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LINEAR COMBINATIONS OF q-STARLIKE FUNCTIONS OF ORDER ALPHA

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Abstract. In this paper, we introduced a new concept of bounded radius rotation to define the class of q-starlike functions of order α using the q-derivative, some geometric properties of linear combination of such functions are studied.

Keywords: q-derivative, q-starlike functions, convex functions, linear combination, bounded radius rotation.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\},$$

and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. Jackson[5] initiated q-calculus and developed the concept of the q-integral and q-derivative. For a function $f \in \mathcal{S}$ given by (1) and $0 < q < 1$, the q-derivative of f is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2}$$

Equivalently (2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

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Note that as $q \to 1$, $[n]_q \to n$.
Now, recall the definition of the class of q-starlike functions of order α, $0 \le \alpha < 1$, denoted by $S_q^*(\alpha)$.

Definition 1.1. [2] *A function $f \in \mathcal{A}$ is said to belong to the class $S_q^*(\alpha)$ if*

$$\left| \frac{\frac{z\partial_q f(z)}{f(z)} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \ z \in \mathcal{U}, \tag{3}$$

where $\partial_q f(z)$ is defined by (2) *and* $0 < q < 1$.

The following is the equivalent form of Definition 1.1

$$f \in S_q^*(\alpha) \Longleftrightarrow \left| \frac{z\partial_q f(z)}{f(z)} - \frac{1-\alpha q}{1-q} \right| \le \frac{1-\alpha}{1-q}. \tag{4}$$

We note that as $q \to 1^-$ the closed disc $|\omega - (1-q)^{-1}| \le (1-q)^{-1}$ becomes the right-half plane and the class $S_q^*(\alpha)$ reduces to $S^*(\alpha)$, the subclass of $\mathcal{A}$ consisting of functions which are starlike of order $\alpha (0 < \alpha < 1)$ in $\mathcal{U}$. In particular, when $\alpha = 0$, the class $S_q^*(\alpha)$ coincides with the class $S_q^* := S_q^*(0)$, which was first introduced by Ismail et al [4] in 1990 and later it has been considered in [1, 8, 10, 6, 7].
Observe that (3) holds if and only if

$$\frac{z\partial_q f(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-qz}, \tag{5}$$

where $\prec$ denotes subordination.
Using the definition of the class of $S_q^*(\alpha)$ and (5) it can be seen that linear transformation $\frac{1+(1-2\alpha)z}{1-qz}$ maps $|z| = r$ onto the circle with center $C(r) = \frac{1+(1-2\alpha)qr}{1-q^2r^2}$ and the radius $\rho(r) = \frac{(1-\alpha)(1+q)r}{1-q^2r^2}$.
Thus using subordination principle, we can write

$$\left| \frac{z\partial_q f(z)}{f(z)} - \frac{1+(1-2\alpha)qr^2}{1-q^2r^2} \right| \le \frac{(1-\alpha)(1+q)r}{1-q^2r^2}. \tag{6}$$

Definition 1.2. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 0$. Then $p \in P_m(q,\alpha)$ if and only if,*

$$P(z) = (\frac{m}{4} + \frac{1}{2})p_1(z) - (\frac{m}{4} - \frac{1}{2})p_2(z).$$

where $p_j(z) \prec \frac{1+(1-2\alpha)z}{1-qz}$, $j = 1,2$, $0 < q < 1$, $m \ge 2$.

For $m = 2$ and $\alpha = 0$, $P_2(q) = P(q)$ consists all functions subordinate to $\frac{1+z}{1-qz}$, $z \in \mathcal{U}$. Also $\lim_{q \to 1^-} P(q) = P$, the class of functions with positive real part.

Definition 1.3. *Let $f \in \mathcal{A}$. Then $f \in R_q^*(m,\alpha)$, if and only if, $\frac{z\partial_q f(z)}{f(z)} \in P_m(q,\alpha)$, $z \in \mathcal{U}$.*

f in this case, is called a function of q-bounded radius rotation.
Observe that $R_q^*(2,0) = S_q^*$ and as $q \to 1^-$, $\alpha = 0$, $R_q^*(m,\alpha) = R_m$, the class of functions with bounded radius rotation.

2. Main results

We need the following lemmas, to prove our main results.

Lemma 2.1. *Let $f \in R_q^*(m,\alpha)$. Then for $m \geq 2$, $0 < q < 1$*

$$\left|\frac{z\partial_q f(z)}{f(z)} - \frac{1+(1-2\alpha)qr^2}{1-q^2r^2}\right| \leq \frac{\frac{m}{2}(1-\alpha)(1+q)r}{1-q^2r^2}. \quad (7)$$

Lemma 2.2. *If $|u-a| \leq d$ and $|v-a| \leq d$ where a and d are real and $a > d \geq 0$, and*

$$\omega = u\frac{1}{1+Ae^{i\beta}} + v\frac{1}{1+A^{-1}e^{-i\beta}},$$

where A is real and $A > 0$ and $\beta \in [0,\beta)$, then

$$\mathcal{R}(\omega) \geq a - d\sec(\frac{\beta}{2}).$$

Lemma 2.3. *Let $f \in R_q^*(m,\alpha)$. Then $f \in S_q^*(\alpha)$ for $|z| < r_q^*(\alpha)$. where*

$$r_q^*(\alpha) = \frac{4(1-2\alpha)}{m(1+q-2\alpha)+\sqrt{m^2(1+q-2\alpha)^2-16(1-2\alpha)q}}. \quad (8)$$

Proof. Since $f \in R_q^*(m,\alpha)$, we have

$$\frac{z\partial_q f(z)}{f(z)} = p(z) \in P_m(q,\alpha).$$

This implies that $p(z)$ can be written as

$$P(z) = (\frac{m}{4}+\frac{1}{2})p_1(z) - (\frac{m}{4}-\frac{1}{2})p_2(z).$$

where $p_j(z) \prec \frac{1+(1-2\alpha)z}{1-qz}$, $j = 1,2$, $0 < q < 1$, $m \geq 2$.
Therefore

$$\mathcal{R}(\frac{z\partial_q f(z)}{f(z)}) = \mathcal{R}(p(z)) \geq (\frac{m}{4}+\frac{1}{2})(\frac{1+(1-2\alpha)r}{1-qr}) - (\frac{m}{4}-\frac{1}{2})(\frac{1-(1-2\alpha)r}{1+qr})$$

$$= \frac{1+\frac{m}{2}(1+q-2\alpha)r+(1-2\alpha)qr^2}{1-q^2r^2},$$

and from this, it follows that $\mathcal{R}(\frac{z\partial_q f(z)}{f(z)}) \geq 0$ for $|z| < r_q^*(\alpha)$. where $r_q^*(\alpha)$ is given by (8). □

Observe that as $\alpha = 0$, $f \in R_q^*(m)$ and in this case $\mathcal{R}(\frac{z\partial_q f(z)}{f(z)}) > 0$ for $|z| < r_q^*$, where $r_q^* = \frac{4}{m(1+q)+\sqrt{m^2(1+q)^2-16q}}$, see [7] and as $q \to 1^-$, $\alpha = 0$, $f \in R_m$ and in this case $\mathcal{R}(\frac{zf'(z)}{f(z)}) > 0$ for $|z| < r^* = \frac{2}{m+\sqrt{m^2-4}}$, see [3].

Lemma 2.4. *Let $f \in R_q^*(m,\alpha)$. Then*
$|\arg f(z)| \leq \frac{m}{2}(1-\alpha)(1+q)\sin^{-1} r$ *and* $|\arg f'(z)| \leq m(1-\alpha)(1+q)\sin^{-1} r$.

Theorem 2.1. *Let $f_1, f_2 \in R_q^*(m,\alpha)$ and let*

$$F(z) = \lambda f_1(z) + (1-\lambda)f_2(z), \quad (9)$$

where $0 \leq \mu = \arg\frac{\lambda}{1-\lambda} < \pi$. Then $F \in S_q^*(\alpha)$ in $|z| < r_{q,m}(\alpha)$ where $r_{q,m}(\alpha)$ is the smallest positive value of r satisfying the equation

$$g(r) = [1+(1-2\alpha)qr^2]\cos(\frac{\mu}{2}+\frac{m}{2}(1-\alpha)(1+q)\sin^{-1} r) - \frac{m}{2}(1-\alpha)(1+q)r\sin^{-1} r = 0.$$

Proof. Using q-difference operator of (15), we obtained
$\partial_q F(z) = \lambda \partial_q f_1(z) + (1-\lambda)\partial_q f_2(z)$,
and therefore
$\frac{\partial_q F(z)}{F(z)} = \frac{\lambda \partial_q f_1(z) + (1-\lambda)\partial_q f_2(z)}{\lambda f_1(z) + (1-\lambda) f_2(z)}$

$$= \frac{z\partial_q f_1(z)}{f_1(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}.\frac{f_1(z)}{f_2(z)}\right)^{-1}\right]^{-1} + \frac{z\partial_q f_2(z)}{f_2(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}.\frac{f_1(z)}{f_2(z)}\right)\right]^{-1}. \quad (10)$$

Put

$$u = \frac{z\partial_q f_2(z)}{f_2(z)},\ v = \frac{z\partial_q f_1(z)}{f_1(z)},\ A = \left|\frac{\lambda}{1-\lambda}.\frac{f_1(z)}{f_2(z)}\right|. \quad (11)$$

From (10) and (17), we obtained

$$\omega(z) = \frac{\partial_q F(z)}{F(z)} = u\frac{1}{1+Ae^{i\beta}} + v\frac{1}{1+A^{-1}e^{-i\beta}}. \quad (12)$$

Using Lemma 2.1 and Lemma 2.2, we obtained

$$\mathcal{R}\left\{\frac{\partial_q F(z)}{F(z)}\right\} \geq \frac{1+(1-2\alpha)qr^2}{1-q^2r^2} - \frac{\frac{m}{2}(1-\alpha)(1+q)r}{1-q^2r^2}\sec(\frac{\beta}{2}), \quad (13)$$

where
$\beta = \arg\left(\frac{\lambda}{1-\lambda}.\frac{f_1(z)}{f_2(z)}\right) = 2n\pi + \mu + \arg f_1(z) - \arg f_2(z)$.
Now by Lemma 15,
$|\beta| \leq \mu + m(1-2\alpha)(1+q)\sin^{-1} r$,
and this gives us $\sec(\frac{\beta}{2}) \leq \frac{1}{\cos\left(\frac{\mu}{2} + \frac{m}{2}(1-\alpha)(1+q)\sin^{-1} r\right)}$.
Therefore
$\mathcal{R}\left\{\frac{\partial_q F(z)}{F(z)}\right\} \geq 0$, if

$$g(r) = [1+(1-2\alpha)qr^2]\cos(\frac{\mu}{2} + \frac{m}{2}(1-\alpha)(1+q)\sin^{-1} r) - \frac{m}{2}(1-\alpha)(1+q)r > 0.$$

We note that
$g(r) = \cos(\frac{\mu}{2})$, for $r = 0$, and
$g(r) = -\frac{m}{2}(1-\alpha)(1+q)\sin\left(\frac{\pi-\mu}{m(1-\alpha)(1+q)}\right) < 0$, when $r = \sin\left(\frac{\pi-\mu}{m(1-\alpha)(1+q)}\right)$.
This implies that $g(r) = 0$ has a root in the interval $\left(0, \sin\left(\frac{\pi-\mu}{m(1-\alpha)(1+q)}\right)\right)$ and right hand side of (13) is positive in the disc $|z| < r_{q,m}(\alpha)$, where $r_{q,m}(\alpha)$ is the least positive value of r satisfying $g(r) = 0$. □

As $\alpha = 0$, we have the following result, proved by Noor et al [6].

Corollary 2.1. *Let* $f_1, f_2 \in R_q^*(m)$ *and let*

$$F(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \quad (14)$$

where $0 \leq \mu = \arg\frac{\lambda}{1-\lambda} < \pi$. *Then* $F \in S_q^*$ *in* $|z| < r_{q,m}$ *where* $r_{q,m}$ *is the smallest positive value of* r *satisfying the equation*

$$g(r) = [1+qr^2]\cos(\frac{\mu}{2} + \frac{m}{2}(1+q)\sin^{-1} r) - \frac{m}{2}(1+q)r = 0.$$

As $q \to 1^-$ and for $\alpha = 0$, we get the following result, introduced by Noor et al [6].

Corollary 2.2. *Let $f_1, f_2 \in R(m)$ and let*

$$F(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \tag{15}$$

where $0 \leq \mu = \arg \frac{\lambda}{1-\lambda} < \pi$. Then $F \in S^*$ in $|z| < r_m^*$ where r_m^* is the smallest positive value of r satisfying the equation $g_m(r) = B(1+r^2) - mr = 0$, $B = \cos\left(\frac{\mu}{2} + m \sin^{-1} r\right)$. This gives us $r_m^* = \frac{m+\sqrt{m^2-4B^2}}{2B}$. As a special case of Corollary 2.2, we take $m = 2$. Therefore $B = B_2 = \cos\left(\frac{\mu}{2} + 2\sin^{-1} r\right)$, and $\lim_{q\to 1^-} R_q^*(2) = S^*$.
From these observations, we deduce the radius of starlikeness of linear combination of two starlike functions is given by $r_2^* = \frac{1-\sqrt{1-B_2^2}}{B^2}$.

Corollary 2.3. *As $\alpha = 0$, and $m = 2$. Then, in Theorem 2.1, $f_1, f_2 \in S_q^*$ and it follows that*

$$\mathcal{R}\left\{\frac{\partial_q F(z)}{F(z)}\right\} \geq 0 \text{ in } |z| < r_q^*.$$

where r_q^* is the least positive root of
$g_q(r) = D_1 q r2 - (1+q)r + D_1 = 0$, where $D_1 = \cos\left(\frac{\mu}{2} + (1+q)\sin^{-1} r\right)$.
and hence $r_q^* = \frac{(1+q)-\sqrt{(1+q)^2-4qD_1^2}}{2qD_1}$.

Theorem 2.2. *Let $f_1, f_2 \in \bigcap_{0<q<1} S_q^*(\alpha)$ and let*

$$F(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \tag{16}$$

where $0 \leq \mu = \arg \frac{\lambda}{1-\lambda} < \pi$. Then F maps the disc $|z| < r_\mu$ onto a convex domain, where r_μ is the least positive value of r that satisfies the equation
$g_\mu(r) = Dr^2 - 2r_1 r + Dr_1^2$, where $r_1 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha}$, $D = \cos(\frac{\mu}{2} + 2(1-\alpha)\sin^{-1}(\frac{r}{r_1}))$.

Proof. It has been shown in [2] that

$$\bigcap_{0<q<1} S_q^*(\alpha) = S^*(\alpha).$$

It is well known [9] that $f \in S^*(\alpha)$ is convex of order α in the disc $|z| < r_1 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha}$. with these facts, we proceed to find the radius of convexity for the function F following the technique used in Theorem 2.1.
We can write

$$1+\frac{zF''(z)}{F'(z)} = \left[1+\frac{zf_1''(z)}{f_1'(z)}\right]\left[1+\left(\frac{\lambda}{1-\lambda}.\frac{f_1'(z)}{f_2'(z)}\right)^{-1}\right]^{-1} + \left[1+\frac{zf_2''(z)}{f_2'(z)}\right]\left[1+\left(\frac{\lambda}{1-\lambda}.\frac{f_1(z)}{f_2(z)}\right)\right]^{-1}.$$

Put

$$u = \left[1+\frac{zf_1''(z)}{f_1'(z)}\right],\ v = \left[1+\frac{zf_2''(z)}{f_2'(z)}\right],\ A = \left|\frac{\lambda}{1-\lambda}.\frac{f_1'(z)}{f_2'(z)}\right|,\ \beta = \arg\left(\frac{\lambda}{1-\lambda}.\frac{f_1'(z)}{f_2'(z)}\right). \tag{17}$$

Now, for $r_1 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha}$, we have

$$\left|u - \frac{r_1^2+r^2}{r_1^2-r^2}\right| \leq \frac{2rr_1}{r_1^2-r^2},$$
$$\left|v - \frac{r_1^2+r^2}{r_1^2-r^2}\right| \leq \frac{2rr_1}{r_1^2-r^2}.$$

Therefore

$$\omega(z) = 1+\frac{zF''(z)}{F'(z)} = u\frac{1}{1+Ae^{i\beta}} + v\frac{1}{1+A^{-1}e^{-i\beta}}. \tag{18}$$

where
$\beta = \arg\left(\frac{\lambda}{1-\lambda}.\frac{f_1'(z)}{f_2'(z)}\right) = 2n\pi + \mu + \arg f_1'(z) - \arg f_2'(z)$, and so
$|\beta| \leq \mu + 4(1-\alpha)\sin^{-1}(\frac{r}{r-1})$,
Therefore
$\mathcal{R}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} > 0$, if $T_\mu(r) = (r_1^2 + r^2)\cos(\frac{\mu}{2} + 2(1-\alpha)\sin^{-1}(\frac{r}{r-1}) - 2r_1 r = 0$, where $r_1 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha}$.
That is
$T_\mu(r) = Dr^2 - 2r_1 r + Dr_1^2$, $D = \cos(\frac{\mu}{2} + 2(1-\alpha)\sin^{-1}(\frac{r}{r_1}))$.
Hence

$$r_\mu = \frac{r_1 - \sqrt{r_1^2 - D^2 r_1^2}}{D}. \tag{19}$$

Hence F maps the disc $|z| < r_\mu$ onto a convex of order α domain, where r_μ is given by (19). □

Remark As $\alpha = 0$ Theorem 2.1 reduces to Theorem 2 in [6].

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