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# INTEGRAL INEQUALITIES VIA log $m$-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce and investigate a new concept of log $m$-convex functions. We establish some new Hermite-Hadamard type integral inequalities via log $m$-convex functions. Our results represent refinement and improvement of the previously known results. Several special cases are also discussed. The concept and technique of this paper may stimulate further research in this field.

Keywords: $m$-convex functions; *log* $m$-convex functions; Hermite-Hadamard type inequalities.

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## 1. INTRODUCTION

Numerous applications in business, industry, art and medicine has turned convex analysis as one of the most interesting and useful field of mathematics for last few decades. The concept of convexity has played a crucial and significant part in the development of several branches of mathematical and engineering sciences. Consequently several new classes of convex functions and convex sets have been introduced and investigated, which make this area of research very attractive and useful, see [1, 2, 5, 6, 13, 14, 17, 20, 22]. It is well known that a function is convex if and only if, it satisfies the inequality which is known as Hermite-Hadamard inequality. It is one of the most important inequality, see [9, 10]. Several Hermite-Hadamard type inequalities have been derived for various classes of convex function using different techniques, see [7, 8, 11, 12, 15, 16, 17, 19, 20, 22].

Toader [23] introduced a new class of convex functions which is known as m-convex functions. This class is an intermediate form between the usual convexity and star shaped property. In this paper, we introduce a new class of convex functions relative to a constant $m \in (0, 1]$, which is known as log $m$-convex function. It is worth mentioning that this class of log $m$-convex functions is distinctly different from the log $m$-convex functions which were considered in [3]. These log m-convex functions are nonconvex functions. Log convex functions are of interest in many areas of mathematics and science. They play an important role in mathematical statistics and the theory of special functions, see [4, 21].

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We derive some new Hermite-Hadamard integral inequalities for these nonconvex function. Our results include a wide class of known and new inequalities as special cases.

## 2. Preliminaries

Let $I$ be an interval in real line $\mathbb{R}$. Let $f : I = [ma, b] \to \mathbb{R}$ be a continuous function and $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction. First of all, we recall the following well known results and concepts.

**Definition 2.1.** [23] *A set $\mathcal{S}$ is said to be m-convex set, if there exists a fixed constant $m \in (0, 1]$, such that*

$$((1-t)ma + tb) \in \mathcal{S}, \quad \forall a, b \in \mathcal{S}, t \in [0, 1].$$

**Definition 2.2.** [23] *A function $f : I = [0, b] \to \mathbb{R}$ is said to be m-convex, where m ∈(0,1], if*

$$f(tma + (1-t)f(b)) \leq tmf(a) + (1-t)f(b), \quad \forall a, b \in I, t \in [0, 1].$$

It is denoted by $K_m(b)$, the class of all m-convex function on $[0, b]$.

If $t = 1$, then $f(ma) \leq m(f)$, which is called sub-homogeneous function.

We now introduce a new definition of log $m$-convex functions and derive various integral inequalities for these log $m$-convex functions.

**Definition 2.3.** *A function $f : I \to [0, \infty)$ is said to be log m-convex or multiplicatively m-convex if log (f) is convex, or equivalently if for all $a, b \in I$, $m \in (0, 1)$ and $t \in [0, 1]$, one has the inequality*

$$f(tma + (1-t)f(b)) \leq [mf(a)]^t [f(b)]^{1-t}, \quad \forall a, b \in I, t \in [0, 1]. \tag{1}$$

*If $t = \frac{1}{2}$ in (1), then*

$$f\frac{(ma + b)}{2} \leq \sqrt{mf(a)][f(b)]}, \quad \forall a, b \in I. \tag{2}$$

*which is known as Jensen* log *$m$-convex function.*

*From definition 2.3, we have*

$$f(tma + (1-t)b) \leq [mf(a)]^t [f(b)]^{1-t}. \leq tmf(a) + (1-t)f(b).$$

*and*

$$\log f(tma + (1-t)b) \quad \leq \quad tm \log f(a) + (1-t) \log f(b) \quad \forall a, b \in I, t \in [0, 1].$$

This means that log $m$-convex functions are m-convex functions. However the converse is not true.

It is worth mention that Bai et al. [3] introduced log$m$- convex function as:

**Definition 2.4.** *A function $f : I \to [0, \infty)$ is said to be m-logarithmically convex if the inequality*

$$f(ta + m(1-t)b) \leq [f(a)]^t [f(b)]^{m(1-t)}, \quad \forall a, b \in I, t \in [0, 1], \quad m \in (0, 1].$$

From definition 2.3 and Definition 2.4, it is clear that log m-convex functions are distinctly different from the log $m$convex functions defined by Bai et al. [3].

We will use the following notations throughout this paper.

(1) Arithmetic Mean: $A(a,b) = \frac{a+b}{2}, \quad \forall a,b \in \mathbb{R}_+.$

(2) Geometric Mean: $G(a,b) = \sqrt{ab}, \quad \forall a,b \in \mathbb{R}_+.$

(3) Logarithmic mean: $L(a,b) = \frac{b-a}{\log b - \log a}, \quad \forall a,b \in \mathbb{R}_+, a \neq b.$ .

(4) Quadratic Mean: $K(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad \forall a,b \in \mathbb{R}_+..$

## 3. Main results

In this section, we establish several new integral inequalities of Hermite-Hadamard type for log $m$-convex functions.

**Theorem 3.1.** *Let $f : I = [ma, b] \to (0, \infty)$ be a* log *$m$-convex function. Then*

$$\frac{1}{\sqrt{m}} f(\frac{ma+b}{2}) \leq \exp \frac{1}{(b-ma)} \int_{ma}^{b} \log f(x) \mathrm{d}x \leq \sqrt{mf(a)f(b)}.$$

*Proof.* Let $f$ be log $m$-convex function on $I$. Then

$$f(tma + (1-t)f(b)) \leq [mf(a)]^{1-t}[f(b)]^t, \tag{3}$$

which can be written as

$$\log f(tma + (1-t)f(b)) \leq (1-t)\log[mf(a)] + t\log[f(b)]. \tag{4}$$

Integrating (4) with respect to $t$ on [0,1], we have

$$\begin{aligned} \int_0^1 \log f(tma + (1-t)f(b)) \mathrm{d}t &\leq \int_0^1 (1-t)\log[mf(a)]\mathrm{d}t + \int_0^1 t\log[f(b)]\mathrm{d}t \\ &= \log\sqrt{[mf(a)][f(b)]}. \end{aligned}$$

Thus

$$\frac{1}{b-ma} \int_{ma}^{b} \log f(x) \mathrm{d}x \leq \log\sqrt{[mf(a)][f(b)]}. \tag{5}$$

Consider

$$\begin{aligned} f(\frac{ma+b}{2}) &= \frac{f((1-t)ma + tb + tma + (1-t)b)}{2} \\ &\leq \sqrt{[mf((1-t)ma + tb)][(f(tma + (1-t)b)]}. \end{aligned} \tag{6}$$

From this it follows that

$$\log f(\frac{ma+b}{2}) \leq \frac{1}{2}\big\{ \log[mf((1-t)ma + tb)] + \log[f((tma + (1-t)b)\big\}. \tag{7}$$

Integrating (7) with respect to $t$ on [0,1], we have

$$\begin{aligned}\log f(\frac{ma+b}{2}) &\leq \frac{1}{2}\left[\frac{1}{(b-ma)}\int_{ma}^{b}\log mf(x)\mathrm{d}x+\frac{1}{(ma-b)}\int_{b}^{ma}\log f(x)\mathrm{d}x\right]\\ &= \left[\frac{1}{2(b-ma)}\int_{ma}^{b}\log m\mathrm{d}x+\frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)\mathrm{d}x\right]\\ &= \left[\frac{\log m}{2(b-ma)}(b-ma)+\frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)\mathrm{d}x\right]\\ &= \left[\log\sqrt{m}+\frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)\mathrm{d}x\right].\end{aligned}$$

Thus

$$\log\frac{f(\frac{ma+b}{2})}{\sqrt{m}} \leq \frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)\mathrm{d}x. \tag{8}$$

Combining(5) and (8), we have

$$\log\frac{f(\frac{ma+b}{2})}{\sqrt{m}} \leq \frac{1}{(b-ma)}\int_{ma}^{b}\log mf(x)\mathrm{d}x \leq \log\sqrt{mf(a)[f(b)]},$$

which implies that

$$\frac{1}{\sqrt{m}}f(\frac{ma+b}{2}) \leq \exp\frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)\mathrm{d}x \leq \sqrt{mf(a)f(b)}.$$

This completes the proof. □

**Corollary 3.1.** [7] *If $m=1$, then under the assumption of Theorem 3.1, we have*

$$f(\frac{a+b}{2}) \leq \exp\frac{1}{(b-a)}\int_{a}^{b}\log f(x)\mathrm{d}x \leq \sqrt{[f(a)][f(b)]},$$

*which is called the Hermite-Hadamard inequality for log convex function.*

**Theorem 3.2.** *Let $f,g : I=[ma,b]\to(0,\infty)$ be log m-convex functions. Then*

$$\begin{aligned}&\frac{1}{b-ma}\int_{am}^{b}f(x)g(ma+b-x)\mathrm{d}x\\ \leq\ & \frac{mg(a)f(b)+mf(a)g(b)}{2}\\ \leq\ & \frac{1}{2}\Big\{A[f(b),mf(a)]L[f(b),mf(a)]+A[mg(a),g(b)]L[mg(a),g(b)]\Big\}\\ \leq\ & \frac{[mf(a)+(f(b))]^2}{16}+\frac{[mg(a)+g(b)]^2}{16}+\frac{mg(a)f(b)+mf(a)g(b)}{4}.\end{aligned}$$

*where $A$ and $L$ are Arithmetic and Logarithmic means respectively.*

*Proof.* Let $f$, $g$ be log m-convex function. Then

$$\begin{aligned}f(tma+(1-t)b) &\leq [mf(a)]^t[f(b)]^{1-t}\\ g((1-t)ma+tb) &\leq [mg(a)]^{1-t}[g(b)]^t.\end{aligned}$$

Consider

$$
\begin{aligned}
&\frac{1}{b-ma}\int_{ma}^{b} f(x)g(ma+b-x)\mathrm{d}x \\
= \quad & \int_0^1 f(tma+(1-t)b)g((1-t)ma+tb)\mathrm{d}t \\
\leq \quad & \int_0^1 \Big[[\{mf(a)\}^t\{f(b)\}^{1-t}][\{mg(a)\}^{1-t}\{g(b)\}^t]\Big]\mathrm{d}t \\
= \quad & [mg(a)f(b)]\int_0^1 \left[\frac{mf(a)g(b)}{mg(a)f(b)}\right]^t \mathrm{d}t \\
= \quad & \frac{\big[mf(a)g(b)-mg(a)f(b)\big]}{\log\big[mf(a)g(b)\big]-\log\big[mg(a)f(b)\big]} \\
\leq \quad & \frac{mg(a)f(b)+mf(a)g(b)}{2} \\
\leq \quad & \frac{1}{2}\int_0^1 [\{f(tma+(1-t)b)\}^2+\{g((1-t)ma+tb))\}^2]\mathrm{d}t \\
\leq \quad & \frac{1}{2}\int_0^1 \Big\{[\{mf(a)\}^t\{f(b)\}^{1-t}]^2+[\{mg(a)\}^{1-t}\{g(b)\}^t]^2\Big\}\mathrm{d}t \\
= \quad & \frac{1}{2}\left\{[f(b)]^2\int_0^1\left[\frac{mf(a)}{f(b)}\right]^{2t}\mathrm{d}t+[mg(a)]^2\int_0^1\left[\frac{g(b)}{mg(a)}\right]^{2t}\mathrm{d}t\right\} \\
= \quad & \frac{1}{4}\left\{\left[\frac{[mf(a)]^2-[f(b)]^2}{\log[mf(a)]-\log[f(b)]}\right]+\left[\frac{[g(b)]^2-[mg(a)]^2}{\log[g(b)]-\log[mg(a)]}\right]\right\} \\
\leq \quad & \frac{1}{2}\Big\{A[f(b),mf(a)]L[f(b),mf(a)]+A[mg(a),g(b)]L[mg(a),g(b)]\Big\} \\
\leq \quad & \frac{1}{4}\int_0^1 [f(tma+(1-t)b+g((1-t)ma+tb)]^2\mathrm{d}t \\
\leq \quad & \frac{1}{4}\int_0^1 \Big\{[\{mf(a)\}^t\{f(b)\}^{1-t}]^2+[\{mg(a)\}^{1-t}\{g(b)\}^t]^2 \\
& +2[\{mf(a)\}^t\{f(b)\}^{1-t}][\{mg(a)\}^{1-t}\{g(b)\}^t]\Big\}\mathrm{d}t \\
= \quad & \frac{1}{4}\left\{[f(b)]^2\int_0^1\left[\frac{mf(a)}{f(b)}\right]^{2t}\mathrm{d}t+[mg(a)]^2\int_0^1\left[\frac{g(b)}{mg(a)}\right]^{2t}\mathrm{d}t\right. \\
& \left.+2[mg(a)f(b)]\int_0^1\left[\frac{mf(a)g(b)}{mg(a)f(b)}\right]^t\mathrm{d}t\right\} \\
= \quad & \frac{1}{8}\left\{\left[\frac{[mf(a)]^2-[f(b)]^2}{\log[mf(a)]-\log[f(b)]}\right]+\left[\frac{[g(b)]^2-[mg(a)]^2}{\log[g(b)]-\log[mg(a)]}\right]\right\} \\
& +\frac{1}{2}\left\{\frac{mf(a)g(b)-mg(a)f(b)}{\log[mf(a)g(b)]-\log\big[mg(a)f(b)\big]}\right\} \\
\leq \quad & \frac{[mf(a)+f(b)]^2}{16}+\frac{[mg(a)+g(b)]^2}{16}+\frac{mg(a)f(b)+mf(a)g(b)}{4}.
\end{aligned}
$$

This completes the proof. $\square$

**Corollary 3.2.** [19] *If $m = 1$, then, under the assumption of Theorem 3.2, we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(a+b-x)\mathrm{d}x &\le \frac{g(a)f(b)+f(a)g(b)}{2}\\ &\le \frac{1}{2}\Big\{A[f(b),f(a)]L[f(b),f(a)]+A[g(a),g(b)]L[g(a),g(b)]\Big\}\\ &\le \frac{[f(a)+f(b)]^2}{16}+\frac{[g(b)+g(a)]^2}{16}+\frac{g(a)f(b)+f(a)g(b)}{4}.\end{aligned}$$

**Corollary 3.3.** *If $f = g$ and $m = 1$, then, under the assumption of Theorem 3.2, we obtain the result given in* [24].

**Theorem 3.3.** *Let $f, g : I = [ma, b] \to (0, \infty)$ be log m-convex functions. If $\alpha + \beta = 1$, Then*

$$\begin{aligned}\frac{1}{b-ma}\int_{ma}^b f(x)g(ma+b-x)\mathrm{d}x &\le \alpha^2\left[\frac{[mf(a)]^{\frac{1}{\alpha}}-[f(b)]^{\frac{1}{\alpha}}}{[mf(a)-f(b)]}\right]L[f(b),mf(a)]\\ &\quad+\beta^2\left[\frac{[g(b)]^{\frac{1}{\beta}}-[mg(a)]^{\frac{1}{\beta}}}{[g(b)-mg(a)]}\right]L[mg(a),g(b)],\end{aligned}$$

*where $L$ is the Logarithmic mean.*

*Proof.* Let $f$ and $g$ be log m-convex function on $I$. Then

$$\begin{aligned}f(tma+(1-t)tb) &\le [mf(a)]^t[f(b)]^{1-t}\\ g((1-t)ma+tb) &\le [mg(a)]^{1-t}[g(b)]^t.\end{aligned}$$

Using young's inequality, that is, $ab \le \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}}$, $\forall \alpha, \beta > 0, \alpha + \beta = 1$, we consider

$$\begin{aligned}&\frac{1}{b-ma}\int_{ma}^b f(x)g(ma+b-x)\mathrm{d}x\\ =\ &\int_0^1 f(tma+(1-t)b))g((1-t)ma+tb)\mathrm{d}t\\ \le\ &\int_0^1\left\{\alpha[f(tma+(1-t)b))]^{\frac{1}{\alpha}}+\beta[g((1-t)ma+tb)]^{\frac{1}{\beta}}\right\}\mathrm{d}t\\ \le\ &\int_0^1\left\{\alpha\{[mf(a)]^t[f(b)]^{1-t}\}^{\frac{1}{\alpha}}+\beta\{[mg(a)]^{1-t}[g(b)]^t\}^{\frac{1}{\beta}}\right\}\mathrm{d}t\\ =\ &\alpha[f(b)]^{\frac{1}{\alpha}}\int_0^1\left[\frac{mf(a)}{f(b)}\right]^{\frac{t}{\alpha}}\mathrm{d}t+\beta[mg(a)]^{\frac{1}{\beta}}\int_0^1\left[\frac{g(b)}{mg(a)}\right]^{\frac{t}{\beta}}\mathrm{d}t\\ =\ &\alpha^2[f(b)]^{\frac{1}{\alpha}}\left[\frac{(\frac{mf(a)}{f(b)})^u}{\log\frac{mf(a)}{f(b)}}\right]_0^{\frac{1}{\alpha}}+\beta^2[mg(a)]^{\frac{1}{\beta}}\left[\frac{(\frac{g(b)}{mg(a)})^u}{\log\frac{g(b)}{mg(a)}}\right]_0^{\frac{1}{\beta}}\\ =\ &\alpha^2\left[\frac{[mf(a)]^{\frac{1}{\alpha}}-[f(b)]^{\frac{1}{\alpha}}}{[\log[mf(a)]-\log f(b)]}\right]+\beta^2\left[\frac{[g(b)]^{\frac{1}{\beta}}-[mg(a)]^{\frac{1}{\beta}}}{\log[g(b)]-\log[mg(a)]}\right]\\ =\ &\alpha^2\left[\frac{[mf(a)]^{\frac{1}{\alpha}}-[f(b)]^{\frac{1}{\alpha}}}{[mf(a)-f(b)]}\right]L[f(b),mf(a)]\\ &+\beta^2\left[\frac{[g(b)]^{\frac{1}{\beta}}-[mg(a)]^{\frac{1}{\beta}}}{[g(b)-mg(a)]}\right]L[mg(a),g(b)].\end{aligned}$$

This completes the proof. $\square$

**Corollary 3.4.** [19] *If* $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, *and* $m=1$, *then, under the assumption of Theorem 3.3, we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(a+b-x)\mathrm{d}x \quad &\leq \quad \frac{1}{4}\Big\{[\frac{[f(a)]^2-[f(b)]^2}{[f(a)-f(b)]}]L[f(b),f(a)] \\ &+[\frac{[g(b)]^2-[g(a)]^2}{[g(b)-g(a)]}]L[g(a),g(b)]\Big\}.\end{aligned}$$

**Corollary 3.5.** *If* $\alpha=\frac{1}{4}$, $\beta=\frac{3}{4}$ *and* $m=1$, *then, under the assumption of Theorem 3.3, we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(a+b-x)\mathrm{d}x \\ \leq\ &\frac{1}{16}\left[\frac{f^4(a)-f^4(b)}{f(b)-f(a)}\right]L[f(b),f(a)]+\frac{9}{16}\left[\frac{g^{\frac{4}{3}}(b)-g^{\frac{4}{3}}(a)}{g(b)-g(a)}\right]L[g(a),g(b)].\end{aligned}$$

**Theorem 3.4.** *Let* $f,g:I=[ma,b]\to(0,\infty)$ *be increasing and log m-convex functions. Then*

$$\begin{aligned}8L[mg(a),g(b)]\Big\{\frac{2}{m+1}f(\frac{ma+b}{2})\Big\} \quad &\leq \quad \frac{1}{b-ma}\int_{ma}^b f^4(x)\mathrm{d}x+K^2[mg(a),g(b)] \\ &A[mg(a),g(b)]L[mg(a),g(b)]+8.\end{aligned}$$

*Proof.* Let $f$ and $g$ be log m-convex functions. Then

$$\begin{aligned}f(tma+(1-t)b) \quad &\leq \quad [mf(a)]^t[f(b)]^{1-t} \\ g((1-t)ma+tb) \quad &\leq \quad [mg(a)]^{1-t}[g(b)]^t.\end{aligned}$$

Using the inequality,

$$8xy\leq x^4+y^4+8. \quad \forall x,y\in\mathbb{R},$$

we have

$$\begin{aligned}&8f(tma+(1-t)b)[mg(a)]^{1-t}t[g(b)]^t \\ \leq\ &f^4(tma+(1-t)b)+[mg(a)]^{4(1-t)}[g(b)]^t+8.\end{aligned}$$

Now integrating the above inequality with respect to $t$ on $[0,1]$, we have

$$\begin{aligned}&8\int_0^1 f(tma+(1-t)b)[mg(a)]^t[g(b)]^{1-t}\mathrm{d}t \\ \leq &\int_0^1 f^4(tma+(1-t)b)\mathrm{d}t+\int_0^1[mg(a)]^{4t}[g(b)]^{4-4t}\mathrm{d}t+8.\end{aligned}$$

As $f$ and $g$ are increasing functions, we have

$$\begin{aligned}&8\int_0^1 f(tma+(1-t)b)\mathrm{d}t\int_0^1[mg(a)]^t[g(b)]^{1-t}\mathrm{d}t \\ \leq\ &\int_0^1 f^4(tma+(1-t)b)\mathrm{d}t+\int_0^1[mg(a)]^{4t}[g(b)]^{4-4t}\mathrm{d}t+8. \qquad (9)\end{aligned}$$

From the above inequality, it is easy to observe that

$$\begin{aligned}\frac{8L[mg(a),g(b)]}{b-ma}\int_{ma}^{b} f(x)\mathrm{d}x \quad &\leq \quad \frac{1}{b-ma}\int_{ma}^{b} f^4(x)\mathrm{d}x + K^2[mg(a),g(b)] \\ & \quad\quad A[mg(a),g(b)]L[mg(a),g(b)] + 8.\end{aligned} \tag{10}$$

Now using the L.H.S of Hermite Hadamard's inequality in (10), we have

$$\begin{aligned}8L[mg(a),g(b)]\left\{\frac{2}{m+1}f(\frac{ma+b}{2})\right\} \quad &\leq \quad \frac{1}{b-ma}\int_{ma}^{b} f^4(x)\mathrm{d}x + K^2[mg(a),g(b)] \\ & \quad\quad A[mg(a),g(b)]L[mg(a),g(b)] + 8,\end{aligned}$$

where A, L and K are Arithmetic, Logarithmic and Quadratic means respectively. □

**Corollary 3.6.** *If $m = 1$, then, under the assumption of Theorem 3.4, we have*

$$\begin{aligned}& 8L[g(a),g(b)]f(\frac{a+b}{2}) \\ \leq \quad & \frac{1}{b-a}\int_{a}^{b} f^4(x)\mathrm{d}x + K^2[g(a),g(b)]A[g(a),g(b)]L[g(a),g(b)] + 8.\end{aligned}$$

**Corollary 3.7.** *If $f = g$ and $m = 1$, then, under the assumption of Theorem 3.2, we obtain the result given in* [24].

**Theorem 3.5.** *Let $f, g : I = [ma, b] \to (0, \infty)$ be increasing and log m-convex functions. Then*

$$\begin{aligned}&(\frac{2}{m+1})\left[f(\frac{ma+b}{2})L[mg(a),g(b)] + g(\frac{ma+b}{2})L[f(b),mf(a)]\right] \\ &\leq \frac{1}{b-ma}\int_{ma}^{b} f(x)g(x)\mathrm{d}x + L\left[f(b)mg(a),[mf(a)g(b)]\right].\end{aligned}$$

*Proof.* Let $f$ and $g$ be log m-convex functions. Then, $\forall a, b \in I, t \in [0,1]$ , we have

$$\begin{aligned}f(tma + (1-t)b) \quad &\leq \quad [mf(a)]^t[f(b]^{1-t} \\ g((1-t)ma + tb) \quad &\leq \quad [mg(a)]^{1-t}[g(b)]^t.\end{aligned}$$

Using the inequality,

$$(a-b)(c-d) \geq 0. \quad \forall a, b, c, d \in \mathbb{R}, a < b, c < d,$$

we have

$$\begin{aligned}f(tma + (1-t)b)\big[[mg(a)]^{1-t}[g(b)]^t\big] \quad &+ \quad [g((1-t)ma + tb)]\big[[f(b)]^{1-t}[mf(a)]^t\big] \\ &\leq \quad f(tma + (1-t)b)g((1-t)ma + tb) \\ & \quad\quad +\big[[mg(a)]^{1-t}[g(b)]^t[f(b)]^{1-t}[mf(a)]^t\big].\end{aligned}$$

Now integrating the above inequality with respect to $t$ on $[0,1]$, we have

$$\begin{aligned}&\int_0^1 [f(tma + (1-t)b)]\big[[mg(a)]^{1-t}[g(b)]^t\big]\mathrm{d}t + \int_0^1 [g((1-t)ma + tb)]\big[[f(b)]^{1-t}[mf(a)]^t\big]\mathrm{d}t \\ &\leq \int_0^1 [f(tma + (1-t)b)]g((1-t)ma + tb)\mathrm{d}t + \int_0^1 \big[[mg(a)]^{1-t}[g(b)]^t[f(b)]^{1-t}[mf(a)]^t\big]\mathrm{d}t.\end{aligned}$$

As $f$ and $g$ are increasing functions, we have

$$\begin{aligned}&\int_0^1 [f(tma+(1-t)b)]\mathrm{d}t \int_0^1 \big[[mg(a)]^{1-t}[g(b)]^t\big]\mathrm{d}t\\ &+\int_0^1 [g((1-t)ma+tb)]\mathrm{d}t \int_0^1 \big[[f(b)]^{1-t}[mf(a)]^t\big]\mathrm{d}t\\ &\leq \int_0^1 [f(tma+(1-t)b)]\mathrm{d}t \int_0^1 [g((1-t)ma+tb)]\mathrm{d}t\\ &+\int_0^1 [f(b)mg(a)]^{1-t}[mf(a)g(b)]^t\mathrm{d}t.\end{aligned}$$

Now after some simple integration , we have

$$\begin{aligned}&(\frac{2}{m+1})\Big[f(\frac{ma+b}{2})L[mg(a),g(b)]+g(\frac{ma+b}{2})L[f(b),mf(a)]\Big]\\ &\leq \frac{1}{b-ma}\int_{ma}^b f(x)g(x)\mathrm{d}x + L\Big[f(b)mg(a),[mf(a)g(b)]\Big].\end{aligned}$$

This completes the proof. □

**Corollary 3.8.** [24] *If $m=1$, then, under the assumption of Theorem 3.5, we have*

$$\begin{aligned}&f(\frac{a+b}{2})L[g(a),g(b)]+g(\frac{a+b}{2})L[f(b),f(a)]\\ &\leq \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x + L\big[f(b)g(a),f(a)g(b)\big].\end{aligned}$$

**Theorem 3.6.** *Let $f,g : I=[ma,b]\to(0,\infty)$ be an increasing and log m-convex functions. Then*

$$\begin{aligned}&\frac{1}{b-ma}\int_{ma}^b f^2(x)\mathrm{d}x + A[f(b),mf(a)]L[f(b),mf(a)] + A[mg(a),g(b)]\\ &L[mg(a),g(b)]\\ &\geq \frac{2}{m+1}f(\frac{ma+b}{2})L[f(b),mf(a)] + L[mg(a)f(b),mf(a)g(b)]\\ &+\frac{2}{m+1}f(\frac{ma+b}{2})L[mg(a),g(b)].\end{aligned}$$

*Proof.* Let $f$ and $g$ are log - convex functions. Then

$$\begin{aligned}f(tma+(1-t)b) &\leq [mf(a)]^t[f(b)]^{1-t}\\ g((1-t)ma+tb) &\leq [mg(a)]^{1-t}[g(b)]^t.\end{aligned}$$

Using the inequality,

$$x^2+y^2+z^2\geq xy+yz+zx. \quad \forall x,y,z\in\mathbb{R}, \tag{11}$$

we have

$$\begin{aligned}f^2(tma+(1-t)b) &+ [mf(a)]^{2t}[f(b)]^{2(1-t)}+[mg(a)]^{2(1-t)}[g(b)]^{2t}\\ &\geq \big[f(tma+(1-t)b)[[mf(a)]^t[f(b)]^{1-t}\big]\\ &\quad+\big[[mf(a)]^t[f(b)]^{1-t}[mg(a)]^{1-t}[g(b)]^t\big]\\ &\quad+\big[[mg(a)]^{1-t}[g(b)]^t f(tma+(1-t)b)\big].\end{aligned}$$

Now integrating the above inequality with respect to $t$ on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f^2(tma+(1-t)b)\mathrm{d}t + \int_0^1 \big[[mf(a)]^{2t}[f(b)]^{2(1-t)}\big]\mathrm{d}t \\
&+\int_0^1 \big[[mg(a)]^{2(1-t)}[g(b)]^{2t}\big]\mathrm{d}t \\
&\geq \int_0^1 \big[f(tma+(1-t)b)[mf(a)]^{t}[f(b)]^{1-t}\big]\mathrm{d}t \\
&+\int_0^1 \big[[mf(a)]^{t}[f(b)]^{1-t}[mg(a)]^{1-t}[g(b)]^{t}\big]\mathrm{d}t \\
&+\int_0^1 \big[[mg(a)]^{1-t}[g(b)]^{t} f(tma+(1-t)b)\big]\mathrm{d}t. \qquad (12)
\end{aligned}$$

To solve the integral in (12), let

$$\begin{aligned}
\mathbb{A} &= \int_0^1 f^2(tma+(1-t)b)\mathrm{d}t + \int_0^1 \big[[mf(a)]^{2t}[f(b)]^{2(1-t)}\big]\mathrm{d}t \\
&\quad +\int_0^1 \big[[mg(a)]^{2(1-t)}[g(b)]^{2t}\big]\mathrm{d}t \\
&= \frac{1}{b-ma}\int_{ma}^{b} f^2(x)\mathrm{d}x + A[f(b),mf(a)]\text{Ł}[f(b),mf(a)] + A[mg(a),g(b)] \\
&\quad L[mg(a),g(b)],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{B} &= \int_0^1 \big[f(tma+(1-t)b)[mf(a)]^{t}[f(b)]^{1-t}\big]\mathrm{d}t \\
&\quad +\int_0^1 \big[[mf(a)]^{t}[f(b)]^{1-t}[mg(a)]^{1-t}[g(b)]^{t}\big]\mathrm{d}t \\
&\quad +\int_0^1 [mg(a)]^{1-t}[g(b)]^{t} f(tma+(1-t)b)\mathrm{d}t \\
&\quad \geq \frac{2}{m+1} f(\frac{ma+b}{2})L[f(b),mf(a)] + L[mg(a)f(b),mf(a)g(b)] \\
&\quad +\frac{2}{m+1} f(\frac{ma+b}{2})L[mg(a),g(b)].
\end{aligned}$$

Subsituting the values of $\mathbb{A}$ and $\mathbb{B}$ in (12), we have

$$\begin{aligned}
&\frac{1}{b-ma}\int_{ma}^{b} f^2(x)\mathrm{d}x + A[f(b),mf(a)]\text{Ł}[f(b),mf(a)] + A[mg(a),g(b)] \\
&L[mg(a),g(b)] \\
&\geq \frac{2}{m+1} f(\frac{ma+b}{2})L[f(b),mf(a)] + L[mg(a)f(b),mf(a)g(b)] \\
&+\frac{2}{m+1} f(\frac{ma+b}{2})L[mg(a),g(b)].
\end{aligned}$$

This completes the proof. $\square$

**Corollary 3.9.** *If $m = 1$, then, under the assumption of Theorem 3.6, we have*

$$\frac{1}{b-a}\int_a^b f^2(x)\mathrm{d}x + A[f(b), f(a)]L[f(b), f(a)] + A[g(a), g(b)]L[g(a), g(b)]$$
$$\geq f(\frac{a+b}{2})L[f(b), f(a)] + L[g(a)f(b), f(a)g(b)] + f(\frac{a+b}{2})L[g(a), g(b)].$$

## CONCLUSION

In this paper, we have introduced a new class of convex function relative to a constant $m \in (0, 1]$, which is known as log m-convex function. New integral inequalities are obtained via these nonconvex functions. Some special cases are also discussed which have been obtained from our results. The technique of this paper may motivate new research.

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## References

[1] Anderson, G.D., Vamanamurthy, and M. Vuorinen,M.,(2007) Generalized convexity and inequalities, J. Math. Anal. Appl, 335(2), pp. 1294-1308.
[2] Alomari, M., Darus, M., and Dragomir, S. S., (2009), New inequalities of Simpson's type for s-convex functions with applications, RGMIA Res. Rep. Coll, 12 (4).
[3] Bai, R. F., Qi, F. and Xi, B. Y., (2013), Hermite-Hadamard type inequalities for the $m$ and $(\alpha, m)$ logarithmically convex functions, Filomat, 27(1), pp. 1-7.
[4] Carlson, B. C., (1977), Special Functions of Applied Mathematics, Academic Press, New York.
[5] Cristescu, G. and Lupsa, L., (2002), Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrechet, Holland, (2002).
[6] Cristescu, G., (2005), Improved integral inequalities for products of convex functions, J. Inequal. Pure and Appl. Math, 6(2), Article: 35.
[7] Dragomir, S. S. and Mond, B., (1998), Integral inequalities of Hadamard type for log-convex functions, Demonstratio, 31, pp. 354-364.
[8] Dragomir, S. S. and Pearce, C. E. M.,(2000), Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia, (2000).
[9] Hadamard, J., (1893), Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann, J. Math. Pure. Appl, 58, pp. 171-215.
[10] C. Hermite, C., (1883), Sur deux limites d'une integrale definie, Mathesis, 3, pp. 82.
[11] Hyers, D. H., and Ulam, S. M., (1952), Approximately convex functions, Proc. Amer. Math. Soc, 3, 821-828, (1952).
[12] Mihesan, V. G., (1993), A generalization of the convexity, Seminar on Functional Equations, Approximation and Convex, Cluj-Napoca (Romania), (1993).
[13] Niculescu, C. P. and Persson, L. E.,(2018), Convex Functions and Their Applications. Springer-Verlag, New York.
[14] Niculescu, C. P., (2012), The Hermite-Hadamard inequality for log-convex functions, Nonlinear Analysis, 75, pp. 662-669.
[15] Noor, M. A., Qi, F. and Awan, M. U.,(2013), Hermite-Hadamard type inequalities for log-$h$-convex functions, Analysis, 33, pp. 367-375.
[16] M. A., Noor, K. I. and Awan, M. U., (2014), Hermite-Hadamard inequalities for relative semi-convex functions and applications, Filomat, 28(2), pp. 221-230.
[17] M. A., Noor, K. I. and Awan, M. U., (2015), Generalized convexity and integral inequalities, Appl. Math. Inf. Sci, 9(1), pp. 233-243.
[18] Noor, M. A.,(2007), On Hadamard integral inequalities involving two log-prenvex functions, J. Inequal. Pure and App. Math, 8(3).
[19] M. A., Noor, K. I. and Awan, M. U., (2014), Some characterizations of harmonically log-convex functions, Proc. Jangjeon. Math. Soc, 17, pp. 51-61.

[20] M. A., Noor, K. I., Iftikhar, S. and F. Safdar, F.,(2015), Integral inequaities for relative harmonic ($s$,$\eta$)-convex functions, Appl. Math. Inf. Sci, 1(1), pp. 27-34.
[21] Pecaric, J. E., Proschan, F. and Y.L. Tong, Y. L.,(1992), Convex functions, partial orderings and statistical applications, Academic Press, Boston.
[22] M. A., Noor, K. I. and Safdar, F., (2017), Integral inequaities via ($\alpha$,$m$)-convex functions, J. Nonlinear. Func. Anal, 2017, pp.
[23] G. Toader, G., (1985), Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, pp. 329-338.
[24] M. Tunc, M.,(2014), Some integral inequlities for logarithmically convex functions, J. Egyp. Math. Soc, 22, pp. 177-181.

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