1}$. The vertex set of Y_n the prism graph having $2n$ vertices and $3n$ edges is defined as $V(Y_n) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ (taken in clockwise) and $E(Y_n) = \{e_i : 1 \leq\} \{i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e_{ii} : 1 \leq\} \{i \leq n\}$ where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $u_i u_{i+1} (1 \leq i \leq n-1)$, e'_n is the edge $u_n u_1$ and e_{ii} is the edge $v_i u_i$ $(1 \leq i \leq n)$.

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called *central graph* [9] of G denoted by $C(G)$.

The *line graph* [2, 5] of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $uv \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

The *middle graph* [4] of a graph G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

The *total graph* [2, 4, 5] of a graph G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G and it is denoted by $T(G)$.

Additional graph theory terminology used in this paper can be found in [2, 5].

In the following sections, we find the star chromatic number for the central graph of prism graph $C(Y_n)$, line graph of prism graph $L(Y_n)$, middle graph of prism graph $M(Y_n)$ and the total graph of prism graph $T(Y_n)$.

Definition 2.1. [3] *A star coloring of G is a proper coloring such that the union of any two color classes induces a forest whose components are stars.*

Theorem 2.1. [6] *If C_n is a cycle with $n \geq 3$ vertices, then*

$$\chi_s(C_n) = \begin{cases} 4 & when \quad n = 5 \\ 3 & otherwise. \end{cases}$$

3. Star Coloring on Central Graph of Prism Graph

Theorem 3.1. *If Y_n is a prism graph with $2n$ vertices and $3n$ edges, then*

$$\chi_s(C(Y_n)) = n + 2 \ \ for \ \ n \geq 3.$$

Proof. Let Y_n be a prism graph with $2n$ vertices and $3n$ edges for $n \geq 3$. By the definition of central graph

$$\begin{aligned} V(C(Y_n)) = V(Y_n) \cup E(Y_n) &= \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \left\{v'_i : 1 \leq i \leq n\right\} \\ &\cup \left\{u'_i : 1 \leq i \leq n\right\} \cup \{w_i : 1 \leq i \leq n\}, \end{aligned}$$

where v'_i, u'_i and w_i represents the edge e_i, e'_i and e_{ii} $(1 \leq i \leq n)$ respectively.

Define the mapping $\sigma \colon V(C(Y_n)) \to c_i$, $1 \leq i \leq n+2$ as follows:

- $\sigma(u_i) = \sigma(v_i) = c_i$, $1 \leq i \leq n$.
- $\sigma(w_i) = c_{n+1}$, $1 \leq i \leq n$.
- $\sigma(v'_{i+1}) = c_i$, $1 \leq i \leq n-1$ and $\sigma(v'_1) = c_n$.
- $\sigma(u'_i) = c_{n+2}$, $1 \leq i \leq n$.

The mapping σ is shown as a proper star coloring by discussing the following cases:

Case 1: Consider the colors c_i and c_j, where $1 \leq i < j \leq n$. The color class of c_i is $\{v_i, u_i\}$ and that of c_j is $\{v_j, u_j\}$. The induced subgraphs of these color classes are star graphs and isolated vertices.

Case 2: Consider the colors c_i, $1 \leq i \leq n$ and c_{n+1}. The color class of c_i is $\{v_i, u_i\}$ for $1 \leq i \leq n$ and that of c_{n+1} is $\{w_i : 1 \leq i \leq n\}$. The union of above said color classes induce a forest whose components are star graphs.

Case 3: Consider the colors c_i, $1 \leq i \leq n$ and c_{n+2}. The color class of c_i is $\{v_i', v_i, u_i\}$ for $1 \leq i \leq n$ and that of c_{n+2} is $\{u_i' : 1 \leq i \leq n\}$. The union of above said color classes induce a forest (the induced sugraphs are acyclic) whose components are star graphs.

Case 4: Consider the colors c_{n+1} and c_{n+2}. The color class of c_{n+1} is $\{w_i : 1 \leq i \leq n\}$ and that of c_{n+2} is $\{u_i' : 1 \leq i \leq n\}$. The induced subgraphs of these color classes are K_1 complete graphs, there exists no bicolored paths.

By definition 2.1, the above said mapping σ is a proper star coloring.
Hence, $\chi_s(C(Y_n)) = n + 2, \forall\, n \geq 3$. □

4. Star Coloring on Line Graph of Prism Graph

Theorem 4.1. *Let $n \geq 3$ be a positive integer, then*

$$\chi_s(L(Y_n)) = \begin{cases} 7 & if \quad n = 5 \\ 6 & otherwise. \end{cases}$$

Proof. By the definition of the line graph,

$$V(L(Y_n)) = E(Y_n) = \left\{v_i' : 1 \leq i \leq n\right\} \cup \left\{u_i' : 1 \leq i \leq n\right\} \cup \{w_i : 1 \leq i \leq n\},$$

where v_i' ,u_i' and w_i represents the edge e_i, e_i' and e_{ii} $(1 \leq i \leq n)$ respectively.

Case 1: When $n = 5$

Define the mapping $\sigma\colon V(L(Y_n)) \to c_i$, $1 \leq i \leq 7$ as follows:

- $\sigma(w_i) = \{c_1c_2c_3c_2c_3\}$, $1 \leq i \leq 5$
- $\sigma(u_i') = \sigma(v_i') = \{c_4c_5c_6c_7c_5\}$, $1 \leq i \leq 5$

Thus, $\chi_s(L(Y_n)) = 7$, $n = 5$.

Suppose to the contrary $\chi_s(L(Y_n)) < 7$, say 6. Then the vertices u_4' and v_4' should be colored with one of the existing colors $\{c_1, c_2, c_3\}$ which results in bi-colored paths on four vertices, else the vertices u_i' and v_i' are colored $\{c_4, c_5, c_6\}$, a contradiction to theorem 2.1 (since u_i' and v_i' form cycles with 5 vertices each). Hence, the mapping σ is a proper star coloring and $\chi_s(L(Y_n)) = 7$.

Case 2: When $n \neq 5$

Define the mapping $\sigma\colon V(L(Y_n)) \to c_i$, $1 \leq i \leq 6$ and as follows:

- For $1 \leq i \leq n$, $\sigma(w_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \ldots c_1c_2c_3\}$, and $\sigma(u_i') = \sigma(v_i') = \{c_4c_5c_6 \quad c_4c_5c_6 \ldots c_4c_5c_6\}$ if $n \equiv 0 \bmod 3$.
- For $1 \leq i \leq n$, $\sigma(w_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \ldots c_1c_2c_3 \quad c_2\}$, and $\sigma(u_i') = \sigma(v_i') = \{c_4c_5c_6 \quad c_4c_5c_6 \ldots c_4c_5c_6 \quad c_5\}$ if $n \equiv 1 \bmod 3$.
- For $1 \leq i \leq n$, $\sigma(w_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \ldots c_1c_2c_3 \quad c_1c_2\}$, and $\sigma(u_i') = \sigma(v_i') = \{c_4c_5c_6 \quad c_4c_5 \quad c_4c_5c_6 \ldots c_4c_5c_6\}$ if $n \equiv 2 \bmod 3$.

Thus, $\chi_s(L(Y_n)) = 6$ for $n \neq 5$.

Suppose to the contrary $\chi_s(L(Y_n)) < 6$, say 5. The vertices u_i' and v_i' are colored with $\{c_4, c_5\}$, a contradiction to theorem 2.1 (since u_i' and v_i' form cycles with 5 vertices each) and assigning one of the existing colors $\{c_1, c_2, c_3\}$ to u_i' and v_i' instead of c_6 results in bicolored paths. Hence, the mapping σ is a proper star coloring and $\chi_s(L(Y_n)) = 6$ for $n \neq 5$. □

5. Star Coloring on Middle Graph of Prism Graph

Theorem 5.1. *Let $n \geq 3$ be a positive integer, then*

$$\chi_s(M(Y_n)) = \begin{cases} 8 & \text{if } n = 5 \\ 7 & \text{otherwise.} \end{cases}$$

Proof. By the definition of middle graph

$$\begin{aligned} V(M(Y_n)) = V(Y_n) \cup E(Y_n) &= \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \left\{v_i' : 1 \leq i \leq n\right\} \\ &\cup \left\{u_i' : 1 \leq i \leq n\right\} \cup \{w_i : 1 \leq i \leq n\}, \end{aligned}$$

where v_i' ,u_i' and w_i represents the edge e_i, e_i' and e_{ii} $(1 \leq i \leq n)$ respectively.

Case 1: When $n = 5$

Define the mapping $\sigma\colon V(M(Y_n)) \to c_i$, $1 \leq i \leq 8$ as follows:

- $\sigma(u_i) = \sigma(v_i) = c_1$ for $1 \leq i \leq 5$
- $\sigma(w_i) = \{c_2c_3c_4c_2c_3\}, 1 \leq i \leq 5$
- $\sigma(u_i') = \sigma(v_i') = \{c_5c_6c_7c_8\}, 1 \leq i \leq 5.$

That is, $\chi_s(M(Y_n)) = 8$, $n = 5$.

Suppose to the contrary $\chi_s(M(Y_n)) < 8$, say 7. Then, the vertices u_i' and v_i' are colored with $\{c_5, c_6, c_7\}$, a contradiction to theorem 2.1 (since u_i' and v_i' form cycles with 5 vertices each) and assigning one of the existing colors $\{c_1, c_2, c_3, c_4\}$ instead of c_8 to u_i' and v_i' results in bicolored paths on four vertices. Hence, $\chi_s(M(Y_n)) = 8$ for $n = 5$.

Case 2: When $n \neq 5$

Define the mapping $\sigma\colon V(M(y_n)) \to c_i$, $1 \leq i \leq 7$ as follows:

- For $1 \leq i \leq n$, $\sigma(u_i) = \sigma(v_i) = c_1$.
- For $1 \leq i \leq n$, $\sigma(w_i) = \{c_2c_3c_4.\quad c_2c_3c_4 \dots c_2c_3c_4\}$
- $\sigma(u_i') = \sigma(v_i') = \{c_5c_6c_7 \quad c_5c_6c_7 \dots c_5c_6c_7\}$ if $n \equiv 0 \bmod 3$ for $1 \leq i \leq n$.
- $\sigma(u_i') = \sigma(v_i') = \{c_5c_6c_7 \quad c_5c_6c_7 \dots c_5c_6c_7 \quad c_6\}$ if $n \equiv 1 \bmod 3$ for $1 \leq i \leq n$.
- For $1 \leq i \leq n$, $\sigma(u_i') = \sigma(v_i') = \{c_5c_6c_7 \quad c_5c_6 \dots c_5c_6c_7 \quad c_5c_6c_7\}$ if $n \equiv 2 \bmod 3$.

Thus, $\chi_s(M(Y_n)) = 7$, $n \neq 5$.

Suppose to the contrary $\chi_s(M(Y_n)) < 7$, say 6. Then, the vertices u_i' and v_i' are colored with $\{c_5, c_6\}$, a contradiction to theorem 2.1 (since u_i' and v_i' form cycles with n vertices each) and assigning one of the existing colors $\{c_1, c_2, c_3, c_4\}$ to u_i' and v_i' instead of c_7 results in bicolored paths on four vertices. Hence, $\chi_s(M(Y_n)) = 7$ for $n \neq 5$. □

6. Star Coloring on Total Graph of Prism Graph

Theorem 6.1. *Let Y_n be a prism graph with $2n$ vertices and $3n$ edges then for $n \geq 3$,*

$$\chi_s(T(Y_n)) = 3n.$$

Proof. By the definition of total graph

$$\begin{aligned} V(T(Y_n)) = V(Y_n) \cup E(Y_n) &= \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \left\{v_i' : 1 \leq i \leq n\right\} \\ &\cup \left\{u_i' : 1 \leq i \leq n\right\} \cup \{w_i : 1 \leq i \leq n\} \end{aligned}$$

where v_i' ,u_i' and w_i represents the edge e_i, e_i' and e_{ii} $(1 \leq i \leq n)$ respectively.
Define the mapping $\sigma\colon V(T(Y_n)) \to c_i$, $1 \leq i \leq 3n$ as follows:

- $\sigma(v_i) = c_i$, $1 \leq i \leq n$.
- $\sigma(u_i') = c_{i-1}$, $2 \leq i \leq n$ and $\sigma(u_1') = c_n$.
- $\sigma(w_i) = c_{n+i}$, $1 \leq i \leq n$.
- $\sigma(v_i') = c_{n+i-1}$, $2 \leq i \leq n$ and $\sigma(v_1') = c_{2n}$.
- $\sigma(u_i) = c_{2n+i}$, $1 \leq i \leq n$.

The mapping σ is showed as a proper star coloring, by discussing the following cases:

Case 1: Consider the colors c_i and c_j, $1 \leq i < j \leq n$. The color class of c_i is $\{v_i, u_i' : 1 \leq i \leq n\}$ and that of c_j is $\left\{v_j, u_j' : 1 \leq j \leq n\right\}$. The induced subgraph of these color classes result in star graphs and isolated vertices.

Case 2: Consider the colors c_i and c_{n+i}, $1 \leq i \leq n$. The color class of c_i is got as $\{v_i, u_i' : 1 \leq\}$ $\{i \leq n\}$ and that of c_{n+i} is $\{w_i, v_i' : 1 \leq i \leq n\}$. The induced subgraphs of these color classes result in forest whose components are star graphs.

Case 3: Consider the colors c_{n+i} and c_{2n+i}, $1 \leq i \leq n$. The color class of c_{n+i} is $\{w_i, v_i' : 1 \leq i \leq n\}$ and that of c_{2n+i} is $\{u_i : 1 \leq i \leq n\}$. Clearly the induced subgraphs of these color classes result in forest whose components are star graphs.

Case 4: Consider the colors c_i and c_{2n+i}, $1 \leq i \leq n$. The color class of c_i is obtained as $\{v_i, u_i' : 1 \leq i \leq n\}$ and that of c_{2n+i} is $\{u_i : 1 \leq i \leq n\}$. The components of induced subgraphs of these color classes are K_1 and $K_{1,2}$, that is, forest with components as star graphs.

By the definition 2.1, the above said mapping σ is a proper star coloring.
Hence, $\chi_s(T(Y_n)) = 3n$ for all $n \geq 3$. □

References

[1] Albertson, Michael O., Chappell, Glenn G., Kierstead, Hal A., Kündgen, André and Ramamurthi, R., (2004), Coloring with no 2-Colored P_4's, The Electronic Journal of Combinatorics 11, Paper # R26, 13.

[2] Bondy, J. A, and Murty U. S. R, (1976), Graph theory with Applications, MacMillan, London.

[3] Branko Grünbaum, (1973), Acyclic colorings of planar graphs, Israel J. Math 14, pp. 390-408.

[4] Danuta Michalak, (1981), On middle and total graphs with coarseness number equal 1, Springer Verlag Graph Theory, Lagow proceedings, Berlin Heidelberg, New York, Tokyo, pp. 139-150.

[5] Frank Harary, (1969), Graph Theory, Narosa Publishing home, New Delhi.

[6] Fertin, Guillaume, Raspaud, André and Reed, Bruce, (2004), Star coloring of graphs, Journal of Graph Theory 47 (3), pp. 163-182.

[7] Thomas F.Coleman and Jin-Yi Cai, (1986), The cyclic coloring problem and estimation of sparse Hessian matrices, SIAM J. Alg. Disc. Math 7, pp. 221-235.

[8] Thomas F.Coleman and Jorge J.Moré, (1984), Estimation of sparse Hessian matrices and graph coloring problems, Mathematical Programming, 2 (3), pp. 243-270.

[9] Vernold Vivin.J, Ph.D Thesis, (2007), Harmonious coloring of total graphs, n-leaf, central graphs and circumdetic graphs, Bharathiar University, Coimbatore, India.

Vernold Vivin. J received his B.Sc (2000) and M.Sc (2002) degree in Mathematics from Government Arts College (Bharathiar University), Coimbatore. He completed his M.Phil (2004) degree in Mathematics and his Doctoral degree (2007) in the field of Graph Theory, particularly in Graph Colorings from Kongunadu Arts and Science College under Bharathiar University, Coimbatore. Presently, he is working as Assistant Professor in the Department of Mathematics, University College of Engineering Nagercoil, (Anna University Constituent College), Konam, Nagercoil. He has 12 years of teaching experience. He has published 70 research papers in International, National Journals and Conference Proceedings. He is a reviewer for American Mathematical Society and has served as a referee for reputed international journals.

Kowsalya.V is working as Assistant Professor in the Department of Mathematics, Sri Ramakrishna College of Arts and Science, Coimbatore, India. She has 11 years of teaching experience. Her area of interest is coloring of graphs and she has published seven research papers in refereed international journals and conferences. She continues her research in star coloring and acyclic coloring of graphs in graph theory.

Vimal Kumar. S is working as Assistant Professor in the Department of Mathematics, RVS Technical Campus - Coimbatore, India. He received his B.Sc. degree in Mathematics from C. Abdul Hakeem College (Affiliated to University of Madras), Melvisharam, 2004; M.Sc. degree in Mathematics from Periyar University, Salem, 2007. He completed the M.Phil. degree in Mathematics from The Gandhigram Rural Institute, Dindigul, 2011; Ph.D. degree in Mathematics from Anna University, Chennai, India, 2018. His research area includes Graph Theory, Control Systems and Differential Equations. He has published 13 research articles in reputed International Journals of Mathematical and Engineering Sciences.