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SOME NEW MULTI-STEP DERIVATIVE-FREE ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, we use the system of coupled equation involving auxiliary function with decomposition technique. We also use finite difference technique to suggest and analyze some new derivative-free iterative methods for solving nonlinear equations. Several examples are given to check the performance of developed methods numerically as well as graphically. This technique can be implemented to suggest a wide class of new derivative-free iterative methods for solving nonlinear equations.

Keywords: Iterative methods, Nonlinear equations, Convergence, Steffensen's method.

AMS Subject Classification: 65Axx, 65Bxx, 65Hxx

1. Introduction

Solution of nonlinear equations is an important area of research in numerical analysis and optimization. Nonlinear problems arise in various fields of sciences and have applications in several branches of pure and applied sciences. Such type of nonlinear problems can be studied in the general framework of nonlinear equation $f(x) = 0$ (see [1-15]). Several methods exist in the literature for finding the approximate solution of nonlinear problems. Classical Newton method [1] which possesses quadratic order of convergence is well known method. Newton method and similar existing methods have drawbacks. For best implementation of these methods, there is a necessary condition that $f'(x_n)$ should not be zero or not be very small. One can easily observe this draw back in following problem. Let us consider

$$f(x) = x^3 - 0.003x^2 + 2.4 \times 10^{-6}. \tag{1}$$

The Newton method [1] reduces the following form for their problem as:

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4 \times 10^{-6}}{3x_i^2 - 0.06x_i} \tag{2}$$

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The drawback of the methods we can be observed when x approaches to 0 or 0.02 Newton method and variants of Newton method diverge. For these types of problems derivative-free methods are required.
To avoid such type of situation Steffensen [1, 12] introduced new method which do not employ derivatives of function. To obtain this method Steffensen modified Newton method with the help of finite difference scheme. Derivative free Steffensen's method is described as:

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f[x_n + f(x_n)] - f(x_n)}, n = 0, 1, 2, 3, ... \tag{3}$$

Steffensen's method is free from derivatives of the function, because sometimes the applications of the iterative methods which depend upon derivatives are restricted in engineering. Here we will find higher-order convergent derivative-free methods. In this work, we will implement the technique of Gijji [3] to decompose the nonlinear equation for obtaining the derivative free iterative methods along with finite difference scheme. In Section2, we construct family of iterative methods for obtaining the approximate solution of nonlinear equations. In Section 3, we analyze the order of convergence of these purposed methods. In Section 4, Numerical and graphical results are exhibited to show the performance of developed iterative methods.

2. Construction of iterative methods

In this section, we will develop multi-step iterative methods by using decomposition technique together with finite difference scheme.
We consider the nonlinear equation

$$f(x) = 0 \tag{4}$$

Using Taylor series we can obtain the following expression

$$f(\gamma)g(\gamma) + (x - \gamma)[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)] + h_1(x) = 0, \tag{5}$$

where $g(x)$ is a auxiliary arbitrary function and γ is the initial guess which is in the neighborhood of x.
We implement the following approximation

$$f'(\gamma) \approx \frac{f(\gamma + f(\gamma)) - f(\gamma)}{f(\gamma)}. \tag{6}$$

Combining (4) and (5), we obtain

$$f^2(\gamma)g(\gamma) + (x - \gamma)\left\{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f^2(\gamma)g'(\gamma)\right\} + h(x) = 0. \tag{7}$$

From (6), we can write the following form

$$x = \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} + \frac{h(x)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)}. \tag{8}$$

Where

$$h(x_0) = f(x_0)f(\gamma)g(\gamma) \tag{9}$$

We express (7), in the following form as

$$x = c + N(x), \tag{10}$$

where

$$x_0 = c = \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \tag{11}$$

and

$$N(x) = \frac{h(x)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)}. \tag{12}$$

Now we construct a sequence of higher-order iterative methods by using the following decomposition technique, which is mainly due to Gejji and Jafari [3]. This decomposition is quite different from that of Adomian decomposition. The main idea of this technique is to look for a solution having the series form

$$x = \sum_{i=0}^{\infty} x_i \tag{13}$$

The nonlinear operator N can be decomposed as:

$$N(x) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right) \right\} \tag{14}$$

Combining (11), (12) and (13), we have

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right) \right\} \tag{15}$$

Thus we have the following iterative scheme:

$$\begin{aligned} x_0 &= c \\ x_1 &= N(x_0) \\ x_2 &= N(x_1) - N(x_0) \\ &\vdots \\ x_{m+1} &= N\left(\sum_{j=0}^{m} x_j\right) - N\left(\sum_{j=0}^{m-1} x_j\right). \end{aligned} \tag{16}$$

then

$$x_0 + x_2 + x_3 \cdots x_{m+1} = N(x_0 + x_1 + x_2 + x_3 \cdots x_m). \tag{17}$$

and

$$x = c + \sum_{i=0}^{\infty} x_i \tag{18}$$

It can be shown that series $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution.

Using (7) and (17), we get the following

$$x_0 = c = \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \tag{19}$$

and

$$x_{1=}N(x) = \frac{h(x)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)}. \tag{20}$$

Note that, x is approximated by

$$X_m = x_0 + x_1 + x_2 + x_3 \cdots x_m$$

where,

$$lim_{m\to\infty} X_m = x \tag{21}$$

for $m = 0$.

$$x = X_0 = x_0 = c = \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \tag{22}$$

This formulation allows us to suggest the following one-step iterative method for solving nonlinear equations.

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f^2(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f(x_n)^2 g'(x_n)}, n = 1, 2, 3, \cdots .$$

For $m = 1$

$$\begin{aligned} x = X_1 = x_0 + x_1 = c + N(x) &= \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \\ &\quad \frac{h(x)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \\ &= \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \\ &\quad \frac{f(x_0)f(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \end{aligned} \tag{23}$$

The formulation allows us to suggest the following iterative method for solving nonlinear equations.

Algorithm 2.2. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f^2(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f(x_n)^2 g'(x_n)} \\ x_{n+1} &= x_n - \frac{f(y_n)f(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f[(x_n)]^2 g'(x_n)} \end{aligned}$$

For $m = 2$

$x = X_2 = x_0 + x_1 + x_2 = c + N(x_0 + x_1)$

$$\begin{aligned} &= \gamma - \frac{f^2(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \\ &\quad + \frac{f(x_0 + x_1)f(\gamma)g(\gamma)}{[f(\gamma + f(\gamma)) - f(\gamma)]g(\gamma) + f(\gamma)^2 g'(\gamma)} \end{aligned} \tag{24}$$

This formulation allows us to suggest the following iterative method for solving nonlinear equations.

Algorithm 2.3. For a given x_n, find the approximate solution x_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f^2(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f(x_n)^2 g'(x_n)} \\ z_n &= y_n - \frac{f(y_n)f(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f(x_n)^2 g'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)f(x_n)g(x_n)}{[f(x_n + f(x_n)) - f(x_n)]g(x_n) + f(x_n)^2 g'(x_n)} \end{aligned}$$

Algorithm 2.1, 2.2 and 2.3 are the main iterative schemes. We can generate various iterative methods by using auxiliary function. For example if we chose $g(x_n) = e^{-\alpha x_n}$ where $\alpha \in \Re$. Then we will obtain following methods for best implementation.

Algorithm 2.4. For a given x_n, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$

Algorithm 2.4 produces a class of second order convergent derivative-free iterative methods for different values of α.

Algorithm 2.5. For a given x_n, finding the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$
$$x_{n+1} = y_n - \frac{f(y_n)f(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$

Algorithm 2.6. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$
$$z_n = y_n - \frac{f(y_n)f(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$
$$x_{n+1} = z_n - \frac{f(z_n)f(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2}$$

Remark 2.1. *For the best implementation of the newly derived methods, we have to select a value of $\alpha \in \Re$, which makes denominator nonzero and largest.*

3. Convergence Analysis

In this section, we consider the convergence criteria of the iterative method described as Algorithm 2.6 developed in section 2. In similar way we can check the order of convergence of other purposed methods.

Theorem 3.1. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has simple root $p \in D$. Let $f(x)$ be smooth sufficiently in some neighborhood of the root and then the Algorithm 2.6 has four order convergence.*

Proof. let p be the simple root of $f(x)$. Since f is sufficiently differential, then expanding $f(x_n), f(x_n - f(x_n))$, and $f(x_n + f(x_n))$ in Taylor's series about p , we get

$$f(x_n) = [c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)] \tag{25}$$

and

$$f(x_n + f(x_n)) = c_1(1 - c_1)e_n + (3c_1^2c_2 + c_1c_2 + c_1^2c_2)e_n^2 + O(e_n^3) \tag{26}$$

Where $c_k = \frac{f^{(k)}(p)}{f(p)}$, $k = 2, 3, 4, \cdots$ and $c_1 = f(p)$

$$e_n = x_n - p$$

Now using (24) and (25), we obtain

$$f(x_n + f(x_n)) - f(x_n) = -p - c_1^2e_n + (3c_1^2c_2 - c_2c_1^3)e_n^2 + (2c_2^2c_1^2$$
$$+2c_2^2c_1^3 + 4c_2^2c_3 + 3c_3c_1^3 + c_3c_1^4)e_n^3 + O(e_n^4)$$

$$f(x_n + f(x_n)) - f(x_n) - \alpha f(x_n)^2 = -p - c_1^2 e_n + (3c_1^2 c_2 - c_2 c_1^3 - \alpha c_1^2) e_n^2 \\ +(-2\alpha c_2^2 c_1^2 + 2c_2^2 c_1^2 + 2c_2^2 c_1^3 + 4c_2^2 c_3 + 3c_3 c_1^3 + c_3 c_1^4) e_n^3 + O(e_n^4). \tag{27}$$

Using (24),(25) and (26) in Algorithm 2.6 for y_n, we have

$$y_n = x_n - \frac{f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2} = (c_2 + c_2 c_1 - \alpha) e_n^2 \\ - (-2c_3 + 2c_2^2 - 2\alpha c_3 + 2c_2^2 c_1 - 3c_1 c_3 - c_1 c_3 - c_3 c_1^2 + c_2^2 c_1^2 \\ - 2\alpha c_1 c_2 + \alpha^2 c_1^2) e_n^3 + O(e_n^4) \tag{28}$$

and

$$f(x_n) = -p + (c_2 c_1 + c_2 c_1^2 - \alpha c_1) e_n^2 \\ - c_1(-2c_3 + 2c_2^2 - 2\alpha c_3 + 2c_2^2 c_1 - 3c_1 c_3 - c_1 c_3 - c_3 c_1^2 \\ + c_2^2 c_1^2 - 2\alpha c_1 c_2 + \alpha^2 c_1^2) e_n^3 + O(e_n^4) \tag{29}$$

By using (27) and (28) in z_n of Algorithm 2.6 for y_n we have

$$z_n = y_n - \frac{f(y_n) f(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2} \\ = (2c_2^2 - 3\alpha c_2 + 3c_2^2 c_1 + c_2^2 c_1^2 - 2\alpha c_1 c_2 + \alpha^2) e_n^3 + O(e_n^4) \tag{30}$$

and

$$f(z_n) = -p + c_1(2c_2^2 - 3\alpha c_2 + 3c_2^2 c_1 + c_2^2 c_1^2 - 2\alpha c_1 c_2 + \alpha^2) e_n^3 + O(e_n^4) \tag{31}$$

By using (29) and (30), in Algorithm 2.6 y_n, we have

$$x_{n+1} = z_n - \frac{f(z_n) f(x_n)}{[f(x_n + f(x_n)) - f(x_n)] + \alpha f(x_n)^2} \\ = -p + (-8\alpha c_2^2 - 10c_1 c_2^2 + 5\alpha^2 c_2 - 3\alpha c_2^2 c_1^2 + 3\alpha^2 c_1 c_2 + 8c_2^3 c_1 \\ + c_2^3 c_1^3 - \alpha^3 + 4c_2^3) e_n^4 + O(e_n^5) \tag{32}$$

We obtain the final result from (31)

$$e_{n+1} = (-8\alpha c_2^2 - 10c_1 c_2^2 + 5\alpha^2 c_2 - 3\alpha c_2^2 c_1^2 + 3\alpha^2 c_1 c_2 + 8c_2^3 c_1 + c_2^3 c_1^3 - \alpha^3 + 4c_2^3) e_n^4 + O(e_n^5) \tag{33}$$

This error equation shows that Algorithm 2.6 is fourth order convergent iterative method. On the same pattern one can analyze the other methods. □

4. Numerical Comparison

In this section, we test some examples to illustrate the efficiency of all derived methods which are developed in section 2. We make computational comparison with the Stefensen's method as well as graphical comparison by solving some examples. We also verify the computational order of convergence to reconfirm the order of convergence and behaviour of the method for specific examples by using the following formulation

$$COC \approx \frac{\ln(|x_{n+1} - x_n| / |x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}| / |x_n - x_{n-2}|)},$$

For a different value of $\alpha \in \Re$ using in Algorithms 2.4, 2.5 and 2.6 we can obtain various classes of iterative methods. Now we use the following examples for them comparison of the methods for $\alpha = 0, 1/4$.

Example 4.1. *A trunnion has to be cooled before it is shrink fitted into a steel hub. The equation that gives the temperature x to which the trunnion has to be cooled to obtain the desired contraction is given by the following equation.*

FIGURE 1

Trunnion to be slid through the hub after contracting

$$f_1(x) = -0.50598 \times 10^{-10}x^3 + 0.38292 \times 10^{-7}x^2 + 0.74363 \times 10^{-4}x + 0.88318 \times 10^{-2} = 0$$

- To find the temperature x to which the trunnion has to be cooled.
- Find the absolute approximate error at the end of each iteration.
- To observe the graph we chose $x_0 = -100$.

The numerical comparison of these methods is shown in this table.

Table 1: Numerical results for example 4.1

Method	α	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
SM	0	4	-128.75486	$3.39768e^{-14}$	$0.00741e^{-12}$	1.99982
Alg 2.5	0	3	-128.75486	$1.02216e^{-12}$	$1.39912e^{-11}$	2.97039
Alg2.6	0	3	-128.75486	$4.21066e^{-20}$	$1.02741e^{-12}$	3.97002
Alg2.4	1/4	4	-128.75486	$1.05921e^{-10}$	$1.14258e^{-14}$	1.57171
Alg 2.5	1/4	4	-128.75486	$1.03966e^{-17}$	$3.80822e^{-15}$	3.02572
Alg2.6	1/4	3	-128.75486	$1.13554e^{-06}$	$3.35592e^{-08}$	3.99281

Now the graphical comparison can be observed for the derived methods for the computations of above example. We use $\log|x_n - x_{n-1}|$ for iterations obtained from all methods.

Example 4.2. *You are working for a company that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water. The equation that gives the depth x in meters to which the ball is submerged under water is given by*

$$f_2(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

We use the developed methods of finding roots of equations to find

- The depth x to which the ball is submerged under water.
- The absolute relative approximate error at the end of each iteration.

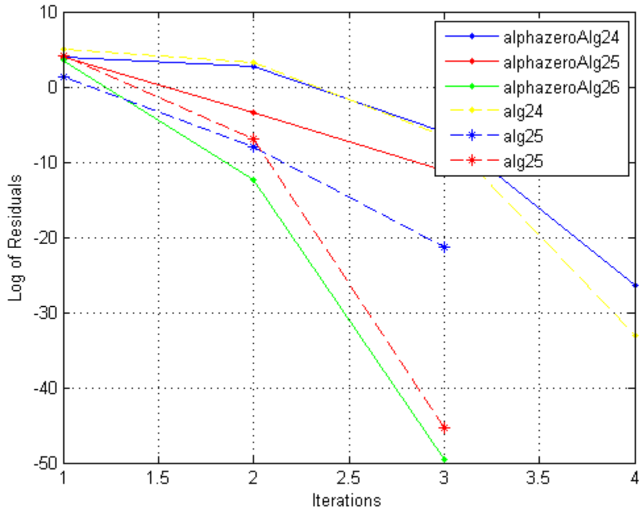

FIGURE 2
Log of residual for example 4.1

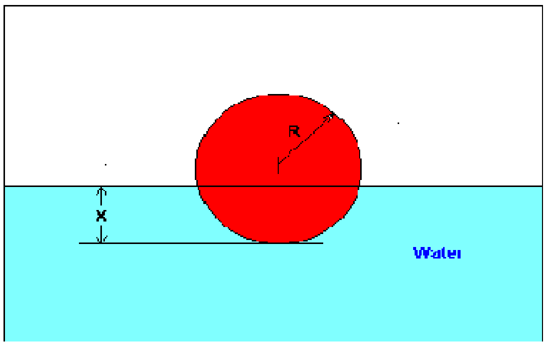

FIGURE 3
Floating ball

- Let us assume the initial guess of the root of is 0.05. This is a reasonable guess is good. As the extreme values of the depth x would be 0 and the diameter (0.11 m) of the ball.

The computational comparison of these methods is shown in this table.

Table 2 : Numerical results for example 4.2

Method	α	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
SM	0	4	0.06237	$4.83281e^{-17}$	$3.77312e^{-14}$	2.00041
Alg 2.5	0	3	0.06237	$9.15112e^{-19}$	$3.81011e^{-7}$	2.61175
Alg2.6	0	3	0.06237	$4.27274e^{-09}$	$1.44490e^{-7}$	3.38788
Alg2.4	1/4	4	0.06237	$4.83150e^{-17}$	$2.60871e^{-4}$	2.00075
Alg 2.5	1/4	4	0.06237	$2.21288e^{-18}$	$4.96601e^{-25}$	2.62298
Alg2.6	1/4	3	0.06237	$7.27688e^{-09}$	$6.48490e^{-7}$	3.66652

Now for the graphical comparison of these developed methods, we use $\log|x_n - x_{n-1}|$ for all methods to create the following graph.

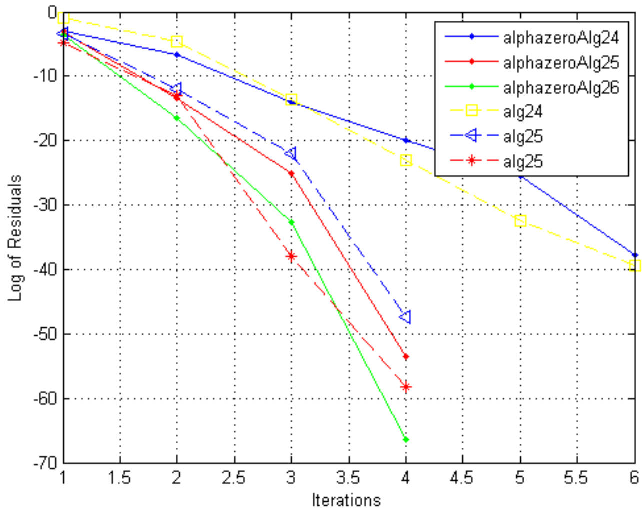

FIGURE 4
Log of residuals for example 4.2

Example 4.3. *We have function* $f_3(x) = e^{x^2} + \sin x - 1$*,* $x_0 = 0.25$ *and we find the root of this function by using the above techniques. We have following results shown in table.*

Table 3:Numerical results for example 4.3

Method	α	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
SM	0	6	0.0000	$4.63247e^{-14}$	$1.2476e^{-09}$	1.99999
Alg 2.5	0	4	0.00000	$8.00190e^{-11}$	$2.2599e^{-14}$	2.76667
Alg2.6	0	4	0.00000	$3.61892e^{-23}$	$3.2546e^{-19}$	3.90225
Alg2.4	1/4	6	0.00000	$7.805531e^{-29}$	$1.0006e^{-09}$	1.99999
Alg 2.5	1/4	4	0.00000	$1.19855e^{-20}$	$1.0500e^{-14}$	2.99777
Alg2.6	1/4	4	0.00000	$1.857974e^{-07}$	$2.2521e^{-24}$	4.00111

Comparing the value of $\log|x_{n+1} - x_n|$ obtained by different techniques, we represent these value in this graph.

Example 4.4. *We have function* $f_4(x) = x + \sin(\cos(x)) - 1$*, and* $x_0 = 1.6$ *we find the root of this function by using the above techniques. We have following results shown in tables*

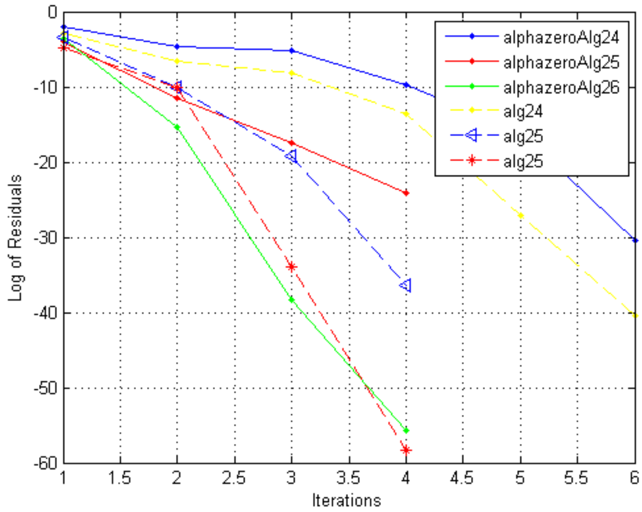

FIGURE 5

Log of Residual for Example 4.3

Table 4: Numerical Results for Example 4.4

Method	α	IT	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$	COC
SM	0	5	1.28146	$3.37822e^{-17}$	$1.00043e^{-08}$	1.99989
Alg 2.5	0	3	1.28146	$2.410098e^{-11}$	$1.01243e^{-17}$	2.58676
Alg2.6	0	3	1.28146	$4.19832e^{-21}$	$2.32121e^{-21}$	3.2467
Alg2.4	1/4	6	1.28146	$8.11423e^{-22}$	$2.012581e^{-11}$	1.99988
Alg 2.5	1/4	4	1.28146	$2.69845e^{-11}$	$1.32243e^{-13}$	2.99789
Alg2.6	1/4	3	1.28146	$2.35350e^{-13}$	$2.31243e^{-23}$	4.21778

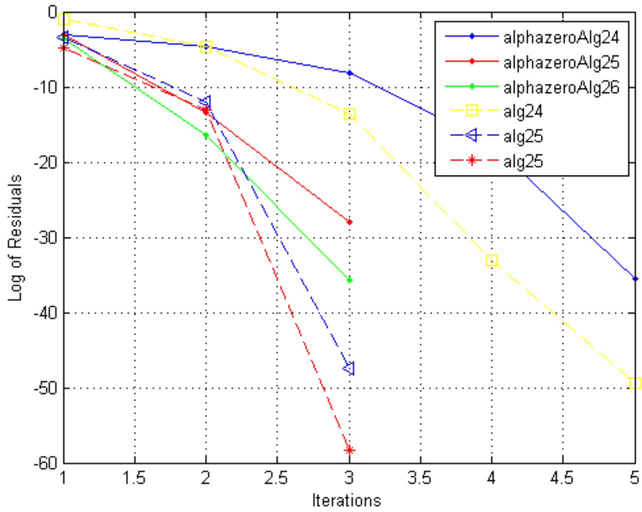

FIGURE 6

Log of Residuals for example 4.4

Example 4.5. *We have function $f_5(x) = x + 3\log(x)$, and $x_0 = 0.5$ we find the root of this function by using the above techniques. We have following results shown in tables.*

Table 5: Numerical results for example 4.5

Method	α	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
SM	0	11	4.536403	$1.117892e^{-19}$	$1.1002e^{-05}$	1.99999
Alg 2.5	0	8	4.536403	$1.2064144e^{-15}$	$1.2005e^{-09}$	2.98991
Alg2.6	0	3	4.536403	$4.9984e^{-23}$	$1.0154e^{-11}$	3.76386
Alg2.4	1/4	5	4.536403	$4.9159414e^{-14}$	$1.1140e^{-13}$	1.99988
Alg 2.5	1/4	4	4.536403	$5.15742e^{-11}$	$1.0646e^{-14}$	3.029852
Alg2.6	1/4	4	4.536403	$5.3577e^{-16}$	$1.2546e^{-24}$	4.066916

Comparing the value of different techniques of $\log|x_{n+1} - x_n|$ we represent this graph.

5. CONCLUSION

In this article, we have suggested some new iterative methods for solving nonlinear equations. We have also analyzed the behavior of several suggested higher order iterative methods for nonlinear equations when the derivative is replaced by some approximation. Numerical results are given to observe the performance and convergence of these methods. Using this technique a wide class of methods for finding simple roots of nonlinear equations can be suggested and implemented.

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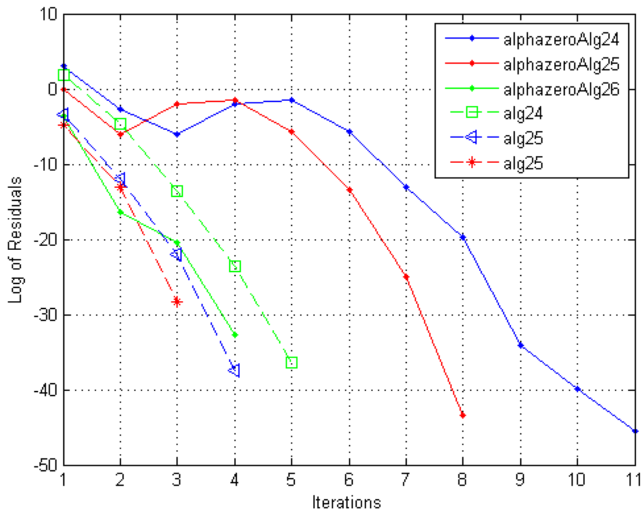

Figure 7
Log of Residuals for example 4.5

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