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ON THE GENERALIZED VECTOR EQUILIBRIUM PROBLEM

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Abstract. In this paper, first a maximal element lemma is proven, then by using properties of the nonlinear function, an existence theorem for generalize vector equilibirium problems is proven. Finally, an example in order to support the main results is given.

Keywords: KKM-function, nonlinear scalarization function, vector equilibrium problem.

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1. Introduction

Throughout in this paper, following notations were used. Let X_1, X_2 and Y_1, Y_2 stand for topological vector spaces(for short, t.v.s.) and P_i be a proper, closed and convex cone of Y_i with $intP_i \neq \emptyset$, where $intP_i$ denotes the topological interior of P_i, for $i = 1, 2$. Let K_i be a nonempty convex subset of X_i,

$$F_i : K_1 \times K_2 \times K_i \to 2^{Y_i}$$

be a set-valued function with nonempty values, where 2^{Y_i} denots the class of all subsets of Y_i, for $i = 1, 2$. Now, we are ready to introduce the following problem which is called generalized vector equilibrium problem(in short, GVEP):
Find $x^* = (x_1^*, x_2^*) \in K_1 \times K_2$ such that

$$F_i(x_1^*, x_2^*, y_i) \cap -intP_i = \emptyset, \tag{1}$$

for each $i = 1, 2$, and for all $y_i \in K_i$.

Remark 1.1. *The GVEP is a generalization of all the following problems:*

(i) *If we take* $P_1 = P_2$*;*

$$f_1 : K_1 \times K_2 \to X_1$$

and

$$f_2 : K_1 \times K_2 \to X_2$$

are two single-valued mappings, and

$$F_1(x, y, z) = \{f_1(z, y) - f_1(x, y)\},$$

and

$$F_2(x, y, z) = \{f_2(x, z) - f_2(x, y)\},$$

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for all $(x, y, z) \in K_1 \times K_2 \times K_2$*, then (1) reduces to the symmetric vector equilibrium problem which was studied in* [11].

(ii) *If* $f_1 : K_1 \times K_1 \to 2^{Y_1}, F_2 = \{0_{Y_2}\}$ *and*

$$F_1(x, y, z) = \{f_1(x, z)\},$$

where $(x, y, z) \in K_1 \times K_2 \times K_1$*, then (1) is the vector equilibrium problem which was introduced by Blum and Oettli* [4]*. For more details please refer to* [7, 5, 13, 14] *and the references therein.*

(iii) *We can state the classical vector variational inequality problem which was introduced by Giannessi* [5] *in the form of GVEP as follows:*
Let $T : K_1 \to L(X_1, X_2)$*, we define*

$$F_1(x_1, x_2, y_1) = T(x_1)(y_1 - x_1),$$

for every $(x_1, x_2, y_1) \in K_1 \times K_2 \times K_1$ *and*

$$F_2 = \{0_{Y_2}\},$$

where $L(X_1, X_2)$ *denotes the space of all continuous linear operators from* X_1 *to* X_2.

(iv) *Finally, if we take* $P_1 = [0, +\infty], Y_1 = \mathbb{R}, K_1 \subseteq \mathbb{R}$*, then (1) reduces to the scalar equilibrium problem for*

$$F_1 : K_1 \times K_1 \to 2^{\mathbb{R}},$$

which was studied by many authors (see, for example,[15, 12] *and the references therein).*

2. **Preliminaries**

In this section, the nonlinear scalarization function and some of its important properties are introduced. Also, the maximal element lemma and the notion of KKM function and Ky Fan's lemma are stated.

Definition 2.1. [3, 1] *Let* X *be a topological vector space with the closed, convex and pointed cone* P*. The nonlinear scalarization function (with respect to* P *and* e*) is defined as follows*

$$\xi_e(x) := inf\{r \in \mathbb{R} : re - x \in P\},$$

where $x \in X$, $e \in intP$ *and* ξ_e *is a function from* X *in to* $\mathbb{R}$.

The following lemma characterizes some of the important properties of the nonlinear scalarization function which are used in the sequel.

Lemma 2.1. [8, 3] *Let* X *be a t.v.s. and* P *a proper, closed and convex cone of* X *with* $e \in intP$*. Then for each* $r \in \mathbb{R}$ *and* $x \in X$ *the following statements are satisfied.*

(i) $\xi_e(x) = min\{r \in \mathbb{R} : re - x \in P\}$.
(ii) $\xi_e(x) \le r \Longleftrightarrow re - x \in P$.
(iii) $\xi_e(x) < r \Longleftrightarrow re - x \in intP$.
(iv) $\xi_e(x) = r \Longleftrightarrow x \in re - \partial P$*, where* ∂P *is the topological boundary of* P.
(v) $y_2 - y_1 \in P \Longrightarrow \xi_e(y_1) \le \xi_e(y_2)$.
(vi) *The function* ξ_e *is continuous, positively homogeneous and subadditive(that is sublinear) on* X.

For proving an existence result of the equilibrium problems Ky Fan's lemma plays a key role. We are going now to state it.

Definition 2.2. [9] *Let K be a nonempty subset of topological vector space X. A set-valued function $T : K \to 2^X$ is called a KKM-function if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $conv\{x_1, x_2, ..., x_n\}$ is contained in $\cup_{i=1}^n T(x_i)$, where conv denotes the convex hull.*

Ky Fan in 1984 obtained the following result, which is known as Ky Fan's lemma.

Lemma 2.2. *(Ky Fan-1984)* [9] *Let K be a nonempty convex subset of topological vector space X and $T : K \to 2^X$ be a KKM function with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\cap_{x\in B} T(x)$ is compact. Then*

$$\cap_{x\in K} T(x) \neq \emptyset.$$

Definition 2.3. *Let T be a multivalued function on a set K. The element $x \in K$ is called "maximal", if $T(x)$ is empty.*

The existence of maximal elements for multivalued function in topological vector spaces and its important applications to mathematical economies have been studied by many authors in both mathematics and economies, see, for example, [10, 6]. Moreover, maximal element lemma plays a crucial role in the establishment of the existence of solutions for GVEP.

3. Main results

In this section, new existence results for the Equilibrium Problem is proven. An immediate consequence of the Ky Fan's lemma, is the following result.

Lemma 3.1. *Let K_i be a nonempty convex subset of a t.v.s. X_i and $T_i : K_1 \times K_2 \longrightarrow 2^{K_i}$ be a set-valued function, $K = K_1 \times K_2$, for $i = 1, 2$, such that*

(i) *for each $(x_1, x_2) \in K_1 \times K_2$, $T_i(x_1, x_2)$ is convex, for $i = 1, 2$,*
(ii) *for each $(x_1, x_2) \in K_1 \times K_2$, $x_1 \notin T_1(x_1, x_2)$ and $x_2 \notin T_2(x_1, x_2)$,*
(iii) *for each $y_i \in K_i$, $T_i^{-1}(y_i) = \{x \in K : y_i \in T_i(x)\}$ is open in K, for $i = 1, 2$,*
(iv) *there exist a nonempty compact convex subset N of K and a nonempty compact subset E_i of K_i, for $i = 1, 2$, such that*
$$T_1(x) \cap E_1 \neq \emptyset \quad or \quad T_2(x) \cap E_2 \neq \emptyset,$$
for all $x \in K \setminus N$.

Then there exists $x^* \in K$ such that $T_i(x^*) = \emptyset$, for $i = 1, 2$.

Proof. $g : K = K_1 \times K_2 \to 2^K$ is defined as follows

$$g(x_1, x_2) = K \setminus (T_1^{-1}(x_1) \cup T_2^{-1}(x_2)),$$

where $(x_1, x_2) \in K_1 \times K_2$. By (iii), $g(x_1, x_2)$ is closed for each $(x_1, x_2) \in K_1 \times K_2$. It can be verified that g is a KKM function. To verify this, let $A = \{z_1, z_2, ..., z_n\} \subseteq K_1 \times K_2$, where $z_i = (z_i^1, z_i^2)$ and show that

$$convA \subset \cup_{i=1}^n g(z_i).$$

Let $z = (z^1, z^2) \in convA$ and $z \notin \cup_{i=1}^n g(z_i)$, then

$$z = (z^1, z^2) = \sum_{i=1}^n \lambda_i z_i = (\sum_{i=1}^n \lambda_i z_i^1 \ , \ \sum_{i=1}^n \lambda_i z_i^2),$$

where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$ for all i. Therefore

$$z^1 = \sum_{i=1}^n \lambda_i z_i^1 \quad and \quad z^2 = \sum_{i=1}^n \lambda_i z_i^2.$$

On the other word

$$z \notin \cup_{i=1}^{n} g(z_i) = \cup_{i=1}^{n} (K \setminus (T_1^{-1}(z_i^1) \cup T_2^{-1}(z_i^2))),$$

hence $z \in T_1^{-1}(z_i^1)$ or $z \in T_2^{-1}(z_i^2)$, for $i = 1, 2, ..., n$. Therefore $z_i^1 \in T_1(z)$ or $z_i^2 \in T_2(z)$, for $i = 1, 2, ..., n$. Now

$$z^1 = \sum_{i=1}^{n} \lambda_i z_i^1 \in T_1(z) = T_1(z^1, z^2)$$

or

$$z^2 = \sum_{i=1}^{n} \lambda_i z_i^2 \in T_2(z) = T_2(z^1, z^2),$$

which is contradicted by (ii) and this completes the proof of the assertion. Moreover, it follows from condition (iv) that

$$\bigcap_{x=(x^1,x^2)\in E_1\times E_2} g(x) \subseteq N.$$

Because, if $y \in \bigcap_{x\in E_1\times E_2} g(x)$, then

$$y \notin T_1^{-1}(x^1) \cup T_2^{-1}(x^2),$$

for every $x = (x^1, x^2) \in E_1 \times E_2$, that is $x^1 \notin T_1(y)$, $x^2 \notin T_2(y)$. Therefore

$$E_1 \cap T_1(y) = \emptyset \ , \ E_2 \cap T_2(y) = \emptyset$$

and we have $y \in N$.

Since $\bigcap_{x\in E_1\times E_2} g(x)$ is a closed subset of the compact set N(note that the values of g are closed), then $\bigcap_{x\in E_1\times E_2} g(x)$ is a compact subset of N and so g satisfies all the assumptions of Lemma 2.2. Hence it follows from Lemma 2.2, that $\cap_{x\in K} g(x) \neq \emptyset$. Let $y^* \in \cap_{x\in K} g(x)$, then $y^* \notin T_1^{-1}(x^1) \cup T_2^{-1}(x^2)$, for every $x = (x^1, x^2) \in K = K_1 \times K_2$. Therefore

$$x^1 \notin T_1(y^*) \ , \ x^2 \notin T_2(y^*),$$

for every $x = (x^1, x^2) \in K = K_1 \times K_2$. Hence $T_1(y^*) = \emptyset$ and $T_2(y^*) = \emptyset$. This completes the proof. □

Remark 3.1. *Let X be a topological vector space and $e \in intP$ with $P \subseteq X$. Then for any $B \subseteq X$, we have*

$$B \cap -intP = \emptyset, \Longleftrightarrow \xi_e(B) \subseteq [0, +\infty),$$

where $\xi_e(B)$ is the image of B under ξ_e.

Now, we are ready to present an existence result of a solution for GVEP, by using scalarization method and the maximal element lemma.

Theorem 3.1. *Let X_1, X_2, Y_1 and Y_2 be topological vector spaces(for short, t.v.s.). For each $i = 1, 2$, let K_i be a nonempty closed convex subset of X_i, P_i be a proper, closed and convex cone in Y_i and $F_i : K_1 \times K_2 \times K_i \longrightarrow 2^{Y_i}$ be a set-valued function with nonempty values. Assume that the following conditions hold:*

(i) *for all $(x_1, x_2) \in K_1 \times K_2$, $F_i(x_1, x_2, x_i) \cap -intP_i = \emptyset$, for $i = 1, 2$*

(ii) *for $i = 1, 2$ and for all $(x_1, x_2) \in K_1 \times K_2$, the set*

$$A_i = \{y_i \in K_i : F_i(x_1, x_2, y_i) \cap -intP_i \neq \emptyset\},$$

is convex.

(iii) *for* $i = 1, 2$ *and for all* $y_i \in K_i$ *, the set*

$$G_{y_i} = \{(x_1, x_2) \in K_1 \times K_2 : F_i(x_1, x_2, y_i) \cap -intP_i \neq \emptyset\}$$

is open.

(iv) *there exist a nonempty compact convex subset* B_i *of* K_i *and a nonempty compact subset* N_i *of* K_i*, for* $i = 1, 2$*, such that for each*

$$(x_1, x_2) \in K_1 \times K_2 \setminus N_1 \times N_2,$$

there exists $i = 1, 2,\ y_i \in B_i$ *satisfying*

$$\xi_{e_i}(F_i(x_1, x_2, y_i)) \nsubseteq [0, +\infty),$$

for $i = 1, 2$*. Then the solution set of* $GVEP$ *is nonempty.*

Proof. Let $\Gamma_i : K_1 \times K_2 \longrightarrow 2^{K_i}$ be defined by

$$\Gamma_i(x_1, x_2) = \{y_i \in K_i : \xi_{e_i}(F_i(x_1, x_2, y_i)) \nsubseteq [0, +\infty)\},$$

for $i = 1, 2$, $e_i \in P_i$ and $(x_1, x_2) \in K_1 \times K_2$. It is easy to see that the function Γ_i satisfies all the conditions of Lemma 3.1. First, applying Remark 3.1 and condition (ii), it is clear that $\Gamma_i(x_1, x_2)$ is a convex set for any $(x_1, x_2) \in K_1 \times K_2$, $x_1 \notin \Gamma_1(x_1, x_2)$ and $x_2 \notin \Gamma_2(x_1, x_2)$. Indeed, if $x_i \in \Gamma_i(x_1, x_2)$, then

$$\xi_{e_i}(F_i(x_1, x_2, x_i)) \nsubseteq [0, +\infty).$$

It follows from Remark 3.1 that

$$F_i(x_1, x_2, x_i) \cap -intP_i \neq \emptyset,$$

which is contradicted by condition (i). Moreover

$$\begin{aligned}\Gamma_i^{-1}(y_i) &= \{(x_1, x_2) \in K_1 \times K_2 : y_i \in \Gamma_i(x_1, x_2)\}\\ &= \{(x_1, x_2) \in K_1 \times K_2 : \xi_{e_i}(F_i(x_1, x_2, y_i)) \nsubseteq [0, +\infty)\}\\ &= \{(x_1, x_2) \in K_1 \times K_2 : F_i(x_1, x_2, y_i) \cap -intP_i \neq \emptyset\}\end{aligned}$$

is an open set by (iii), for every $y_i \in K_i$ $(i = 1, 2)$.
Finally, applying condition (iv), there exists a nonempty compact convex subset B_i of K_i and a nonempty compact subset N_i of K_i such that for each

$$(x_1, x_2) \in K_1 \times K_2 \setminus N_1 \times N_2,$$

there exists $i = 1, 2,\ y_i \in B_i$ satisfying

$$\xi_{e_i}(F_i(x_1, x_2, y_i)) \nsubseteq [0, +\infty),$$

therefore $y_i \in \Gamma_i(x_1, x_2)$. Hence

$$B_i \cap \Gamma_i(x_1, x_2) \neq \emptyset.$$

Thus all the conditions of Lemma 3.1 are satisfied and then there exists $(x_1^*, x_2^*) \in K_1 \times K_2$ such that

$$\Gamma_i(x_1^*, x_2^*) = \emptyset,$$

for $i = 1, 2$. This means that for all $i = 1, 2,\ y_i \in K_i$,

$$\xi_{e_i}(F_i(x_1^*, x_2^*, y_i)) \subseteq [0, +\infty).$$

Then, applying Remark 3.1, (x_1^*, x_2^*) is a solution of $GVEP$. It is straightforward to see that, the solution set of GVEP is convex, if for all $(y_1, y_2) \in K_1 \times K_2$, the set

$$\{(x_1, x_2) \in K_1 \times K_2 : \xi_{e_i}(F_i(x_1, x_2, y_i)) \subseteq \mathbb{R}^2_+\}$$

is convex, for $i = 1, 2$. □

Example 3.1. *Let* $X_1 = X_2 = Y_1 = Y_2 = \mathbb{R}, P_1 = P_2 = [0, +\infty), K_1 = K_2 = [0, 1]$ *and* $e = (1, 1)$. *I define the functions*
$F_1 : K_1 \times K_2 \times K_1 \longrightarrow 2^{\mathbb{R}}$ *and* $F_2 : K_1 \times K_2 \times K_2 \longrightarrow 2^{\mathbb{R}}$ *by*

$$F_1(x_1, x_2, z_1) = [x_1 - z_1, e^{x_1+x_2}],$$

and

$$F_2(x_1, x_2, z_2) = [x_2 - z_2, e^{x_1+x_2}].$$

The functions F_1 *and* F_2 *fulfill the conditions of Theorem 3.1. Condition (i) trivially holds. For* $i = 1, 2$,

$$F_i(x_1, x_2, x_i) \cap -intP_1 = [0, e^{x_1+x_2}] \cap (-\infty, 0) = \emptyset.$$

Therefore condition (i) is valid.
To verify condition (ii), let $(x_1, x_2) \in [0, 1] \times [0, 1]$, *then for* $i = 1, 2$,

$$\begin{aligned} A_i &= \{y_i \in K_i : F_i(x_1, x_2, y_i) \cap -intP_i \neq \emptyset\} \\ &= \{y_i \in [0, 1] : [x_i - y_i, e^{x_1+x_2}] \cap (-\infty, 0) \neq \emptyset\} \\ &= \{y_i \in [0, 1] : x_i < y_i\} = (x_i, 1] \end{aligned}$$

is convex set. Now, let $y_1 \in K_1 = [0, 1]$, *then*

$$\begin{aligned} G_{y_1} &= \{(x_1, x_2) \in [0, 1] \times [0, 1] : F_1(x_1, x_2, y_1) \cap -intP_1 \neq \emptyset\} \\ &= \{(x_1, x_2) \in [0, 1] \times [0, 1] : [x_1 - y_1, e^{x_1+x_2}] \cap (-\infty, 0) \neq \emptyset\} \\ &= \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1 < y_1\} = (x_1, 1] \times [0, 1] \end{aligned}$$

is open set. Also the set G_{y_2} *is open. Hence condition (iii) is valid. Finally, to show condition (iv), it is enough to take* $B_i = K_i = [0, 1]$, *and* $N_i = \{1\}$, *for* $i = 1, 2$. *Indeed for every* $(x_1, x_2) \in K_1 \times K_2 \setminus N_1 \times N_2$, *take* $y_i \in B_i$ *such that* $y_i > x_i$, *then*

$$F_i(x_1, x_2, y_i) \cap -intP_i = [x_i - y_i, e^{x_1+x_2}] \cap (-\infty, 0) \neq \emptyset.$$

Therefore,

$$\xi_{e_i}(F_i(x_1, x_2, x_i)) \nsubseteq [0, +\infty),$$

for $i = 1, 2$. *Hence, applying Theorem 3.1, GVEP has a solution. It is obvious that* $(x_1^*, x_2^*) = (1, 1)$ *is a solution of GVEP. In other words, for each* $i = 1, 2$,

$$F_i((1, 1), z_i) \cap (-\infty, 0) = \emptyset,$$

for all $z_i \in K_i$.

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