TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 203-215

REVAN WEIGHTED PI INDEX ON SOME PRODUCT OF GRAPHS

M. PRIYADHARSHINI [1], P. KANDAN[2,*], E. CHANDRASEKARAN [3], A. JOSEPH KENNEDY [4], §

Abstract. In chemical graph thoery, PI index is an additive topological index which has been used to measure the characteristics of chemical compounds. In this paper we introduce the weighted version of PI index of graph called the Revan Weighted PI index and we have obtained it for the hierarchical product of graphs, cartesian product, subdivision and join of two graphs. Also we have derived this index for some molecular graphs.

Keywords: Hierarchical Product, Cartesian Product, Subdivision, Join, PI index, Weighted PI index, Revan index.
AMS Subject Classification: 05C12;05C76

1. Introduction

Let $G = G(V, E)$ be the graph where $V = V(G)$ and $E = E(G)$ denotes the vertex set and edge set of the graph G. Throughout this paper, the graphs considered here are simple and connected. A molecular graph is a graph such that its vertices correspond to the atoms and the edges to the bonds.

A vertex $x \in V(G)$ is said to be equidistant from the edge $e = uv$ of G if $d_G(u, x) = d_G(v, x)$, where $d_G(u, x)$ denotes the distance between u and x in G; otherwise, x is a non equidistant vertex. The degree of a vertex $x \in V(G)$ is number of edges incident with x and is denoted by $d_G(x)$. Let $\Delta(G)$ and $\delta(G)$ denotes the maximum and minimum degree of the vertices of G respectively.

For an edge $e = uv \in E(G)$, the number of vertices of G whose distance to the vertex u is less than the distance to the vertex v in G is denoted by $n_u^G(e) = n_u(e, G)$; analogously, $n_v^G(e) = n_v(e, G)$; is the number of vertices of G whose distance to the vertex v in G is less than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted.

[1] Department of Mathematics,Presidency College, Chennai 600 005, India.
e-mail: priyaphd19@gmail.com; ORCID: https://orcid.org/0000-0002-2674-859X.
[2] Department of Mathematics,Annamalai University, Annamalainagar, 608 002, India.
[*] PG and Research Department of Mathematics, Government Arts College, Chidambaram 608102, India.
e-mail: kandan2k@gmail.com; ORCID: https://orcid.org/0000-0002-3088-2185.
[3] Department of Mathematics in Vel-Tech Rangarajan Dr Sagunthala R and D Institute of Science and Technology, Chennai,India.
e-mail: e_chandrasekaran@yahoo.com; ORCID: https://orcid.org/0000-0001-9393-4184.
[4] Department of Mathematics, Pondicherry University, 605 014, India.
e-mail: kennedy.pondi@gmail.com; ORCID: https://orcid.org/0000-0002-5405-4317.
§ Manuscript received: September 29, 2019; accepted: April 08, 2020.

Graph operations produce new graphs from the given simpler graphs. Frucht and Harary [6] introduced the corona of two graphs. In Graph theory, the operation of graphs play an important role in the decomposition of graphs into isomorphic subgraphs. Graph theory provided variety of tools to the chemist [7]. The tools for chemical graph theory focuses on topological indices of chemical graphs. These topological indices have many applications in QSPR and QSAR as well see [4].

The Padmakar-Ivan(PI) index is a topological descriptor. The two topological indices namely the PI index and weighted PI index of G denoted by $PI(G)$ and $PI_w(G)$ respectively, are defined as follows

$$PI(G) = \sum_{e=uv\in E(G)} \left(n_u^G(e) + n_v^G(e)\right)$$

$$PI_w(G) = \sum_{e=uv\in E(G)} (d_G(u) + d_G(v)) \left(n_u^G(e) + n_v^G(e)\right)$$

.

The PI index [20] of the graph G is a topological index related to equidistant vertices. Illić et al. [8] introduced weighted vertex PI index to increase the diversity of bipartite graphs, that is for bipartite graphs,we have $n_u^G(e) + n_v^G(e) = |V(G)| = n$ where n denotes the number of vertices in G and therefore the diversity of the PI index is not satisfying. For a graph G with n vertices and m edges, the inequality that $PI(G) \leq nm$ holds and the same equality holds if and only if G is bipartite. Khadikar et al. [11] were the first to introduce edge Padmakar – Ivan index of graphs. K. Pattabiraman et al. [19] studied the vertex and edge Padmakar – Ivan indices of the generalized hierarchical product of graphs. On this extension by K. Pattabiraman et al. [16–18] developed the exact formula for weighted PI index of graphs, corona product and Szeged indices of some graph operations. To continuation of work on weighted version of PI index, Kandan et al. [9, 10] developed the Revan Weighted PI and Szeged index of graphs.

The Revan vertex degree of a vertex u in G is defined as $r_G(u) = \Delta(G) + \delta(G) - d_G(u)$. Recently [13] Kulli's concept in graph theory: the first and second Revan indices of a graph G are defined as $R_1(G) = \sum_{uv\in E(G)} (r_G(u) + r_G(v))$ and $R_2(G) = \sum_{uv\in E(G)} (r_G(u)r_G(v))$ and derived the exact value for the various molecular structure see [14, 15]. Motivated by the invariants like the Weighted PI indices and Revan indices, In [10] Kandan et al. introduced the Revan Weighted PI index $PI_r(G)$ of a graph G as

$$PI_r(G) = \sum_{e=uv\in E(G)} (r_G(u) + r_G(v)) \left(n_u^G(e) + n_v^G(e)\right)$$

.

The hierarchical product were defined by Barriere et al. [2, 3]. Let G and H be two connected graphs and $U \subseteq V(G)$ i.e., U be a non empty subset of $V(G)$. Then the Hierarchical Product of G and H denoted by $G(U) \sqcap H$ is the graph with the vertex set $V(G) \times V(H)$ and any two verices (u, v) and (u', v') are adjacent if and only if $u = u' \in U$ and $(v, v') \in E(H)$ or $(u, u') \in E(G)$ and $v = v'$.

The cartesian product of graphs G and H, denoted by $G\Box H$ is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and any two vertices (u, x) and (v, y) are adjacent if and only if $u = v \in V(G)$ and $(x, y) \in E(H)$ or $x = y \in V(H)$ and $(u, v) \in E(G)$. It is clear that $G(U) \sqcap H$ is isomorphic to $G\Box H$, if $U = V(G)$ then $G\Box H = G(U) \sqcap H$.

In this paper, we obtain the Revan Weighted PI index of hierarchical product, of graphs using the two already known topological indices,namely the PI index and Weighted PI index.Likewise for cartesian product, subdivision and join of two graphs. In addition, we present explicit results for different classes of molecular graphs.

2. Revan Weighted PI index of $G(U) \sqcap H$

Let G be a graph. The distance between u and v is denoted by $d_G(u,v)$ where $u, v \in V(G)$ and $U \subseteq V(G)$ is a $U-V$ path in G containing some vertex $w \in U$ (vertex w could be the vertex u or v). The distance between u and v through U is the length of the shortest path between u and v through U and is denoted by $d_{G(U)}(u,v)$. Note that if one of the vertices u and v belongs to U, then $d_{G(U)}(u,v) = d_G(u,v)$.

A vertex $a \in V(G(U))$ is said to be equidistant from $e = uv \in E(G(U))$ through U in $G(U)$, if $d_{G(U)}(u,a) = d_{G(U)}(v,a)$. For an edge $e \in G(U)$, let $N_{G(U)}(e)$ denote the number of equidistant vertices of e through U in $G(U)$. Then $PI_r(G(U))$ is defined as follows,

$$PI_r(G(U)) = \sum_{e=uv\in E(G)} \left(r_{G(U)}(u) + r_{G(U)}(v)\right)\left(n_u^{G(U)}(e) + n_v^{G(U)}(e)\right)$$

For an edge $e = uv \in E(G)$, let $T_G(e,u)$ be the set of vertices closer to u than v and $T_G(e,v)$ be the set of vertices closer to v than u. That is $T_G(e,u) = \{a \in V(G) \mid d_G(u,a) < d_G(v,a)\}$, $T_G(e,v) = \{a \in V(G) \mid d_G(u,a) > d_G(v,a)\}$.

Lemma 2.1. *[18] Let G and H be graphs with $U \subseteq V(G)$ then*

(i). $|V(G(U) \sqcap H)| = |V(G)|\,|V(H)|$ *and* $|E(G(U) \sqcap H)| = |E(G)|\,|V(H)| + |E(H)|\,|U|$.
(ii). *The degree of the vertex $(x,y) \in V(G(U) \sqcap H)$ is $d_{G(U)}(x) + \phi_U(x) d_H(y)$, where ϕ_U denotes the characteristic function on the set U.*
(iii). $d_{G(U)\sqcap H}((x,y)(x',y')) = \begin{cases} d_{G(U)}(x,x') + d_H(y,y') \ if \ y \neq y' \\ d_G(x,x') \ if \ y = y' \end{cases}$

Next we compute the Revan Weighted PI index of the hierarchical product of two connected graphs G and H, expressed interms of the known indices, namely the PI index and the Weighted PI index.

Theorem 2.1. *Let G and H be two connected graphs and U be a non empty subset of $V(G)$. Then*

$$\begin{aligned} PI_r(G(U) \sqcap H) &= 2(\Delta(G(U) \sqcap H) + \delta(G(U) \sqcap H))[|U||V(G)|PI_v(H) \\ &\quad + |V(H)|^2 PI_v(G(U))] - \Bigg[2|V(G)|PI_v(H)\left(\sum_{u_r\in U} d_{G(U)}(u_r)\right) \\ &\quad + |U||V(G)|PI_w(H) + |V(H)|PI_w(G) \\ &\quad + |V(H)|((|V(H)|-1)PI_w(G(U)) \\ &\quad + 2|E(H)| \sum_{u_iu_k\in E(G)} (\phi_U(u_i) + \phi_U(u_k))\left(n_{u_i}^G(e) + n_{u_k}^G(e)\right) \\ &\quad + 2|E(H)|((|V(H)|-1) \\ &\quad \sum_{u_iu_k\in E(G)} (\phi_U(u_i) + \phi_U(u_k))\left(n_{u_i}^{G(U)}(e) + n_{u_k}^{G(U)}(e)\right)\Bigg] \end{aligned}$$

Proof.

$$\begin{aligned}
PI_r(G(U)\sqcap H) &= \sum_{(u,v)\in E(G(U)\sqcap H)} \left(r_{G(U)\sqcap H}(u)+r_{G(U)\sqcap H}(v)\right) \\
&\quad \left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right) \\
PI_r(G(U)\sqcap H) &= \sum_{(u,v)\in E(G(U)\sqcap H)} (\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H) \\
&\quad -d_{G(U)\sqcap H}(u)+\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H) \\
&\quad -d_{G(U)\sqcap H}(v))\left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(u,v)\in E(G(U)\sqcap H)} (2((\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H)))\left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right) \\
&\quad -\sum_{(u,v)\in E(G(U)\sqcap H)} (d_{G(U)\sqcap H}(u)+d_{G(U)\sqcap H}(v))\left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right) \\
&= 2\left(\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H)\right)\sum_{(u,v)\in E(G(U)\sqcap H)}\left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right) \\
&\quad -\sum_{(u,v)\in E(G(U)\sqcap H)} (d_{G(U)\sqcap H}(u)+d_{G(U)\sqcap H}(v))\left(n_u^{G(U)\sqcap H}(e)+n_v^{G(U)\sqcap H}(e)\right) \\
&= 2\left(\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H)\right)PI_v(G(U)\sqcap H)-PI_w(G(U)\sqcap H)
\end{aligned}$$

Using [18] and [19], we have

$$\begin{aligned}
PI_r(G(U)\sqcap H) &= 2\left(\Delta(G(U)\sqcap H)+\delta(G(U)\sqcap H)\right)[|U||V(G)|PI_v(H) \\
&\quad +|V(H)|^2PI_v(G(U))]-\Bigg[2|V(G)|PI_v(H)\Bigg(\sum_{u_r\in U} d_{G(U)}(u_r)\Bigg) \\
&\quad +|U||V(G)|PI_w(H)+|V(H)|PI_w(G) \\
&\quad +|V(H)|\left(|V(H)|-1\right)PI_w(G(U))+2|E(H)|\sum_{u_iu_k\in E(G)}(\phi_U(u_i) \\
&\quad +\phi_U(u_k))\left(n_{u_i}^G(e)+n_{u_k}^G(e)\right)+2|E(H)|\left(|V(H)|-1\right) \\
&\quad \sum_{u_iu_k\in E(G)}(\phi_U(u_i)+\phi_U(u_k))\left(n_{u_i}^{G(U)}(e)+n_{u_k}^{G(U)}(e)\right)\Bigg]
\end{aligned}$$

□

In the above theorem, if we set $U=V(G)$, we obtain the following corollary.

Corollary 2.1. *Let G and H be connected graphs. Then*

$$\begin{aligned}
PI_r(G\Box H) &= 2(\Delta(G\Box H)+\delta(G\Box H))(|V(H)|^2PI_v(G)+|V(G)|^2PI_v(H)) \\
&\quad -(|V(H)|^2PI_w(G)+|V(G)|^2PI_w(H)+4(|V(G)||E(G)|PI_v(H) \\
&\quad +|V(H)||E(H)|PI_v(G)))
\end{aligned}$$

Let $G_1, G_2, \cdots G_n$ be graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, $1\le i\le n$. Denote by $\mathop{\Box}\limits_{i=1}^{n} G_i$ the cartesian product of graphs $G_1, G_2, \cdots G_n$. Clearly, $|V(\mathop{\Box}\limits_{i=1}^{n} G_i)| =$

$\prod_{i=1}^{n}|V(G_i)|$. By induction on n, we have $|E(\mathop{\square}_{i=1}^{n} G_i)| = \prod_{i=1}^{n}|V(G_i)|\sum_{i=1}^{n}\frac{|E(G_i)|}{|V(G_i)|}$. Khalifeh et al. [12] have proved $PI_v(\mathop{\square}_{i=1}^{n} G_i) = \sum_{i=1}^{n} PI_v(G_i) \prod_{j=1,j\neq i}^{n} |V(G_j)|^2$.
Now we shall prove the Revan Weighted PI index for n graphs.

Theorem 2.2. *Let $G_1, G_2, \cdots G_n$ be connected graphs. Then*

$$\begin{aligned} PI_r(\mathop{\square}_{i=1}^{n} G_i) &= 2\left(\sum_{i=1}^{n}\Delta(G_i) + \sum_{i=1}^{n}\delta(G_i)\right)\sum_{i=1}^{n} PI_v(G_i) \prod_{j=1,j\neq i}^{n} |V(G_j)|^2 \\ &\quad -\Bigg(\sum_{i=1}^{n} PI_w(G_i) \prod_{j=1,j\neq i}^{n} |V(G_j)|^2 \\ &\quad +4\sum_{i,j=1,i\neq j}^{n} PI_v(G_i)|V(G_j)||E(G_j)| \prod_{k=1,i\neq k\neq j}^{n} |V(G_k)|^2\Bigg) \end{aligned}$$

Proof:

The proof is by induction on n. For $n = 2$, the proof follows from Theorem 2.1 and let us assume that the result hold for n graphs. then

$$\begin{aligned} PI_r(\mathop{\square}_{i=1}^{n+1} G_i) &= PI_r(\mathop{\square}_{i=1}^{n} G_i \square G_{n+1}) \\ &= 2[\Delta(\mathop{\square}_{i=1}^{n} G_i \square G_{n+1}) + \delta(\mathop{\square}_{i=1}^{n} G_i \square G_{n+1})](|V(G_{n+1})|^2 \\ &\quad PI_v(\mathop{\square}_{i=1}^{n} G_i) + |V(\mathop{\square}_{i=1}^{n} G_i)|^2 PI_v(G_{n+1})) \\ &\quad -(|V(G_{n+1})|^2 PI_w(\mathop{\square}_{i=1}^{n} G_i) + |V(\mathop{\square}_{i=1}^{n} G_i)|^2 \\ &\quad PI_w(G_{n+1}) + 4(|V(\mathop{\square}_{i=1}^{n} G_i)||E(\mathop{\square}_{i=1}^{n} G_i)|PI_v(G_{n+1}) \\ &\quad +|V(G_{n+1})||E(G_{n+1})|PI_v(\mathop{\square}_{i=1}^{n} G_i))) \end{aligned}$$

$$
\begin{aligned}
= \quad & 2\Big[\sum_{i=1}^{n}\Delta(G_i)+\sum_{i=1}^{n}\delta(G_i)\Big]\Big(|V(G_{n+1})|^2 \\
& \Big(\sum_{i=1}^{n}PI_v(G_i)\prod_{j=1,j\neq i}^{n}|V(G_j)|^2\Big)+\prod_{i=1}^{n}|V(G_i)|^2PI_v(G_{n+1})\Big) \\
& -\Big(|V(G_{n+1})|^2\sum_{i=1}^{n}PI_w(G_i)\prod_{j=1,j\neq i}^{n}|V(G_j)|^2 \\
& +4\sum_{i,j=1,i\neq j}^{n}PI_v(G_i)|V(G_j)||E(G_j)|\prod_{k=1,i\neq k\neq j}^{n}|V(G_k)|^2\Big) \\
& +\prod_{i=1}^{n}|V(G_i)|^2PI_w(G_{n+1})+4\Big(PI_v(G_{n+1})\sum_{i=1}^{n}|V(G_i)||E(G_i)| \\
& \prod_{j=1,j\neq i}^{n}|V(G_j)|^2+|V(G_{n+1})||E(G_{n+1})| \\
& \sum_{i=1}^{n}PI_v(G_i)\prod_{j=1,j\neq i}^{n}|V(G_j)|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 2\Big[\sum_{i=1}^{n}(\Delta(G_i)+\delta(G_i))\Big]\Big(\sum_{i=1}^{n+1}PI_v(G_i)\prod_{j=1,j\neq i}^{n+1}|V(G_j)|^2\Big) \\
& -\Big(\sum_{i=1}^{n+1}PI_w(G_i)\prod_{j=1,j\neq i}^{n+1}|V(G_j)|^2+4\Big(\sum_{i,j=1,i\neq j}^{n}PI_v(G_i)|V(G_j)||E(G_j)| \\
& \prod_{k=1,i\neq k\neq j}^{n+1}|V(G_k)|^2+\sum_{i\leq j\leq n}PI_v(G_i)|V(G_j)||E(G_j)|\prod_{k=1,i\neq k\neq j}^{n+1}|V(G_k)|^2\Big)\Big) \\
= \quad & 2\Big[\sum_{i=1}^{n}\Delta(G_i)+\sum_{i=1}^{n}\delta(G_i)\Big]\sum_{i=1}^{n}PI_v(G_i)\prod_{j=1,j\neq i}^{n}|V(G_j)|^2 \\
& -\Big(\sum_{i=1}^{n}PI_w(G_i)\prod_{j=1,j\neq i}^{n}|V(G_j)|^2 \\
& +4\sum_{i,j=1,i\neq j}^{n}PI_v(G_i)|V(G_j)||E(G_j)|\prod_{k=1,i\neq k\neq j}^{n}|V(G_k)|^2\Big).
\end{aligned}
$$

This completes the proof.

Example 2.1. *An equivalent description of hypercube $n-$cube Q_n is that it is the n^{th} power of K_2. That is $Q_n = K_2^n$. By Theorem 2.2, we have*

$$PI_r(Q_n) = PI_r(K_2^n) = n2^{2n}\left[\frac{1}{2}(\Delta(G(U)\sqcap H)) + \delta(G(U)\sqcap H)(n-1) - n\right]$$

If $n = 3$ we have $PI_r(K_2^3) = 64$.

Example 2.2. *Using Theorem 2.2 and corollary 2.1, we have* $PI(C_n) = \begin{cases} n^2, & n \textit{ is even} \\ n(n-1), & n \textit{ is odd} \end{cases}$

$PI_r(C_n) = \begin{cases} 4n^2, & n \textit{ is even} \\ 4n(n-1), & n \textit{ is odd.} \end{cases}$ *We obtain the exact Revan Weighted PI index of*

$C^{n_1}\square C^{n_2}\square\cdots\square C^{n_k}$ *is* $PI_r(C^{n_1}\square C^{n_2}\square\cdots\square C^{n_k}) = \begin{cases} 4k^2\prod\limits_{i=1}^{k} n_i^2, & n_i \textit{ is even} \\ 4k\prod\limits_{i=1}^{k} n_i^2 \sum\limits_{i=1}^{k}\left(1-\dfrac{1}{n_i}\right), & n_i \textit{ is odd} \end{cases}$

where $\mathop{\square}\limits_{i=1}^{n} C^{n_i}$ *denotes the cartesian product of n cycles.*

If each $n_i = n$, *then* $PI_r(\square C_k^n) = \begin{cases} 4k^2n^{2k}, & n_i \textit{ is even} \\ 4k^2(n-1)n^{2k-1}, & n_i \textit{ is odd} \end{cases}$

3. Revan Weighted PI Index of $S(G)(U)\sqcap H$

Let G be a connected graph. The sub divison graph $S = S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex into each edge of G. Clearly, $V(S(G)$ contains all the vertices of G.

The edge a - Zagreb index is defined as, $Z_a(G) = \sum\limits_{uv\in E(G)} (d_G(u)^a + d_G(v)^a)$. It is clear that $Z_1(G) = M_1(G)$, where $M_1(G)$ is the first Zagreb index and is defined as $M_1(G) = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$. The second Zagreb index is defined as $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. The above indices are going to be used in the section 4.

Next we compute the $PI_r(S(G)(U)\sqcap H)$, for suitable U.

Theorem 3.1. *Let G and H be connected graphs. If $U = V(G) \subseteq V(S(G))$, then*

$$\begin{aligned} PI_r\left(S(G)(U)\sqcap H\right) &= 2(\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H))[(|V(G)| \\ &\quad +|E(G)|)(2|V(H)|^2|E(G)| + |V(G)||PI_v(H)|)] \\ &\quad -\big[4(|V(G)| + |E(G)|)|E(G)|PI_v(H) + (|V(G)| \\ &\quad +|E(G)|)|V(G)|PI_w(H) + |V(H)|^2(|V(G)| \\ &\quad +|E(G)|)(M_1(G) + 4|E(G)|) \\ &\quad +4|E(H)||V(H)||E(G)|(|V(G)| + |E(G)|)\big] \end{aligned}$$

Proof:

$$\begin{aligned}
PI_r(S(G)(U) \sqcap H) &= \sum_{uv=e\in E(S(G)(U)\sqcap H)} \Big(r_{S(G)(U)\sqcap H}(u) + r_{S(G)(U)\sqcap H}(v)\Big) \\
&\quad \Big(n_u^{S(G)(U)\sqcap H}(e) + n_v^{S(G)(U)\sqcap H}(e)\Big) \\
&= \sum_{uv=e\in E(S(G)(U)\sqcap H)} \Big[(\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H) \\
&\quad -d_{S(G)(U)\sqcap H}(u)) + (\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H) \\
&\quad -d_{S(G)(U)\sqcap H}(v))\Big]\Big(n_u^{S(G(U))\sqcap H}(e) + n_v^{S(G(U))\sqcap H}(e)\Big)
\end{aligned}$$

$$\begin{aligned}
PI_r(S(G)(U) \sqcap H) &= 2(\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H)) \\
&\quad \sum_{uv=e\in E(S(G)(U)\sqcap H)} \Big(n_u^{S(G)(U)\sqcap H}(e) + n_v^{S(G)(U)\sqcap H}(e)\Big) \\
&\quad - \sum_{uv=e\in E(S(G)(U)\sqcap H)} \big(d_{S(G)(U)\sqcap H}(u) + d_{S(G)(U)\sqcap H}(v)\big) \\
&\quad \Big(n_u^{S(G)(U)\sqcap H}(e) + n_v^{S(G)(U)\sqcap H}(e)\Big) \\
&= 2(\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H))PI_v(S(G)(U)\sqcap H) \\
&\quad -PI_w(S(G)(U)\sqcap H)
\end{aligned}$$

Using Theorem [18] and [19], we have

$$\begin{aligned}
PI_r(S(G)(U) \sqcap H) &= 2(\Delta(S(G)(U)\sqcap H) + \delta(S(G)(U)\sqcap H))[(|V(G)| \\
&\quad +|E(G)|)(2|V(H)|^2|E(G)| + |V(G)||PI_v(H)|)] \\
&\quad -[4(|V(G)| + |E(G)|)|E(G)|PI_v(H) + (|V(G)| \\
&\quad +|E(G)|)|V(G)|PI_w(H) + |V(H)|^2(|V(G)| \\
&\quad +|E(G)|)(M_1(G) \\
&\quad +4|E(G)|) + 4|E(H)||V(H)||E(G)|(|V(G)| + |E(G)|)]
\end{aligned}$$

Clearly, we observe that $S(C_n) \cong C_{2n}$ and $S(P_n) \cong P_{2n-1}$ where C_n and P_n denote the cycle and path on $n-$ vertices. By using Theorem 3.1, we obtain the exact PI_r of the graphs $S(C_r)(U) \sqcap P_s$ and $S(P_r)(U) \sqcap P_s$.

Corollary 3.1. *Let $r \geq 3$ and $s \geq 2$ be two integers. Then for $U = V(C_r)$, $PI_r(S(C_r)(U) \sqcap P_S) = 20r^2s^2 + 8r^2s$.*

Corollary 3.2. *Let $r \geq 2$ and $s \geq 2$ be two integers. Then for $U = V(P_r)$, $PI_r(S(P_r)(U) \sqcap P_S) = 20r^2s^2 - 14rs^2 + 8r^2s - 36rs + 2s^2 + 16s$.*

The oldest one of the topological indices is the Weiner index. Weiner and hyper Wiener indices, vertex, edge and Weighted PI indices were obtained in [1, 5, 18, 19].

Example 3.1. *If G is the zig-zag polyhex nanotube $TUHC_6[2n, 2]$ then $PI_r(G) = 96n^2$.*

The zig-zag polyhex nanotube is the graph $S(C_n)(U) \sqcap P_2$ where $U = V(C_n) \subseteq V(S(C_n))$. Since by Corollary 3.1 we obtain the formula as above in Example 3.1.

Example 3.2. *If L_n is the hexagonal chain then $PI_r(L_n) = 96n^2 + 64n + 8$.*

The hexagonal chain L_n is the graph $S(P_{n+1})(U)\sqcap P_2$, where $U = V(P_{n+1}) \subseteq V(S(P_{n+1}))$. Since by Corollary 3.2 we obtain the formula as above in Example 3.2.

4. Revan Weighted PI Indices of Join of Graphs

In this section , we compute the Revan Weighted PI indices of join of two graphs. The join graph $G+H$ of graphs G and H is obtained from the disjoint union of the graphs G and H, where each vertex of G is adjacent to each vertex of H.

For an edge $e = uv$ of a graph G, let $S_G(e)$ be the set of common neighbours of u and v and let $|S_G(e)| = s_G(e)$. For our convenience, we partition the edge set of $G+H$ into three sets, $E_1 = \{e \in E(G+H) \mid e \in E(G)\}$, $E_2 = \{e \in E(G+H) \mid e \in E(H)\}$ and $E_3 = \{e \in E(G+H) \mid e = uv, u \in V(G), v \in V(H)\}$. Also we have $|E_1| = |E(G)|$, $|E_2| = |E(H)|$ and $|E_3| = |V(G)||V(H)|$.

Theorem 4.1. *Let G and H be two graphs with n, m vertices and p, q edges respectively. Then*

$$\begin{aligned} PI_r(G+H) &= 2(\Delta_{G+H} + \delta_{G+H} - m)M_1(G) \\ &\quad -4 \sum_{uv=e\in E(G)} (\Delta_{G+H} + \delta_{G+H} - m)s_G(e) \\ &\quad -Z_2(G) - 2M_2(G) + 2M_1(G)s_G(e) \\ &\quad +2(\Delta_{G+H} + \delta_{G+H} - n)M_1(H) \\ &\quad -4 \sum_{xy=e\in E(H)} (\Delta_{G+H} + \delta_{G+H} - n)s_H(e) \\ &\quad -Z_2(H) - 2M_2(H) + 2M_1(H)s_H(e) \\ &\quad +mn(m+n)[2(\Delta_{G+H} + \delta_{G+H}) - (m+n)] \\ &\quad -2(\Delta_{G+H} + \delta_{G+H})2(pm+qn) + nM_1(H) + mM_1(G) + 8pq. \end{aligned}$$

Proof:

Since $deg_{G+H}(x) = \begin{cases} d_G(x) + m, & if\ x \in V(G) \\ d_H(x) + n, & if\ x \in V(H) \end{cases}$ and the join of two graphs has diameter atmost two, that is,

$$d_{G+H}(u,v) = \begin{cases} 1, & if\ uv \in E(G)\ or\ uv \in E(H)\ or\ \ (u \in V(G)\ and\ v \in V(H)) \\ 2, & otherwise \end{cases}$$

For an edge $e = uv \in E(G) \subset E(G+H)$, by the definiton of join we have, $n_u^{G+H}(e) = d_G(u) - S_G(e)$ and $n_v^{G+H}(e) = d_G(v) - S_G(e)$. Similarly we have $n_x^{G+H}(e) = d_H(x) - S_H(e)$ and $n_y^{G+H}(e) = d_H(y) - S_H(e)$. For an edge $e = xy \in E(H) \subset E(G+H)$. Let $e = ux \in E(G+H)$,where $u \in V(G)$ and $x \in V(H)$. Then $n_u^{G+H}(e) = m - d_H(x)$ and

$n_x^{G+H}(e) = n - d_G(u)$. Hence

$$
\begin{aligned}
PI_r(G+H) = & \sum_{uv=e\in E(G)} (\Delta_{G+H} + \delta_{G+H} - d_{G+H}(u) \\
& + \Delta_{G+H} + \delta_{G+H} - d_{G+H}(v))(d_G(u) - s_G(e) + d_G(v) - s_G(e)) \\
& + \sum_{xy=e\in E(H)} (\Delta_{G+H} + \delta_{G+H} - d_{G+H}(u) \\
& + \Delta_{G+H} + \delta_{G+H} - d_{G+H}(v))(d_H(x) - s_G(e) + d_H(y) - s_G(e)) \\
& + \sum_{u\in V(G)} \sum_{x\in V(H)} (\Delta_{G+H} + \delta_{G+H} - d_{G+H}(u) + \Delta_{G+H} + \delta_{G+H} \\
& - d_{G+H}(v))(m - d_H(x) + n - d_G(u)). \\
& = \sum_{e\in E(G)} [2(\Delta_{G+H} + \delta_{G+H} - m)(d_G(u) + d_G(v) - 2s_G(e)) \\
& - ((d_G(u) + d_G(v))(d_G(u) + d_G(v) - 2s_G(e)))] \\
& + \sum_{xy=e\in E(H)} [2(\Delta_{G+H} + \delta_{G+H} - n)(d_H(x) + d_H(y) - 2s_H(e)) \\
& - ((d_H(x) + d_H(y))(d_H(x) + d_H(y) - 2s_H(e)))] \\
& + \sum_{u\in V(G)} \sum_{x\in V(H)} [2(\Delta_{G+H} + \delta_{G+H})(m + n - (d_G(u) + d_H(x))) \\
& - ((d_G(u) + d_H(x) + m + n)(m + n - d_G(u) + d_H(x)))] \\
& = 2(\Delta_{G+H} + \delta_{G+H} - m) \sum_{e\in E(G)} (d_G(u) + d_G(v)) \\
& - 4 \sum_{e\in E(G)} (\Delta_{G+H} + \delta_{G+H} - m)s_G(e) - \sum_{e\in E(G)} (d_G(u) + d_G(v))^2 \\
& + 2 \sum_{e\in E(G)} (d_G(u) + d_G(v))s_G(e) \\
& + 2(\Delta_{G+H} + \delta_{G+H} - n) \sum_{e\in E(H)} (d_H(x) + d_H(y)) \\
& - 4 \sum_{e\in E(H)} (\Delta_{G+H} + \delta_{G+H} - n)s_H(e) \\
& - \sum_{e\in E(H)} (d_H(x) + d_H(y))^2 + 2 \sum_{e\in E(H)} (d_H(x) + d_H(y))s_H(e)
\end{aligned}
$$

$$+2 \sum_{u\in V(G)} \sum_{x\in V(H)} (\Delta_{G+H} + \delta_{G+H})(m+n)$$
$$-2 \sum_{u\in V(G)} \sum_{x\in V(H)} (\Delta_{G+H} + \delta_{G+H})(d_G(u) + d_H(x))$$
$$-\Big[\sum_{u\in V(G)} \sum_{x\in V(H)} (d_G(u) + d_H(x))(m+n)$$
$$- \sum_{u\in V(G)} \sum_{x\in V(H)} (d_G(u) + d_H(x))^2 + \sum_{u\in V(G)} \sum_{x\in V(H)} (m+n)^2$$
$$- \sum_{u\in V(G)} \sum_{x\in V(H)} (m+n)(d_G(u) + d_H(x))\Big]$$

$$\begin{aligned} PI_r(G+H) =& 2(\Delta_{G+H} + \delta_{G+H} - m)(M_1(G)) \\ &- 4 \sum_{uv=e\in E(G)} (\Delta_{G+H} + \delta_{G+H} - m)s_G(e) - Z_2(G) - 2M_2(G) \\ &+ 2M_1(G)s_G(e) + 2(\Delta_{G+H} + \delta_{G+H} - n)M_1(H) \\ &- 4 \sum_{e\in E(H)} (\Delta_{G+H} + \delta_{G+H} - n)s_H(e) \\ &- Z_2(H) - 2M_2(H) + 2M_1(H)s_H(e) \\ &+ mn(m+n)[2(\Delta_{G+H} + \delta_{G+H}) - (m+n)] \\ &- 2(\Delta_{G+H} + \delta_{G+H})2(pm+qn) + nM_1(H) + mM_1(G) + 8pq. \end{aligned}$$

Using the above Theorem we deduce the following corollary and example.

Corollary 4.1. *For any two triangle-free graphs G and H, we have $s_G(e) = 0$ and for an edge uv in G. Then*

$$\begin{aligned} PI_r(G+H) =& 2(\Delta_{G+H} + \delta_{G+H} - m)M_1(G) - Z_2(G) - 2M_2(G) + 2(\Delta_{G+H} + \delta_{G+H} \\ &- n)M_1(H) - Z_2(H) - 2M_2(H) + nm(n+m)[2(\Delta_{G+H} + \delta_{G+H}) - (m+n)] \\ &- 2(\Delta_{G+H} + \delta_{G+H})2(pm+qn) + nM_1(H) + mM_1(G) + 8pq \end{aligned}$$

Example 4.1. $PI_r(K_1 + P_m) =$

$$\begin{cases} m^3 + 2m^2 + 13m - 18 & if m = 2 \\ (m-2)(m^2-1) + 2m(m+2) & \\ +6(2m-1) + 8(m-3)(m-1) & if m > 2 \end{cases}$$

In the above example if P_m is a path graph on 4 vertices then we get Gem graph, it is the Fan graph $F_{4,1}$ and $PI_r(K_1 + P_4) = 144$.

5. Conclusion

In this paper we focus on the Revan Weighted PI index on various graph operation. Further this paper can be extended to some other molecular graphs, infinite graphs and directed or digraphs.

References

1. Ashrafi, A. R. and Loghman, A., (2006), PI index of zig-zag polyhex nanotubes. MATCH Commun. Math. Comput. Chem, 55, 447-452.
2. Barriere, L., Comellas, F., Dalfo, C. and Fiol, M. A., (2009), The hierarchical product of graphs, Discrete Applied Mathematics, 157(1), 36 – 48.
3. Barriere, L., Dalfo, C., Fiol M. A. and Mitjana, M., (2009), The generalized hierarchical product of graphs, Discrete Mathematics, 309(12), 3871 – 3881.
4. Devillers, J. and Balaban, A.T, Eds., (1999), Topological Indices and Related Descriptors in QSAR and QSPR, Gordan and Breach, Amsterdam, The Netherlands.
5. Eliasi, M. and Iranmanesh, A., (2011), The Hyper-Wiener Index of the Generalized Hierarchical Product of Graphs, Discrete Applied Mathematics, 159(8), 866 – 871.
6. Frucht, R. and Harary, F., (1970), On the corona of two graphs, Aequationes Mathematicae, 4, 322 – 325.
7. Gutman, I., (2003), Introduction to Chemical Graph Theory, Faculty of Science, Kragujevac,(in Serbian).
8. Ilić, A. and Milosavljenić, N., (2013), The Weighted Vertex PI Index, Math. Comput. Model., 57, 623 – 631.
9. Kandan, P., Chandrasekaran, E. and Priyadharshini, M., (2018), The Revan Weighted Szeged Index of Graphs, Journal of Emerging Technologies and Innovative Research, 5(9), 358–366.
10. Kandan, P., Chandrasekaran, E. and Priyadharshini, M., (2019), Revan Weighted PI Index of Corona Product of Graphs, International Journal of Research and Analytical Reviews, 6(2), 792–799.
11. Khadikar, P. V., Karmakar, S. and Agarwal, V. K., (2001), A Novel PI Index and its Application to QSPR/QSAR Studies, J. Chem. Inf. Comput. Sci., 41, 934 – 949.
12. Khalifeh, M. H., Yousefi-Azari, H. and Ashrafi, A. R., (2008), Vertex and Edge PI Indices of Cartesian Product Graphs, Discrete Applied Mathematics, 156(10), 1780 – 1789.
13. Kulli, V. R., (2017), Revan Indices of Oxide and Honeycomb Network,International Journal of Mathematics and its Applications, 5(4 – E), 663–667.
14. Kulli, V. R., (2018), F-Revan Index and F-Revan Polynomial of Some Families of Benzenoid Systems, Journal of Global Research in Mathematical Archives, 5(11), 01 – 06.
15. Kulli, V. R., (2017), On the product connectivity Revan Index of Certain Nanotubes, Journal of Computer and Mathematical Sciences, 8(10), 562 – 567.
16. Pattabiraman, K. and Kandan, P., (2014), Weighted PI index of Corona Product of Graphs, Discrete Mathematics, Algorithms and Applications, 6(4), 1450055-1 – 1450055-9.
17. Pattabiraman, K. and Kandan, P., (2016), Weighted Szeged Indices of Some Graph Operations, Transactions on Combinatorics, 5(1), 25-35.
18. Pattabiraman, K. and Kandan, P., (2016), On Weighted PI Index of Graphs, Electronic Notes in Discrete Mathematics, 53, 225 – 238.
19. Pattabiraman, K. and Paulraja, P., (2012), Vertex and Edge Padmakar – Ivan Indices of the Generalised Hierearchical Product of Graphs, Discrete Applied Mathematics, 160, 1376–1384.
20. Wiener, H., (1947), Structural Determination of the Paraffin Boiling Points, J. Am. Chem. Soc., 69, 17–20.

M. Priyadharshini is a research scholar, PG and Research Department of Mathematics, Presidency College, University of Madras, Chennai, India. She received her M.Sc and M.Phil from University of Madras and Thiruvalluvar University respectively.

Dr. P.Kandan is working as an assistant professor in the Department of Mathematics, Faculty of Science, Annamalai University, India. He has recieved M.Sc. an M.Phil. degree from University of Madras,and received his Ph.D. in Graph Theory from Annamalai University. He has published nearly 19 research papers in Graph Theory and Applied Mathematics. His area of interests focus mainly in graph theory, especially Decomposition of graphs, Topological Indexes of graphs and Applied Mathematics. He has delivered special lectures in various conferences.

Dr. E. Chandrasekaran is currently working as professor of Mathematics in Vel-Tech Rangarajan Dr Sagunthala R and D Institute of Science and Technology, Chennai,India. He published more than 30 Research article. His area of interest include Operation Research, Graph Theory and Statistics.

Dr. A. Joseph Kennedy is currently working as a professor of Mathematics in Pondicherry University, India. His area of interest include Algebra and Graph Theory.