APPLICATIONS OF CHEBYSHEV POLYNOMIALS ON λ-PSEUDO BI-STARLIKE AND λ-PSEUDO BI-CONVEX FUNCTIONS WITH RESPECT TO SYMMETRICAL POINTS

ABBAS KAREEM WANAS, §

Abstract. The purpose of this work is to use the Chebyshev polynomial expansions to seek upper bounds for the second and third coefficients of functions belongs to a subclass of λ-pseudo bi-starlike and λ-pseudo bi-convex functions with respect to symmetrical points in the open unit disk.

Keywords: Coefficient bounds, bi-univalent functions, Chebyshev polynomials, λ-pseudo bi-starlike with respect to symmetrical points, λ-pseudo bi-convex with respect to symmetrical points, subordination.

AMS Subject Classification: 30C45, 30C50.

1. Introduction

The importance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Several researchers dealing with orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications one can refer [7, 9, 11]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \qquad and \qquad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1),$$

where n indicates the polynomial degree and $t = \cos n\theta$.

Let $\mathcal{A}$ stand for the family of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are univalent in U. It is well known (see [8]) that every function $f \in S$ has an inverse f^{-1}, defined by

Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq.
e-mail: abbas.kareem.w@qu.edu.iq; ORCID: https://orcid.org/0000-0001-5838-7365.
§ Manuscript received: December 12, 2018; accepted: February 7, 2019.

$f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots . \qquad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ stand for the class of bi-univalent functions in U given by (1). For a brief history and interesting examples of functions that are in (or are not in) the class Σ, together with various other properties of the bi-univalent functions class Σ, one can refer the work of Srivastava et al. [13] and the references stated therein. Recently, many authors introduced various subclasses of the bi-univalent functions class Σ and investigated non sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) (see [1, 2, 3, 4, 6, 10]).

A function $f \in S$ is called starlike with respect to symmetrical points, if (see [12])

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in U.$$

The set of all such functions is denote by S_s^*.
The class of starlike functions with respect to symmetrical points obviously includes the class of convex functions with respect to symmetrical points, C_s, satisfying the following condition:

$$Re\left\{\frac{(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0, \quad z \in U.$$

Recently, Babalola [5] defined the class $\mathcal{L}_\lambda$ of λ-pseudo-starlike functions as follows: Let $f \in \mathcal{A}$ and $\lambda \geq 1$. Then $f \in \mathcal{L}_\lambda$ of λ-pseudo-starlike functions in U if and only if

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right\} > 0, \quad z \in U.$$

With a view to recalling the principal of subordination between analytic functions, let the functions f and g be analytic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g\left(w(z)\right)$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.1. *For $\lambda \geq 1$, $\gamma \geq 0$ and $t \in (\frac{1}{2}, 1]$, a function $f \in \Sigma$ is said to be in the class $T_\Sigma^s(\lambda, \gamma, t)$ if it satisfies the subordinations:*

$$\left(\frac{2z\left(f'(z)\right)^\lambda}{f(z) - f(-z)}\right)^\gamma \left(\frac{2\left((zf'(z))'\right)^\lambda}{(f(z) - f(-z))'}\right)^{1-\gamma} \prec H(z,t) = \frac{1}{1 - 2tz + z^2}$$

and

$$\left(\frac{2w\left(g'(w)\right)^\lambda}{g(w) - g(-w)}\right)^\gamma \left(\frac{2\left((wg'(w))'\right)^\lambda}{(g(w) - g(-w))'}\right)^{1-\gamma} \prec H(w,t) = \frac{1}{1 - 2tw + w^2},$$

where the function $g = f^{-1}$ is given by (2).

We note that if $t = \cos\beta$, where $\beta \in (\frac{-\pi}{3}, \frac{\pi}{3})$, then

$$H(z,t) = \frac{1}{1 - 2\cos\beta z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\beta}{\sin\beta} z^n, \ z \in U.$$

Therefore

$$H(z,t) = 1 + 2\cos\beta z + \left(3\cos^2\beta - \sin^2\beta\right) z^2 + \cdots , \ z \in U.$$

From [15], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \ (z \in U, t \in (-1,1)),$$

where

$$U_{n-1} = \frac{\sin(n\, arc\cos t)}{\sqrt{1-t^2}} \ (n \in \mathbb{N} = \{1,2,\cdots\}),$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t,\ U_2(t) = 4t^2 - 1,\ U_3(t) = 8t^3 - 4t, \cdots . \tag{3}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$ is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2},\ z \in U.$$

The Chebyshev polynomials of first kind $T_n(t)$ and of the second kind $U_n(t)$ are connected by

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),\ T_n(t) = U_n(t) - tU_{n-1}(t),\ 2T_n(t) = U_n(t) - U_{n-2}(t).$$

2. Main Results

Theorem 2.1. *For $\lambda \geq 1$, $\gamma \geq 0$ and $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $T_{\Sigma}^{s}(\lambda,\gamma,t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{\left|(\lambda+2\gamma-3)\,t^2 - \lambda^2\,(\gamma-2)^2\,(2t^2-1)\right|}}$$

and

$$|a_3| \leq \frac{t^2}{\lambda^2\,(\gamma-2)^2} + \frac{2t}{(3\lambda-1)\,|3-2\gamma|}.$$

Proof. Let $f \in T_{\Sigma}^{s}(\lambda,\gamma,t)$. Then there exists two analytic functions $u, v : U \longrightarrow U$ given by

$$u(z) = u_1z + u_2z^2 + u_3z^3 + \cdots \quad (z \in U) \tag{4}$$

and

$$v(w) = v_1w + v_2w^2 + v_3w^3 + \cdots \quad (w \in U), \tag{5}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

$$\left(\frac{2z\,(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\gamma}\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-f(-z))'}\right)^{1-\gamma} = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots \tag{6}$$

and

$$\left(\frac{2w\,(g'(w))^{\lambda}}{g(w)-g(-w)}\right)^{\gamma}\left(\frac{2\left((wg'(w))'\right)^{\lambda}}{(g(w)-g(-w))'}\right)^{1-\gamma} = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots . \tag{7}$$

Combining (4), (5), (6) and (7), we obtain

$$\left(\frac{2z\,(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\gamma}\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-f(-z))'}\right)^{1-\gamma} = 1 + U_1(t)u_1z + \left[U_1(t)u_2 + U_2(t)u_1^2\right]z^2 + \cdots \tag{8}$$

and

$$\left(\frac{2w\,(g'(w))^{\lambda}}{g(w)-g(-w)}\right)^{\gamma}\left(\frac{2\left((wg'(w))'\right)^{\lambda}}{(g(w)-g(-w))'}\right)^{1-\gamma}=1+U_1(t)v_1w+\left[U_1(t)v_2+U_2(t)v_1^2\right]w^2+\cdots. \tag{9}$$

It is well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\le 1 \quad and \quad |v_i|\le 1\; forall\; i\in\mathbb{N}. \tag{10}$$

Comparing the corresponding coefficients in (8) and (9), after simplifying, we have

$$-2\lambda(\gamma-2)a_2=U_1(t)u_1, \tag{11}$$

$$2\left[\lambda^2(\gamma-2)^2+\lambda(3\gamma-4)\right]a_2^2+(3\lambda-1)(3-2\gamma)a_3=U_1(t)u_2+U_2(t)u_1^2, \tag{12}$$

$$2\lambda(\gamma-2)a_2=U_1(t)v_1 \tag{13}$$

and

$$2\left[\lambda^2(\gamma-2)^2+\lambda(5-3\gamma)+(2\gamma-3)\right]a_2^2+(3\lambda-1)(2\gamma-3)a_3=U_1(t)v_2+U_2(t)v_1^2. \tag{14}$$

It follows from (11) and (13) that

$$u_1=-v_1 \tag{15}$$

and

$$8\lambda^2(\gamma-2)^2a_2^2=U_1^2(t)(u_1^2+v_1^2). \tag{16}$$

If we add (12) to (14), we find that

$$2\left(2\lambda^2(\gamma-2)^2+(\lambda+2\gamma-3)\right)a_2^2=U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2). \tag{17}$$

Substituting the value of $u_1^2+v_1^2$ from (16) in the right hand side of (17), we get

$$2\left[2\lambda^2(\gamma-2)^2\left(1-\frac{2U_2(t)}{U_1^2(t)}\right)+(\lambda+2\gamma-3)\right]a_2^2=U_1(t)(u_2+v_2). \tag{18}$$

Further computations using (3), (10) and (18), we obtain

$$|a_2|\le\frac{t\sqrt{2t}}{\sqrt{\left|(\lambda+2\gamma-3)\,t^2-\lambda^2(\gamma-2)^2(2t^2-1)\right|}}.$$

Next, if we subtract (14) from (12), we deduce that

$$2(3\lambda-1)(3-2\gamma)(a_3-a_2^2)=U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{19}$$

In view of (15) and (16), we get from (19)

$$a_3=\frac{U_1^2(t)}{8\lambda^2(\gamma-2)^2}(u_1^2+v_1^2)+\frac{U_1(t)}{2(3\lambda-1)(3-2\gamma)}(u_2-v_2).$$

Thus applying (3), we obtain

$$|a_3|\le\frac{t^2}{\lambda^2(\gamma-2)^2}+\frac{2t}{(3\lambda-1)|3-2\gamma|}.$$

□

For $\gamma=1$, the class $T_{\Sigma}^{s}(\lambda,\gamma,t)$ reduced to the class $T_{\Sigma}^{s}(\lambda,1,t)$ of λ-pseudo bi-starlike functions with respect to symmetrical points. For functions belongs to this class, we conclude the following result.

Corollary 2.1. *For $\lambda \geq 1$ and $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $T_{\Sigma}^{s}(\lambda, 1, t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(\lambda-1)\,t^2 - \lambda^2\,(2t^2-1)|}}$$

and

$$|a_3| \leq \frac{t^2}{\lambda^2} + \frac{2t}{3\lambda - 1}.$$

For $\gamma = 0$, the class $T_{\Sigma}^{s}(\lambda, \gamma, t)$ reduced to the class $T_{\Sigma}^{s}(\lambda, t)$ of λ-pseudo bi-convex functions with respect to symmetrical points. For functions belongs to this class, we conclude the following result.

Corollary 2.2. *For $\lambda \geq 1$ and $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $T_{\Sigma}^{s}(\lambda, t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(\lambda-3)\,t^2 - 4\lambda^2\,(2t^2-1)|}}$$

and

$$|a_3| \leq \frac{t^2}{4\lambda^2} + \frac{2t}{3(3\lambda - 1)}.$$

If we choose $\lambda = 1$ in Corollary 2.2, we obtain the result for well-known class $F_{\Sigma}^{sc}(t)$ which was considered recently by Wanas and Majeed [14].

Corollary 2.3. ([14]) *For $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $F_{\Sigma}^{sc}(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2 - 5t^2|}}$$

and

$$|a_3| \leq \frac{t(3t+4)}{12}.$$

References

[1] Ali, R. M., Lee, S. K., Ravichandran, V. and Supramaniam, S., (2012), Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett., 25, pp. 344–51.

[2] Altinkaya, S. and Yalçin, S., (2015), Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, J. Funct. Spaces, Art. ID 145242, pp. 1-5.

[3] Altinkaya, S. and Yalçin, S., (2016), On the Chebyshev polynomial bounds for classes of univalent functions, Khayyam J. Math., 2 (1), pp. 1-5.

[4] Altinkaya, S. and Yalçin, S., (2017), On a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions, Malaya J. Math., 5 (2), pp. 305-309.

[5] Babalola, K. O., (2013), On λ-pseudo-starlike functions, J. Class. Anal., 3, (2), pp. 137-147.

[6] Caglar, M., Orhan, H. and Yagmur, N., (2013), Coefficient bounds for new subclasses of bi-univalent functions, Filomat, 27, pp. 1165–1171.

[7] Doha, E. H., (1994), The first and second kind Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function, Int. J. Comput. Math., 51, pp. 21–35.

[8] Duren, P. L., (1983), Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo.

[9] Dziok, J., Raina, R. K. and Sokol, J., (2015), Application of Chebyshev polynomials to classes of analytic functions, C. R. Math. Acad. Sci. Paris, Ser I, 353, pp. 433–438.

[10] Magesh, N. and Prameela, V., (2013), Coefficient estimate problems for certain subclasses of analytic and bi-univalent functions, Afr. Mat., 26 (3), pp. 465-470.

[11] Mason, J. C., (1967), Chebyshev polynomials approximations for the L-membrane eigenvalue problem, SIAM J. Appl. Math., 15, pp. 172-186.

[12] Sakaguchi, K., (1959), On a certain univalent mapping, J. Math. Soc. Japan, 11 (1), pp. 72-75.

[13] Srivastava, H. M., Mishra, A. K. and Gochhayat, P., (2010), Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23, pp. 1188–1192.
[14] Wanas, A. K. and Majeed, A. H., (2019), Chebyshev polynomial bounded for analytic and bi-univalent functions with respect to symmetric conjugate points, Applied Mathematics E-Notes, 19, pp. 14-21.
[15] Whittaker, T. and Watson, G. N., (1996), A Course of Modern Analysis, Cambridge: Cambridge Univ. Press.

Abbas Kareem Wanas completed his doctoral studies at the University of Baghdad in Iraq and is currently a lecturer at the College of Science, University of Al-Qadisiyah. His research interests include Complex Analysis, Geometric Function Theory, Univalent and Multivalent Functions, Fractional Calculus and Operator Theory. He also serves the scientific community as Editorial Board Member and a reviewer for some scientific journals.