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LIFTS OF (0,2) TENSOR FIELDS IN THE SEMI-TANGENT BUNDLE

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Abstract. In this paper the vertical, complete and horizontal lifts of tensor fields of type (0, 2) to semi-tangent bundle and their properties are studied.

Keywords: Complete lift, Degenerate metric, Horizontal lift, Pull-back bundle, Semi-tangent bundle.

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1. Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let (M_n, π_1, B_m) be a differentiable bundle over B_m. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, ...$ run from 1 to n, the indices $a, b, ...$ from 1 to $n-m$ and the indices $\alpha, \beta, ...$ from $n-m+1$ to n, x^α are coordinates in B_m, x^a are fibre coordinates of the bundle

$$\pi_1 : M_n \to B_m.$$

Let now $(T(B_m), \widetilde{\pi}, B_m)$ be a tangent bundle [12] over base space B_m, and let M_n be differentiable bundle determined by a natural projection (submersion) $\pi_1 : M_n \to B_m$. The semi-tangent bundle (pull-back [[1],[2],[6],[9]]) of the tangent bundle $(T(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t(B_m), \pi_2, M_n)$ over differentiable bundle M_n with a total space

$$\begin{aligned} t(B_m) &= \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times T_x(B_m) : \pi_1 (x^a, x^\alpha) = \widetilde{\pi} \left(x^\alpha, x^{\overline{\alpha}} \right) = (x^\alpha) \right\} \\ &\subset M_n \times T_x(B_m) \end{aligned}$$

and with the projection map $\pi_2 : t(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T_x(B_m)\,(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ is the tangent space at a point x of B_m, where $x^{\overline{\alpha}} = y^\alpha \; \left(\overline{\alpha}, \overline{\beta}, ... = n+1, ..., 2n\right)$ are fibre coordinates of the tangent bundle $T(B_m)$.

Where the pull-back (Pontryagin [3]) bundle $t(B_m)$ of the differentiable bundle M_n also has the natural bundle structure over B_m, its bundle projection $\pi : t(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$.

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Thus $(t(B_m),\pi_1\circ\pi_2)$ is the composite bundle [[4], p.9] or step-like bundle [5]. Consequently, we notice the semi-tangent bundle $(t(B_m),\pi_2)$ is a pull-back bundle of the tangent bundle over B_m by π_1 [6].

If $(x^{i'})=(x^{a'},x^{\alpha'})$ is another local adapted coordinates in differentiable bundle M_n, then we have

$$\begin{cases} x^{a'}=x^{a'}(x^b,x^\beta), \\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1}$$

The Jacobian of (1) has the components

$$\left(A_j^{i'}\right)=\left(\frac{\partial x^{i'}}{\partial x^j}\right)=\begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_b^{a'}=\frac{\partial x^{a'}}{\partial x^b}$, $A_\beta^{a'}=\frac{\partial x^{a'}}{\partial x^\beta}$, $A_\beta^{\alpha'}=\frac{\partial x^{\alpha'}}{\partial x^\beta}$ [6].

To a transformation (1) of local coordinates of M_n, there corresponds on $t(B_m)$ the change of coordinate

$$\begin{cases} x^{a'}=x^{a'}(x^b,x^\beta), \\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'}=\frac{\partial x^{\alpha'}}{\partial x^\beta}y^\beta. \end{cases} \tag{2}$$

The Jacobian of (2) is:

$$\bar{A}=\left(A_J^{I'}\right)=\begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'}y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix}, \tag{3}$$

where $I=(a,\alpha,\overline{\alpha})$, $J=(b,\beta,\overline{\beta})$, $I,J,....=1,...,2n$; $A_{\beta\varepsilon}^{\alpha'}=\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\varepsilon}$ [6]. Writing the inverse of (2) as

$$\begin{cases} x^{a}=x^{a}(x^{b'},x^{\beta'}), \\ x^{\alpha}=x^{\alpha}\left(x^{\beta'}\right), \\ x^{\overline{\alpha}}=\frac{\partial x^{\alpha}}{\partial x^{\beta'}}y^{\beta'}, \end{cases} \tag{4}$$

we have

$$\left(A_{J'}^{I}\right)=\begin{pmatrix} A_{b'}^{a} & A_{\beta'}^{a} & 0 \\ 0 & A_{\beta'}^{\alpha} & 0 \\ 0 & A_{\beta'\varepsilon'}^{\alpha}y^{\varepsilon'} & A_{\beta'}^{\alpha} \end{pmatrix}. \tag{5}$$

The main purpose of this paper is to study vertical, complete and horizontal lifts of tensor fields of type (0,2) to semi-tangent (pull-back) bundle $(t(B_m),\pi_2)$ and their metric properties [7, 8].

We denote by $\Im_q^p(M_n)$ the set of all tensor fields of class C^∞ and of type (p,q) on M_n, i.e., contravariant degree p and covariant degree q. We now put $\Im(M_n)=\sum_{p,q=0}^{\infty}\Im_q^p(M_n)$, which is the set of all tensor fields on M_n. Smilarly, we denote by $\Im_q^p(B_m)$ and $\Im(B_m)$ respectively the corresponding sets of tensor fields in the base space B_m.

2. Vertical Lifts of tensor field of type (0,2)

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi:t(B_m)\to B_m$ and ${}^{v}f=f\circ\pi_1$, so that

$${}^{vv}f={}^{v}f\circ\pi_2=f\circ\pi_1\circ\pi_2=f\circ\pi.$$

Thus, the vertical lift ${}^{vv}f$ of the function f to $t(B_m)$ satisfies

$$ {}^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha). \tag{6}$$

We note here that value ${}^{vv}f$ is constant along each fibre of $\pi : t(B_m) \to B_m$.

On the other hand, if $f = f(x^a, x^\alpha)$ is a function in M_n, we write ${}^{cc}f$ for the function in $t(B_m)$ defined by

$$ {}^{cc}f = \imath(df) = x^{\overline{\beta}}\partial_\beta f = y^\beta \partial_\beta f \tag{7}$$

and call of the complete lift ${}^{cc}f$ of the function f [6].

Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. On putting

$$ {}^{vv}X = \left({}^{vv}X^I\right) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{8}$$

from (3), we easily see that ${}^{vv}X' = \bar{A}({}^{vv}X)$. The vector field ${}^{vv}X$ is called the vertical lift of X to $t(B_m)$.

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [10] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Now, consider $\widetilde{X} \in \Im_0^1(M_n)$, then ${}^{cc}\widetilde{X}$ (complete lift) has the components on the semi-tangent bundle $t(B_m)$ [6]:

$$ {}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{9}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

Let $G \in \Im_2^0(M_n)$, i.e. $G = G_{\alpha\beta}dx^\alpha \otimes dx^\beta$. On putting

$$ {}^{vv}G = ({}^{vv}G_{IJ}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & G_{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

from (3), we easily see that ${}^{vv}G_{I'J'} = A_{I'}^I A_{J'}^J({}^{vv}G_{IJ})$. The tensor field ${}^{vv}G$ of type (0,2) is called the vertical lift of G to $t(B_m)$.

Since $Det({}^{vv}G) = 0$, we have:

Theorem 2.1. *The semi-tangent bundle $t(B_m)$ has a trivial metric ${}^{vv}G$.*

Theorem 2.2. *If G is tensor field of type (0,2) on B_m, and $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$, then*

(i) ${}^{vv}G({}^{vv}X, {}^{vv}Y) = 0,$

(ii) ${}^{vv}G({}^{vv}X, {}^{cc}Y) = 0,$

(iii) ${}^{vv}G({}^{cc}X, {}^{vv}Y) = 0,$

(iv) ${}^{vv}G({}^{cc}X, {}^{cc}Y) = {}^{vv}(G(X, Y)).$

Proof. (i) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $G \in \Im_2^0(B_m)$, from (8) and (10), then we have

$$\begin{aligned} {}^{vv}G({}^{vv}X, {}^{vv}Y) &= {}^{vv}G_{IJ}{}^{vv}X^I{}^{vv}Y^J \\ &= {}^{vv}G_{ab}\underbrace{{}^{vv}X^a}_{0}{}^{vv}Y^b + {}^{vv}G_{a\beta}\underbrace{{}^{vv}X^a}_{0}{}^{vv}Y^\beta + {}^{vv}G_{a\overline{\beta}}\underbrace{{}^{vv}X^a}_{0}{}^{vv}Y^{\overline{\beta}} \\ &\quad + {}^{vv}G_{\alpha b}\underbrace{{}^{vv}X^\alpha}_{0}{}^{vv}Y^b + {}^{vv}G_{\alpha\beta}\underbrace{{}^{vv}X^\alpha}_{0}{}^{vv}Y^\beta + {}^{vv}G_{\alpha\overline{\beta}}\underbrace{{}^{vv}X^\alpha}_{0}{}^{vv}Y^{\overline{\beta}} \\ &\quad + {}^{vv}G_{\overline{\alpha}b}{}^{vv}X^{\overline{\alpha}}\underbrace{{}^{vv}Y^b}_{0} + {}^{vv}G_{\overline{\alpha}\beta}{}^{vv}X^{\overline{\alpha}}\underbrace{{}^{vv}Y^\beta}_{0} + \underbrace{{}^{vv}G_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{vv}Y^{\overline{\beta}} \\ &= 0. \end{aligned}$$

(ii) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $G \in \Im_2^0(B_m)$, from (8), (9) and (10), then we have

$$\begin{aligned}
{}^{vv}G({}^{vv}X, {}^{cc}Y) &= {}^{vv}G_{IJ}{}^{vv}X^{I}{}^{cc}Y^{J} \\
&= {}^{vv}G_{ab}\underbrace{{}^{vv}X^{a}}_{0}{}^{cc}Y^{b} + {}^{vv}G_{a\beta}\underbrace{{}^{vv}X^{a}}_{0}{}^{cc}Y^{\beta} + {}^{vv}G_{a\overline{\beta}}\underbrace{{}^{vv}X^{a}}_{0}{}^{cc}Y^{\overline{\beta}} \\
&\quad + {}^{vv}G_{\alpha b}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{cc}Y^{b} + {}^{vv}G_{\alpha\beta}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{cc}Y^{\beta} + {}^{vv}G_{\alpha\overline{\beta}}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{cc}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{vv}G_{\overline{\alpha}b}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{cc}Y^{b} + \underbrace{{}^{vv}G_{\overline{\alpha}\beta}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{cc}Y^{\beta} + \underbrace{{}^{vv}G_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{cc}Y^{\overline{\beta}} \\
&= 0.
\end{aligned}$$

(iii) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $G \in \Im_2^0(B_m)$, from (8), (9) and (10), then we have

$$\begin{aligned}
{}^{vv}G({}^{cc}X, {}^{vv}Y) &= {}^{vv}G_{IJ}{}^{cc}X^{I}{}^{vv}Y^{J} \\
&= {}^{vv}G_{ab}{}^{cc}X^{a}\underbrace{{}^{vv}Y^{b}}_{0} + {}^{vv}G_{a\beta}{}^{cc}X^{a}\underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{vv}G_{a\overline{\beta}}}_{0}{}^{cc}X^{a}{}^{vv}Y^{\overline{\beta}} \\
&\quad + {}^{vv}G_{\alpha b}{}^{cc}X^{\alpha}\underbrace{{}^{vv}Y^{b}}_{0} + {}^{vv}G_{\alpha\beta}{}^{cc}X^{\alpha}\underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{vv}G_{\alpha\overline{\beta}}}_{0}{}^{cc}X^{\alpha}{}^{vv}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{vv}G_{\overline{\alpha}b}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{vv}Y^{b} + \underbrace{{}^{vv}G_{\overline{\alpha}\beta}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{vv}Y^{\beta} + \underbrace{{}^{vv}G_{\overline{\alpha}\overline{\beta}}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{vv}Y^{\overline{\beta}} \\
&= 0.
\end{aligned}$$

(iv) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $G \in \Im_2^0(B_m)$, from (6), (8), (9) and (10), then we have

$$\begin{aligned}
{}^{vv}G({}^{cc}X, {}^{cc}Y) &= {}^{vv}G_{IJ}{}^{cc}X^{I}{}^{cc}Y^{J} \\
&= \underbrace{{}^{vv}G_{ab}}_{0}{}^{cc}X^{a}{}^{cc}Y^{b} + \underbrace{{}^{vv}G_{a\beta}}_{0}{}^{cc}X^{a}{}^{cc}Y^{\beta} + \underbrace{{}^{vv}G_{a\overline{\beta}}}_{0}{}^{cc}X^{a}{}^{cc}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{vv}G_{\alpha b}}_{0}{}^{cc}X^{\alpha}{}^{cc}Y^{b} + \underbrace{{}^{vv}G_{\alpha\beta}}_{G_{\alpha\beta}}\underbrace{{}^{cc}X^{\alpha}}_{X^{\alpha}}\underbrace{{}^{cc}Y^{\beta}}_{Y^{\beta}} + \underbrace{{}^{vv}G_{\alpha\overline{\beta}}}_{0}{}^{cc}X^{\alpha}{}^{cc}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{vv}G_{\overline{\alpha}b}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{cc}Y^{b} + \underbrace{{}^{vv}G_{\overline{\alpha}\beta}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{cc}Y^{\beta} + \underbrace{{}^{vv}G_{\overline{\alpha}\overline{\beta}}}_{0}{}^{cc}X^{\overline{\alpha}}{}^{cc}Y^{\overline{\beta}} \\
&= G_{\alpha\beta}X^{\alpha}Y^{\beta} \\
&= {}^{vv}\left(G(X,Y)\right).
\end{aligned}$$

□

3. Complete Lifts of tensor field of type (0,2)

Let $\widetilde{G} \in \Im_2^0(M_n)$ be a projectable tensor field of type (0,2) [10] with projection $G = G_{\alpha\beta}(x^{\alpha})\, dx^{\alpha} \otimes dx^{\beta}$, i.e. $\widetilde{G}$ has the componets

$$\widetilde{G} = \left(\widetilde{G}_{ij}\right) = \begin{pmatrix} 0 & 0 \\ 0 & G_{\alpha\beta}(x^{\alpha}) \end{pmatrix}$$

with respect to the coordinates (x^a, x^α) [11]. On putting

$$^{cc}\widetilde{G} = \left(^{cc}\widetilde{G}_{IJ}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \partial_\varepsilon G_{\alpha\beta} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix}, \tag{11}$$

we easily see that $^{cc}G_{I'J'} = A^I_{I'} A^J_{J'} (^{cc}G_{IJ})$. We call $^{cc}\widetilde{G}$ the complete lift of the tensor field $\widetilde{G}$ of type (0,2) to $t(B_m)$ [11].

Since $Det(^{cc}G) = 0$, we have:

Theorem 3.1. *The semi-tangent bundle $t(B_m)$ has a degenerate metric ^{cc}G [11].*

Theorem 3.2. *If G is projectable tensor field of type (0,2) on M_n, and $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$, then*

(i) $^{cc}\widetilde{G}(^{vv}X, {}^{vv}Y) = 0,$
(ii) $^{cc}\widetilde{G}(^{vv}X, {}^{cc}\widetilde{Y}) = {}^{vv}(G(X,Y)),$
(iii) $^{cc}\widetilde{G}(^{cc}\widetilde{X}, {}^{vv}Y) = {}^{vv}(G(X,Y)),$
(iv) $^{cc}\widetilde{G}(^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}) = {}^{cc}(G(X,Y)).$

Proof. (i) If $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$ and $\widetilde{G} \in \Im^0_2(M_n)$, from (8) and (11), then we have

$$\begin{aligned} ^{cc}\widetilde{G}(^{vv}X, {}^{vv}Y) &= {}^{cc}\widetilde{G}_{IJ}{}^{vv}X^I{}^{vv}Y^J \\ &= {}^{cc}\widetilde{G}_{ab}\underbrace{^{vv}X^a}_{0}{}^{vv}Y^b + {}^{cc}\widetilde{G}_{a\beta}\underbrace{^{vv}X^a}_{0}{}^{vv}Y^\beta + {}^{cc}\widetilde{G}_{a\overline{\beta}}\underbrace{^{vv}X^a}_{0}{}^{vv}Y^{\overline{\beta}} \\ &\quad + {}^{cc}\widetilde{G}_{\alpha b}\underbrace{^{vv}X^\alpha}_{0}{}^{vv}Y^b + {}^{cc}\widetilde{G}_{\alpha\beta}\underbrace{^{vv}X^\alpha}_{0}{}^{vv}Y^\beta + {}^{cc}\widetilde{G}_{\alpha\overline{\beta}}\underbrace{^{vv}X^\alpha}_{0}{}^{vv}Y^{\overline{\beta}} \\ &\quad + {}^{cc}\widetilde{G}_{\overline{\alpha}b}{}^{vv}X^{\overline{\alpha}}\underbrace{^{vv}Y^b}_{0} + {}^{cc}\widetilde{G}_{\overline{\alpha}\beta}{}^{vv}X^{\overline{\alpha}}\underbrace{^{vv}Y^\beta}_{0} + \underbrace{^{cc}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{vv}Y^{\overline{\beta}} \\ &= 0. \end{aligned}$$

(ii) If $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$ and $\widetilde{G} \in \Im^0_2(M_n)$, from (6), (8), (9) and (11), then we have

$$\begin{aligned} ^{cc}\widetilde{G}(^{vv}X, {}^{cc}\widetilde{Y}) &= {}^{cc}\widetilde{G}_{IJ}{}^{vv}X^I{}^{cc}\widetilde{Y}^J \\ &= {}^{cc}\widetilde{G}_{ab}\underbrace{^{vv}X^a}_{0}{}^{cc}\widetilde{Y}^b + {}^{cc}\widetilde{G}_{a\beta}\underbrace{^{vv}X^a}_{0}{}^{cc}\widetilde{Y}^\beta + {}^{cc}\widetilde{G}_{a\overline{\beta}}\underbrace{^{vv}X^a}_{0}{}^{cc}\widetilde{Y}^{\overline{\beta}} \\ &\quad + {}^{cc}\widetilde{G}_{\alpha b}\underbrace{^{vv}X^\alpha}_{0}{}^{cc}\widetilde{Y}^b + {}^{cc}\widetilde{G}_{\alpha\beta}\underbrace{^{vv}X^\alpha}_{0}{}^{cc}\widetilde{Y}^\beta + {}^{cc}\widetilde{G}_{\alpha\overline{\beta}}\underbrace{^{vv}X^\alpha}_{0}{}^{cc}\widetilde{Y}^{\overline{\beta}} \\ &\quad + \underbrace{^{cc}\widetilde{G}_{\overline{\alpha}b}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{cc}\widetilde{Y}^b + \underbrace{^{cc}\widetilde{G}_{\overline{\alpha}\beta}}_{G_{\alpha\beta}}\underbrace{^{vv}X^{\overline{\alpha}}}_{X^\alpha}\underbrace{^{cc}\widetilde{Y}^\beta}_{Y^\beta} + \underbrace{^{cc}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}}{}^{cc}\widetilde{Y}^{\overline{\beta}} \\ &= G_{\alpha\beta}X^\alpha Y^\beta \\ &= {}^{vv}(G(X,Y)). \end{aligned}$$

(iii) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (6), (8), (9) and (11), then we have

$$\begin{aligned}
{}^{cc}\widetilde{G}({}^{cc}\widetilde{X}, {}^{vv}Y) &= {}^{cc}\widetilde{G}_{IJ}\, {}^{cc}\widetilde{X}^{I}\, {}^{vv}Y^{J} \\
&= {}^{cc}\widetilde{G}_{ab}\, {}^{cc}\widetilde{X}^{a} \underbrace{{}^{vv}Y^{b}}_{0} + {}^{cc}\widetilde{G}_{a\beta}\, {}^{cc}\widetilde{X}^{a} \underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{cc}\widetilde{G}_{a\overline{\beta}}}_{0}\, {}^{cc}\widetilde{X}^{a}\, {}^{vv}Y^{\overline{\beta}} \\
&\quad + {}^{cc}\widetilde{G}_{\alpha b}\, {}^{cc}\widetilde{X}^{\alpha} \underbrace{{}^{vv}Y^{b}}_{0} + {}^{cc}\widetilde{G}_{\alpha\beta}\, {}^{cc}\widetilde{X}^{\alpha} \underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{cc}\widetilde{G}_{\alpha\overline{\beta}}}_{G_{\alpha\beta}} \underbrace{{}^{cc}\widetilde{X}^{\alpha}}_{X^{\alpha}} \underbrace{{}^{vv}Y^{\overline{\beta}}}_{Y^{\beta}} \\
&\quad + \underbrace{{}^{cc}\widetilde{G}_{\overline{\alpha}b}}_{0}\, {}^{cc}\widetilde{X}^{\overline{\alpha}}\, {}^{vv}Y^{b} + {}^{cc}\widetilde{G}_{\overline{\alpha}\beta}\, {}^{cc}\widetilde{X}^{\overline{\alpha}} \underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{cc}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}\, {}^{cc}\widetilde{X}^{\overline{\alpha}}\, {}^{vv}Y^{\overline{\beta}} \\
&= G_{\alpha\beta} X^{\alpha} Y^{\beta} \\
&= {}^{vv}\left(G(X,Y)\right).
\end{aligned}$$

(iv) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (7), (8), (9) and (11), then we have

$$\begin{aligned}
{}^{cc}\widetilde{G}({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}) &= {}^{cc}\widetilde{G}_{IJ}\, {}^{cc}\widetilde{X}^{I}\, {}^{cc}\widetilde{Y}^{J} \\
&= \underbrace{{}^{cc}\widetilde{G}_{ab}}_{0}\, {}^{cc}\widetilde{X}^{a}\, {}^{cc}\widetilde{Y}^{b} + \underbrace{{}^{cc}\widetilde{G}_{a\beta}}_{0}\, {}^{cc}\widetilde{X}^{a}\, {}^{cc}\widetilde{Y}^{\beta} + \underbrace{{}^{cc}\widetilde{G}_{a\overline{\beta}}}_{0}\, {}^{cc}\widetilde{X}^{a}\, {}^{cc}\widetilde{Y}^{\overline{\beta}} \\
&\quad + \underbrace{{}^{cc}\widetilde{G}_{\alpha b}}_{0}\, {}^{cc}\widetilde{X}^{\alpha}\, {}^{cc}\widetilde{Y}^{b} + \underbrace{{}^{cc}\widetilde{G}_{\alpha\beta}}_{y^{\varepsilon}\partial_{\varepsilon}G_{\alpha\beta}} \underbrace{{}^{cc}\widetilde{X}^{\alpha}}_{X^{\alpha}} \underbrace{{}^{cc}\widetilde{Y}^{\beta}}_{Y^{\beta}} + \underbrace{{}^{cc}\widetilde{G}_{\alpha\overline{\beta}}}_{G_{\alpha\beta}} \underbrace{{}^{cc}\widetilde{X}^{\alpha}}_{X^{\alpha}} \underbrace{{}^{cc}\widetilde{Y}^{\overline{\beta}}}_{y^{\varepsilon}\partial_{\varepsilon}Y^{\beta}} \\
&\quad + \underbrace{{}^{cc}\widetilde{G}_{\overline{\alpha}b}}_{0}\, {}^{cc}\widetilde{X}^{\overline{\alpha}}\, {}^{cc}\widetilde{Y}^{b} + \underbrace{{}^{cc}\widetilde{G}_{\overline{\alpha}\beta}}_{G_{\alpha\beta}} \underbrace{{}^{cc}\widetilde{X}^{\overline{\alpha}}}_{y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}} \underbrace{{}^{cc}\widetilde{Y}^{\beta}}_{Y^{\beta}} + \underbrace{{}^{cc}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}\, {}^{cc}\widetilde{X}^{\overline{\alpha}}\, {}^{cc}\widetilde{Y}^{\overline{\beta}} \\
&= y^{\varepsilon}\left(\partial_{\varepsilon}G_{\alpha\beta}\right) X^{\alpha} Y^{\beta} + G_{\alpha\beta} X^{\alpha} y^{\varepsilon}\left(\partial_{\varepsilon}Y^{\beta}\right) + G_{\alpha\beta} y^{\varepsilon}\left(\partial_{\varepsilon}X^{\alpha}\right) Y^{\beta} \\
&= y^{\varepsilon}\partial_{\varepsilon}\left(G_{\alpha\beta} X^{\alpha} Y^{\beta}\right) \\
&= {}^{cc}\left(G(X,Y)\right).
\end{aligned}$$

□

In addition, according to (10) and (11), we define new projectable tensor field of type (0,2) i.e. ${}^{cc}\widetilde{G}^{*}$ by

$${}^{cc}\widetilde{G}^{*} = {}^{cc}\widetilde{G} + {}^{vv}G$$

with respect to the coordinates $(x^{a}, x^{\alpha}, x^{\overline{\alpha}})$ in $t(B_m)$, where

$$\begin{aligned}
{}^{cc}\widetilde{G}^{*} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^{\varepsilon}\partial_{\varepsilon}G_{\alpha\beta} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & G_{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^{\varepsilon}\partial_{\varepsilon}G_{\alpha\beta} + G_{\alpha\beta} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix}.
\end{aligned}$$

We call ${}^{cc}\widetilde{G}^{*}$ the deformed complete lift of the tensor field $\widetilde{G}$ of type (0,2) to $t(B_m)$. Taking account of (5), we easily see that ${}^{cc}\widetilde{G}^{*}_{I'J'} = A^{I}_{I'} A^{J}_{J'} ({}^{cc}\widetilde{G}^{*}_{IJ})$.

Proof. For simplicity we take only ${}^{cc}\widetilde{G}^{*}_{\alpha'\beta'}$. In fact, from (5)

$$\begin{aligned}
{}^{cc}\widetilde{G}^{*}_{\alpha'\beta'} &= A^{a}_{\alpha'}A^{b}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{ab}}_{0}+A^{a}_{\alpha'}A^{\beta}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{a\beta}}_{0}+A^{a}_{\alpha'}A^{\overline{\beta}}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{a\overline{\beta}}}_{0} \\
&+A^{\alpha}_{\alpha'}A^{b}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{\alpha b}}_{0}+\underbrace{A^{\alpha}_{\alpha'}}_{A^{\alpha}_{\alpha'}}\underbrace{A^{\beta}_{\beta'}}_{A^{\beta}_{\beta'}}\underbrace{{}^{cc}\widetilde{G}^{*}_{\alpha\beta}}_{y^{\varepsilon'}\partial_{\varepsilon'}G_{\alpha\beta}+G_{\alpha\beta}} \\
&+\underbrace{A^{\alpha}_{\alpha'}}_{A^{\alpha}_{\alpha'}}\underbrace{A^{\overline{\beta}}_{\beta'}}_{A^{\beta}_{\beta'\varepsilon'}y^{\varepsilon'}}\underbrace{{}^{cc}\widetilde{G}^{*}_{\alpha\overline{\beta}}}_{G_{\alpha\beta}} \\
&+A^{\overline{\alpha}}_{\alpha'}A^{b}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{\overline{\alpha}b}}_{0}+\underbrace{A^{\overline{\alpha}}_{\alpha'}}_{A^{\alpha}_{\alpha'\varepsilon'}y^{\varepsilon'}}\underbrace{A^{\beta}_{\beta'}}_{A^{\alpha}_{\alpha'}}\underbrace{{}^{cc}\widetilde{G}^{*}_{\overline{\alpha}\beta}}_{G_{\alpha\beta}}+A^{\overline{\alpha}}_{\alpha'}A^{\overline{\beta}}_{\beta'}\underbrace{{}^{cc}\widetilde{G}^{*}_{\overline{\alpha}\overline{\beta}}}_{0} \\
&= A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}\left(y^{\varepsilon'}\partial_{\varepsilon'}G_{\alpha\beta}+G_{\alpha\beta}\right)+y^{\varepsilon'}(\partial_{\varepsilon'}A^{\beta}_{\beta'})A^{\alpha}_{\alpha'}G_{\alpha\beta}+y^{\varepsilon'}(\partial_{\varepsilon'}A^{\alpha}_{\alpha'})A^{\beta}_{\beta'}G_{\alpha\beta} \\
&= A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}\left(y^{\varepsilon'}\partial_{\varepsilon'}G_{\alpha\beta}\right)+A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}G_{\alpha\beta} \\
&+y^{\varepsilon'}(\partial_{\varepsilon'}A^{\beta}_{\beta'})A^{\alpha}_{\alpha'}G_{\alpha\beta}+y^{\varepsilon'}(\partial_{\varepsilon'}A^{\alpha}_{\alpha'})A^{\beta}_{\beta'}G_{\alpha\beta} \\
&= y^{\varepsilon'}\partial_{\varepsilon'}(A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}G_{\alpha\beta})+G_{\alpha'\beta'} \\
&= y^{\varepsilon'}\partial_{\varepsilon'}G_{\alpha'\beta'}+G_{\alpha'\beta'}.
\end{aligned}$$

Thus, we have ${}^{cc}\widetilde{G}^{*}_{I'J'} = A^{I}_{I'}A^{J}_{J'}({}^{cc}\widetilde{G}^{*}_{IJ})$. We can easily obtain other components of ${}^{cc}\widetilde{G}^{*}_{I'J'}$ by using this way. □

Since $Det({}^{cc}\widetilde{G}^{*}) = 0$, we have:

Theorem 3.3. *The semi-tangent bundle $t(B_m)$ has a degenerate deformed metric ${}^{cc}\widetilde{G}^{*}$.*

4. Horizontal Lifts of tensor field of type (0,2)

Firstly, we will give some preliminary definitions. For any $F \in \Im^1_1(B_m)$, if we take account of (5), we can prove that $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a vector field defined by

$$\gamma F = (\gamma F^I) = \begin{pmatrix} 0 \\ 0 \\ y^{\varepsilon}F^{\alpha}_{\varepsilon} \end{pmatrix} \tag{12}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

For any $S \in \Im^0_3(B_m)$, if we take account of (5), we can prove that $(\gamma S)' = A^I_{I'}A^J_{J'}(\gamma S)$, where γS is a tensor field of type (0,2) defined by

$$\gamma S = (\gamma S_{IJ}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^{\varepsilon}S_{\varepsilon\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ and $(x^b, x^\beta, x^{\overline{\beta}})$.

Let now $\widetilde{X} \in \Im^1_0(M_n)$ be a projectable vector field on M_n with projection $X \in \Im^1_0(B_m)$ [10]. Then we define the horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X}$ by

$${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X})$$

on $t(M_n)$. Where ∇ is a symmetric affine connection in a differentiable manifold B_m. Then, remembering that ${}^{cc}\widetilde{X}$ and $\gamma(\nabla\widetilde{X})$ have, respectively, local componenets

$$ {}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \gamma(\nabla\widetilde{X}) = \left(\gamma(\nabla\widetilde{X})^I\right) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$ (\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma^\varepsilon_{\beta\alpha}. $$

We find that the horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X}$ has the components

$$ {}^{HH}X = \left({}^{HH}X^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma^\alpha_\beta X^\beta \end{pmatrix} \tag{14} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. Where

$$ \Gamma^\alpha_\beta = y^\varepsilon \Gamma^\alpha_{\varepsilon\,\beta}. \tag{15} $$

Suppose now that $\widetilde{G} \in \Im^0_2(M_n)$ and G has local components $G_{\alpha\beta}$ in a neighborhood U of B_m, $G = G_{\alpha\beta}\,(x^\alpha)\, dx^\alpha \otimes dx^\beta$. Then we define the horizontal lift ${}^{HH}\widetilde{G}$ of $\widetilde{G}$ by

$$ {}^{HH}\widetilde{G} = {}^{cc}\widetilde{G} - \nabla_\gamma \widetilde{G} = {}^{cc}\widetilde{G} - \gamma[\nabla\widetilde{G}] \tag{16} $$

on $t(B_m)$. Where $\gamma[\nabla\widetilde{G}]$ is a tensor field of type (0,2) defined by

$$ \gamma[\nabla\widetilde{G}] = y^\varepsilon \nabla_\varepsilon G_{\alpha\beta} dx^\alpha \otimes dx^\beta. \tag{17} $$

From (11), (13), (16) and (17), we see that the horizontal lift ${}^{HH}\widetilde{G}$ has the components of the form

$$ {}^{HH}\widetilde{G} = ({}^{HH}\widetilde{G}_{IJ}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \Gamma^\sigma_{\varepsilon\,\alpha} G_{\sigma\beta} + y^\varepsilon \Gamma^\sigma_{\varepsilon\,\beta} G_{\alpha\sigma} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix} \tag{18} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$, where $G_{\alpha\beta}$ are the local components of G, $\Gamma^\sigma_{\varepsilon\,\alpha}$ componenets of ∇ on $t(B_m)$ and Γ^α_β are defined by (15).

Proof. From (11), (13), (16) and (17), we have

$$ \begin{aligned} {}^{HH}\widetilde{G} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \Gamma^\sigma_{\varepsilon\,\alpha} G_{\sigma\beta} + y^\varepsilon \Gamma^\sigma_{\varepsilon\,\beta} G_{\alpha\sigma} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \partial_\varepsilon G_{\alpha\beta} - y^\varepsilon(\partial_\varepsilon G_{\alpha\beta} - \Gamma^\sigma_{\varepsilon\,\alpha} G_{\sigma\beta} - \Gamma^\sigma_{\varepsilon\,\beta} G_{\alpha\sigma}) & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \partial_\varepsilon G_{\alpha\beta} & G_{\alpha\beta} \\ 0 & G_{\alpha\beta} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^\varepsilon \nabla_\varepsilon G_{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= {}^{cc}\widetilde{G} - \gamma[\nabla\widetilde{G}]. \end{aligned} $$

Thus we have (18). □

Theorem 4.1. *If G is projectable tensor field of type (0,2) on M_n, and $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$, then*

(i) ${}^{HH}\widetilde{G}({}^{vv}X, {}^{vv}Y) = 0,$
(ii) ${}^{HH}\widetilde{G}({}^{HH}\widetilde{X}, {}^{HH}\widetilde{Y}) = {}^{HH}\left(G(X,Y)\right),$
(iii) ${}^{HH}\widetilde{G}({}^{vv}X, {}^{HH}\widetilde{Y}) = {}^{vv}\left(G(X,Y)\right),$
(iv) ${}^{HH}\widetilde{G}({}^{HH}\widetilde{X}, {}^{vv}Y) = {}^{vv}\left(G(X,Y)\right).$

Proof. (i) If $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (8) and (18), then we have

$$\begin{aligned}
{}^{HH}\widetilde{G}({}^{vv}X, {}^{vv}Y) &= {}^{cc}\widetilde{G}_{IJ}{}^{vv}X^{I\,vv}Y^{J} \\
&= {}^{HH}\widetilde{G}_{ab}\underbrace{{}^{vv}X^{a}}_{0}{}^{vv}Y^{b} + {}^{HH}\widetilde{G}_{a\beta}\underbrace{{}^{vv}X^{a}}_{0}{}^{vv}Y^{\beta} + {}^{HH}\widetilde{G}_{a\overline{\beta}}\underbrace{{}^{vv}X^{a}}_{0}{}^{vv}Y^{\overline{\beta}} \\
&\quad + {}^{HH}\widetilde{G}_{\alpha b}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{vv}Y^{b} + {}^{HH}\widetilde{G}_{\alpha\beta}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{vv}Y^{\beta} + {}^{HH}\widetilde{G}_{\alpha\overline{\beta}}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{vv}Y^{\overline{\beta}} \\
&\quad + {}^{HH}\widetilde{G}_{\overline{\alpha}b}{}^{vv}X^{\overline{\alpha}}\underbrace{{}^{vv}Y^{b}}_{0} + {}^{HH}\widetilde{G}_{\overline{\alpha}\beta}{}^{vv}X^{\overline{\alpha}}\underbrace{{}^{vv}Y^{\beta}}_{0} + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}\,vv}Y^{\overline{\beta}} \\
&= 0.
\end{aligned}$$

(ii) If $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (14) and (18), then we have

$$\begin{aligned}
{}^{HH}\widetilde{G}({}^{HH}\widetilde{X}, {}^{HH}\widetilde{Y}) &= {}^{HH}\widetilde{G}_{IJ}{}^{HH}\widetilde{X}^{I\,HH}\widetilde{Y}^{J} \\
&= \underbrace{{}^{HH}\widetilde{G}_{ab}}_{0}{}^{HH}\widetilde{X}^{a\,HH}\widetilde{Y}^{b} + \underbrace{{}^{HH}\widetilde{G}_{a\beta}}_{0}{}^{HH}\widetilde{X}^{a\,HH}\widetilde{Y}^{\beta} + \underbrace{{}^{HH}\widetilde{G}_{a\overline{\beta}}}_{0}{}^{HH}\widetilde{X}^{a\,HH}\widetilde{Y}^{\overline{\beta}} \\
&\quad + \underbrace{{}^{HH}\widetilde{G}_{\alpha b}}_{0}{}^{HH}\widetilde{X}^{\alpha\,HH}\widetilde{Y}^{b} + {}^{HH}\widetilde{G}_{\alpha\beta}{}^{HH}\widetilde{X}^{\alpha\,HH}\widetilde{Y}^{\beta} + {}^{HH}\widetilde{G}_{\alpha\overline{\beta}}{}^{HH}\widetilde{X}^{\alpha\,HH}\widetilde{Y}^{\overline{\beta}} \\
&\quad + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}b}}_{0}{}^{HH}\widetilde{X}^{\overline{\alpha}\,HH}\widetilde{Y}^{b} + {}^{HH}\widetilde{G}_{\overline{\alpha}\beta}{}^{HH}\widetilde{X}^{\overline{\alpha}\,HH}\widetilde{Y}^{\beta} + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}{}^{HH}\widetilde{X}^{\overline{\alpha}\,HH}\widetilde{Y}^{\overline{\beta}} \\
&= \left(y^{\varepsilon}\Gamma_{\varepsilon\,\alpha}^{\sigma}G_{\sigma\beta} + y^{\varepsilon}\Gamma_{\varepsilon\,\beta}^{\sigma}G_{\alpha\sigma}\right)X^{\alpha}Y^{\beta} + G_{\alpha\beta}X^{\alpha}\left(-y^{\varepsilon}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}\right) + G_{\alpha\beta}Y^{\beta}\left(-y^{\varepsilon}\Gamma_{\varepsilon\,\sigma}^{\beta}X^{\sigma}\right) \\
&= {}^{cc}\left(G(X,Y)\right) - \gamma[\nabla\left(G(X,Y)\right)] \\
&= {}^{HH}\left(G(X,Y)\right).
\end{aligned}$$

(iii) If $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (6), (8), (14) and (18), then we have

$$\begin{aligned}
{}^{HH}\widetilde{G}({}^{vv}X, {}^{HH}\widetilde{Y}) &= {}^{HH}\widetilde{G}_{IJ}{}^{vv}X^{I\,HH}\widetilde{Y}^{J} \\
&= {}^{HH}\widetilde{G}_{ab}\underbrace{{}^{vv}X^{a}}_{0}{}^{HH}\widetilde{Y}^{b} + {}^{HH}\widetilde{G}_{a\beta}\underbrace{{}^{vv}X^{a}}_{0}{}^{HH}\widetilde{Y}^{\beta} + {}^{HH}\widetilde{G}_{a\overline{\beta}}\underbrace{{}^{vv}X^{a}}_{0}{}^{HH}\widetilde{Y}^{\overline{\beta}} \\
&\quad + {}^{HH}\widetilde{G}_{\alpha b}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{HH}\widetilde{Y}^{b} + {}^{HH}\widetilde{G}_{\alpha\beta}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{HH}\widetilde{Y}^{\beta} + {}^{HH}\widetilde{G}_{\alpha\overline{\beta}}\underbrace{{}^{vv}X^{\alpha}}_{0}{}^{HH}\widetilde{Y}^{\overline{\beta}} \\
&\quad + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}b}}_{0}{}^{vv}X^{\overline{\alpha}\,HH}\widetilde{Y}^{b} + {}^{HH}\widetilde{G}_{\overline{\alpha}\beta}{}^{vv}X^{\overline{\alpha}\,HH}\widetilde{Y}^{\beta} + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}{}^{vv}X^{\overline{\alpha}\,HH}\widetilde{Y}^{\overline{\beta}} \\
&= G_{\alpha\beta}X^{\alpha}Y^{\beta} \\
&= {}^{vv}\left(G(X,Y)\right).
\end{aligned}$$

(iv) If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $\widetilde{G} \in \Im_2^0(M_n)$, from (6), (8), (14) and (18), then we have

$$\begin{aligned}
{}^{HH}\widetilde{G}({}^{HH}\widetilde{X}, {}^{vv}Y) &= {}^{HH}\widetilde{G}_{IJ}\,{}^{HH}\widetilde{X}^{I\,vv}Y^J \\
&= \underbrace{{}^{HH}\widetilde{G}_{ab}}_{0}\,{}^{HH}\widetilde{X}^{a\,vv}Y^b + \underbrace{{}^{HH}\widetilde{G}_{a\beta}}_{0}\,{}^{HH}\widetilde{X}^{a\,vv}Y^\beta + \underbrace{{}^{HH}\widetilde{G}_{a\overline{\beta}}}_{0}\,{}^{HH}\widetilde{X}^{a\,vv}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{HH}\widetilde{G}_{\alpha b}}_{0}\,{}^{HH}\widetilde{X}^{\alpha\,vv}Y^b + {}^{HH}\widetilde{G}_{\alpha\beta}\,{}^{HH}\widetilde{X}^{\alpha}\,\underbrace{{}^{vv}Y^\beta}_{0} + {}^{HH}\widetilde{G}_{\alpha\overline{\beta}}\,{}^{HH}\widetilde{X}^{\alpha\,vv}Y^{\overline{\beta}} \\
&\quad + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}b}}_{0}\,{}^{HH}\widetilde{X}^{\overline{\alpha}\,vv}Y^b + {}^{HH}\widetilde{G}_{\overline{\alpha}\beta}\,{}^{HH}\widetilde{X}^{\overline{\alpha}}\,\underbrace{{}^{vv}Y^\beta}_{0} + \underbrace{{}^{HH}\widetilde{G}_{\overline{\alpha}\overline{\beta}}}_{0}\,{}^{HH}\widetilde{X}^{\overline{\alpha}\,vv}Y^{\overline{\beta}} \\
&= G_{\alpha\beta}X^\alpha Y^\beta \\
&= {}^{vv}\left(G(X,Y)\right).
\end{aligned}$$

□

5. Conclusions

Using the fiber bundle M over a manifold B, we define a semi-tangent (pull-back) bundle tB. We consider vertical, complete and horizontal lifting problem of tensor fields of type (0, 2) on M to the semi-tangent bundle. Relations between lifted objects are also presented.

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References

[1] Husemoller, D., (1994), Fibre Bundles. Springer, New York.
[2] Lawson, H.B. and Michelsohn, M.L., (1989), Spin Geometry. Princeton University Press., Princeton.
[3] Pontryagin, L.S., (1950), Characteristic cycles on differentiable manifolds, Amer. Math. Soc. Translation., no. 32, pp. 72.
[4] Poor, W.A., (1981), Differential Geometric Structures, New York, McGraw-Hill .
[5] Ostianu, N.M., (1974), Step-fibred spaces, Tr. Geom. Sem. 5, Moscow. (VINITI), pp. 259-309.
[6] Salimov, A.A. and Kadıoğlu, E., (2000), Lifts of Derivations to the Semitangent Bundle, Turk J. Math., pp. 259-266.
[7] Serhan, E., (2019), Seiberg-Witten-like equations on the strictly pseudoconvex CR-3 manifolds, Bull. Korean Math. Soc., 56(6), pp. 1551-1567.
[8] Serhan, E., (2019), Seiberg-Witten-like equations on the strictly-pseudoconvex CR 7-manifolds, Miskolc Mathematical Notes, 20(1) , pp. 233-243.
[9] Steenrod, N., (1951), The Topology of Fibre Bundles, Princeton University Press., Princeton.
[10] Vishnevskii, V.V., (2002), Integrable affinor structures and their plural interpretationsi Geometry, 7.J. Math. Sci., (New York) 108(2), pp. 151-187.
[11] Vishnevskii, V.V., Shirokov, A.P. and Shurygin, V.V., (1985), Spaces over Algebras., Kazan. Kazan Gos. Univ. (in Russian).
[12] Yano, K. and Ishihara, S., (1973), Tangent and Cotangent Bundles. Marcel Dekker, Inc., New York.

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