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SĂLĂGEAN-TYPE ANALYTIC FUNCTIONS ASSOCIATED WITH THE JANOWSKI FUNCTIONS AND $q-$DIFFERENCE OPERATOR

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Abstract. We introduce three new subclasses of Sălăgean-type analytic functions by using Janowski functions and $q-$difference operator. We investigate inclusion theorem, sufficient coefficient estimates and distortion bounds for the functions belonging to these subclasses. Moreover, partial sums of these subclasses were obtained.

Keywords: $q-$calculus, $q-$difference operator, Sălăgean differential operator, Janowski function.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. A function f is said to be subordinate to a function g, written as $f \prec g$ in $\mathbb{D}$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), z \in \mathbb{D}$. We denote by $\mathcal{S}$, the class of univalent functions in $\mathbb{D}$. Denote by $\mathcal{S}^*$, the subclass of functions f in $\mathcal{S}$ are starlike if satisfy the following condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ (z \in \mathbb{D}).$$

Also, let $\mathcal{P}$ be the class of Carathéodory functions $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, $z \in \mathbb{D}$ such that $Re(p(z)) > 0$. For the theory of analytic univalent functions one may refer to [3].

Quantum calculus or $q-$calculus is the traditional calculus without the use of limits. Quantum calculus dates back to Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). But $q-$calculus became popular after Jackson published his papers on $q-$derivative and $q-$integral opeartors (see [6, 7, 8]). The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas

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of ordinary fractional calculus, orthogonal polynomials, basic hypergeometric functions, combinatorics.

The operator

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0 \end{cases}$$

is called $q-$derivative (or $q-$difference operator) of a function f. For a function f of the form (1), we observe that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}, q \in (0,1)$. Clearly, for $q \to 1^-$, $[n]_q \to n$. For definitions and properties of $q-$calculus, one may refer to [2, 6, 7].

In 1990, Ismail *et al.* [5] used quantum calculus in the theory of analytic univalent functions and defined the following class:

Definition 1.1. *A function $f \in \mathcal{A}$ defined by (1) is said to belong to the class $P\mathcal{S}_q$ if*

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

for all $z \in \mathbb{D}$. As $q \to 1^-$, $P\mathcal{S}_q$ reduces to the class $\mathcal{S}^*$ of starlike functions.

The $q-$analogue of Sălăgean differential operator $R_q^m f(z) : \mathcal{A} \to \mathcal{A}$ is formed by (see [4]);

$$\begin{aligned} R_q^0 f(z) &= f(z) \\ R_q^1 f(z) &= z D_q(f(z)) \\ &\vdots \\ R_q^m f(z) &= z D_q^1(R_q^{m-1} f(z)). \end{aligned}$$

From definition $R_q^m f(z)$, we obtain

$$R_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n, \tag{2}$$

where $[n]_q^m = (\frac{1-q^n}{1-q})^m$, $q \in (0,1)$, $m \in \mathbb{N}$. Clearly, as $q \to 1^-$, the equation (2) reduces to Sălăgean differential operator (see [10]).

Definition 1.2. [9] *A given function f with $f(0) = 1$ is said to belong to the class $\mathcal{P}[A,B]$ if and only if*

$$f(z) \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1; z \in \mathbb{D})$$

where $\prec$ denotes subordination symbol. The analytic function class $\mathcal{P}[A,B]$ was introduced by Janowski [9]*, who showed that $f \in \mathcal{P}[A,B]$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$f(z) = \frac{(A+1)p(z)-(A-1)}{(B+1)p(z)-(B-1)}, \quad (-1 \leq B < A \leq 1; z \in \mathbb{D}).$$

Making use of Janowski functions and $q-$difference operator, we introduce three new subclasses of Sălăgean-type analytic functions.

Definition 1.3. *For $q \in (0,1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$ and $z \in \mathbb{D}$, a function f of the form (1) is said to belong to the class $\mathcal{S}_{(q,1)}(m,A,B)$ if*

$$Re\left(\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A+1)}\right) \geq 0, \tag{3}$$

where $R_q^m f$ is defined by (2). This class is called $q-$Sălăgean-type analytic functions type-1 associated with the Janowski functions.

Definition 1.4. *For $q \in (0,1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$ and $z \in \mathbb{D}$, a function f of the form (1) is said to belong to the class $\mathcal{S}_{(q,2)}(m,A,B)$ if*

$$\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q}, \tag{4}$$

where $R_q^m f$ is defined by (2). This class is called $q-$Sălăgean-type analytic functions type-2 associated with the Janowski functions.

Definition 1.5. *For $q \in (0,1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$ and $z \in \mathbb{D}$, a function f of the form (1) is said to belong to the class $\mathcal{S}_{(q,2)}(m,A,B)$ if*

$$\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)} - (A+1)} - 1\right| < 1, \tag{5}$$

where $R_q^m f$ is defined by (2). This class is called $q-$Sălăgean-type analytic functions type-3 associated with the Janowski functions.

For special values of $q \in (0,1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$, we get the following known subclasses:

1) If we put $m = 0$ in Definition 1.3, 1.4 and 1.5, respectively, we get the classes $\mathcal{S}^*_{(q,1)}(A,B)$, $\mathcal{S}^*_{(q,2)}(A,B)$ and $\mathcal{S}^*_{(q,3)}(A,B)$, which was introduced and studied by Srivastava *et al.* (see [11]).

2) If we put $A = 1-2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$ and $m = 0$ in Definition 1.3, 1.4 and 1.5, respectively, we get the classes $\mathcal{S}^*_{q,1}(\alpha)$, $\mathcal{S}^*_{q,2}(\alpha)$ and $\mathcal{S}^*_{q,3}(\alpha)$, which was introduced and studied by Wongsaijai and Sukantamala (see [12]).

3) If we put $A = 1-2\alpha$ $(0 \leq \alpha < 1)$, $B = -1$ and $m = 0$ in Definition 1.4, we get the class $\mathcal{S}^*_q(\alpha)$ which was introduced and studied by Agrawal and Sahoo (see [1]).

4) If we put $A = 1$, $B = -1$ and $m = 0$ in Definition 1.4, we get the class $\mathcal{PS}_q$ which was introduced and studied by Ismail *et al.* (see [5]).

2. Main Results

We first show the inclusion theorem of each classes of $q-$Sălăgean-type analytic functions associated with the Janowski functions.

Theorem 2.1. *If $q \in (0,1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$, then*

$$\mathcal{S}_{(q,3)}(m,A,B) \subset \mathcal{S}_{(q,2)}(m,A,B) \subset \mathcal{S}_{(q,1)}(m,A,B). \tag{6}$$

Proof. Assume that $f \in \mathcal{S}_{(q,3)}(m,A,B)$, then by Definition 1.5 we have

$$\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}-1\right|<1,$$

so that

$$\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}-1\right|+\frac{q}{1-q}<1+\frac{q}{1-q}.$$

By using triangle inequality in the above expression, we obtain

$$\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}. \tag{7}$$

Inequality in (7) shows that $f \in \mathcal{S}_{(q,2)}(m,A,B)$, and we conclude that

$$\mathcal{S}_{(q,3)}(m,A,B) \subset \mathcal{S}_{(q,2)}(m,A,B).$$

Next, we let $f \in \mathcal{S}_{(q,2)}(m,A,B)$, then by Definition 1.4 we have

$$f \in \mathcal{S}_{(q,2)}(m,A,B) \Leftrightarrow \left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

Since

$$\frac{1}{1-q}>\left|\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}-\frac{1}{1-q}\right|$$
$$=\left|\frac{1}{1-q}-\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}\right|,$$

then we get

$$Re\left(\frac{(B-1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A-1)}{(B+1)\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-(A+1)}\right)>0.$$

This last expression shows that $f \in \mathcal{S}_{(q,1)}(m,A,B)$, and we conclude that

$$\mathcal{S}_{(q,2)}(m,A,B) \subset \mathcal{S}_{(q,1)}(m,A,B).$$

This completes the proof. □

Next, we give a sufficient condition of $\mathcal{S}_{(q,3)}(m,A,B)$ via coefficient inequality which guarantees a sufficient condition for $\mathcal{S}_{(q,1)}(m,A,B)$ and $\mathcal{S}_{(q,2)}(m,A,B)$.

Theorem 2.2. *For $q \in (0,1), -1 \le B < A \le 1$, $m \in \mathbb{N}$ and $z \in \mathbb{D}$, if a function f of the form (1) satisfies the inequality*

$$\sum_{n=2}^{\infty}[n]_q^m\{2([n]_q-1)+[n]_q(B+1)+(A+1)\}|a_n|<A-B, \tag{8}$$

then f is belong to the class $\mathcal{S}_{(q,3)}(m,A,B)$.

Proof. Assume that inequality (8) holds. Using Definition 1.5, we obtain

$$\left|\frac{2(R_q^m f(z) - R_q^{m+1} f(z))}{(B+1)R_q^{m+1} f(z) - (A+1)R_q^m f(z)}\right| < 1,$$

and in view of (2), we get

$$\left|\frac{\sum\limits_{n=2}^{\infty} 2[n]_q^m([n]_q - 1)a_n z^n}{(A-B)z - \sum\limits_{n=2}^{\infty}[n]_q^{m+1}(B+1) + \sum\limits_{n=2}^{\infty}[n]_q^m(A+1))a_n z^n}\right| < 1,$$

which gives

$$\frac{\sum\limits_{n=2}^{\infty} 2[n]_q^m([n]_q - 1)|a_n|}{A - B - \sum\limits_{n=2}^{\infty}[n]_q^m([n]_q(B+1) + (A+1))|a_n|} < 1.$$

This last inequality gives (8), it follows that, $f \in \mathcal{S}_{(q,3)}(m, A, B)$. □

We now introduce additional new subclasses of $q-$Sălăgean-type analytic functions associated with the Janowski functions by using negative coefficients. Let $\mathcal{T}$ be a subset of $\mathcal{A}$ containing negative coefficient functions; that is,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{9}$$

We also let

$$\mathcal{TS}_{(q,i)}(m, A, B) := \mathcal{T} \cap \mathcal{S}_{(q,i)}(m, A, B), \quad i = \{1, 2, 3\}.$$

In view of negative coefficients given by (9), we get

$$R_q^m f(z) = z - \sum_{n=2}^{\infty} [n]_q^m |a_n| z^n. \tag{10}$$

Theorem 2.3. *If* $q \in (0, 1), -1 \leq B < A \leq 1$, $m \in \mathbb{N}$, *then*

$$\mathcal{TS}^*_{(q,1)}(m, A, B) \equiv \mathcal{TS}^*_{(q,2)}(m, A, B) \equiv \mathcal{TS}^*_{(q,3)}(m, A, B). \tag{11}$$

Proof. By using Theorem 2.1, it is sufficient to show that $\mathcal{TS}^*_{(q,1)}(m, A, B) \subset \mathcal{TS}^*_{(q,3)}(m, A, B)$. Assuming that $f \in \mathcal{TS}^*_{(q,1)}(m, A, B)$, we have

$$Re\left(\frac{(B-1)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - (A-1)}{(B+1)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - (A+1)}\right) \geq 0.$$

Thus we can obtain

$$Re\left(\frac{(B-1)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - (A-1)}{(B+1)\frac{R_q^{m+1} f(z)}{R_q^m f(z)} - (A+1)} - 1\right) \geq -1.$$

Routine calculations shows that

$$\frac{2(R_q^m f(z) - R_q^{m+1} f(z))}{(B+1)R_q^{m+1} f(z) - (A+1)R_q^m f(z)} \geq -1.$$

By using negative coefficients given by (10), we obtain

$$\frac{\sum_{n=2}^{\infty} 2[n]_q^m([n]_q-1)|a_n|}{A-B-\sum_{n=2}^{\infty}[n]_q^m([n]_q(B+1)+(A+1))|a_n|}<1,$$

which satisfies (8). Theorem 2.2 implies the proof of this theorem. □

By using the result of Theorem 2.3, all types of $q-$Sălăgean-type analytic functions associated with the Janowski functions are the same. For convenience, we state the following distortion theorem by using the notation $\mathcal{TS}^*_{(q,i)}(m,A,B), i=\{1,2,3\}$.

Theorem 2.4. *If $f\in\mathcal{TS}^*_{(q,i)}(m,A,B), i=\{1,2,3\}$, then for $|z|=r<1$ we have*

$$r-\frac{A-B}{\Theta_q(2,m,A,B)}r^2\le|f(z)|\le r+\frac{A-B}{\Theta_q(2,m,A,B)}r^2, \tag{12}$$

where $\Theta_q(2,m,A,B)=[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)$.

Proof. Since $f\in\mathcal{TS}^*_{(q,i)}(m,A,B)$, in view of Theorem 2.3 we obtain

$$\begin{aligned}&[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)|a_n|\sum_{n=2}^{\infty}\\&\le\sum_{n=2}^{\infty}[n]_q^m\big(2([n]_q-1)+[n]_q(B+1)+(A+1)\big)|a_n|\\&<A-B,\end{aligned}$$

which gives

$$\sum_{n=2}^{\infty}|a_n|\le\frac{A-B}{[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)}.$$

Therefore, we easily get

$$\begin{aligned}|f(z)|&\le|z|+\sum_{n=2}^{\infty}|a_n||z|^n\\&\le r+\frac{A-B}{[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)}r^2.\end{aligned}$$

Similarly, we also get

$$\begin{aligned}|f(z)|&\ge|z|-\sum_{n=2}^{\infty}|a_n||z|^n\\&\ge r-\frac{A-B}{[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)}r^2.\end{aligned}$$

This completes the proof. □

Theorem 2.5. *If $f\in\mathcal{TS}^*_{(q,i)}(m,A,B), i=\{1,2,3\}$, then for $|z|=r<1$ we have*

$$1-\frac{2(A-B)}{\Theta_q(2,m,A,B)}r\le|f'(z)|\le1+\frac{2(A-B)}{\Theta_q(2,m,A,B)}r, \tag{13}$$

where $\Theta_q(2,m,A,B)=[2]_q^m\big(2([2]_q-1)+[2]_q(B+1)+(A+1)\big)$.

Proof. From the following expressions given by

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1},$$

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1},$$

and

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{2(A-B)}{[2]_q^m \big(2([2]_q - 1) + [2]_q(B+1) + (A+1)\big)}.$$

we get (13). □

3. Partial Sums

In this section, we examine the ratio of a function of the form (1) to its sequence of partial sums

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (z \in \mathbb{D})$$

when the coefficients of f are sufficiently small to satisfy condition (8). We will determine partial sums of the functions in the classes $\mathcal{S}^*_{(q,i)}(m, A, B), i = 1, 2, 3,$ and obtain sharp lower bounds for the ratios of $f(z)$ to $f_k(z)$.

Theorem 3.1. *If f of the form (1) satisfies the condition (8), then*

$$i) \quad Re\Big(\frac{f(z)}{f_k(z)}\Big) \geq 1 - \frac{1}{\lambda_{k+1}} \tag{14}$$

and

$$ii) \quad Re\Big(\frac{f_k(z)}{f(z)}\Big) \geq \frac{\lambda_{k+1}}{1 + \lambda_{k+1}} \tag{15}$$

where $\lambda_k = \frac{[n]_q^m \big(2([n]_q - 1) + [n]_q(B+1) + (A+1)\big)}{A-B}$.

Proof. i) In order to prove (14), we may write

$$\psi_1(z) = \lambda_{k+1}\bigg(\frac{f(z)}{f_k(z)} - \big(1 - \frac{1}{\lambda_{k+1}}\big)\bigg)$$
$$= 1 + \frac{\lambda_{k+1}\sum_{n=k+1}^{\infty} a_n z^n}{z + \sum_{n=2}^{k} a_n z^n}.$$

It is suffcient to show that $Re\psi_1(z) > 0$, or equivalently

$$\left|\frac{\psi_1(z) - 1}{\psi_1(z) + 1}\right| \leq 1,$$

then we get

$$\left|\frac{\psi_1(z) - 1}{\psi_1(z) + 1}\right| \leq \frac{\lambda_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{k} |a_n| - \lambda_{k+1} \sum\limits_{n=k+1}^{\infty} |a_n|} \leq 1$$

if and only if

$$2\lambda_{k+1}\sum_{n=k+1}^{\infty}|a_n| \leq 2-2\sum_{n=2}^{k}|a_n|,$$

which implies that

$$\sum_{n=2}^{k}|a_n| + \lambda_{k+1}\sum_{n=k+1}^{\infty}|a_n| \leq 1. \tag{16}$$

Finally, to prove the inequality in (14), it is suffices to show that the left-hand side of (16) is bounded above by $\sum\limits_{n=2}^{\infty}\lambda_n|a_n|$, which is equivalent to

$$\sum_{n=2}^{k}(1-\lambda_n)|a_n| + \sum_{n=k+1}^{\infty}(\lambda_{k+1}-\lambda_n)|a_n| \geq 0. \tag{17}$$

By taking into account (17), we get (14).

ii) Next, in order to prove (15), we may write

$$\begin{aligned}\psi_2(z) &= (1+\lambda_{k+1})\left(\frac{f_k(z)}{f(z)} - \big(1-\frac{1}{1+\lambda_{k+1}}\big)\right)\\ &= 1-\frac{(1+\lambda_{k+1})\sum_{n=k+1}^{\infty}a_n z^n}{z+\sum_{n=2}^{\infty}a_n z^n}.\end{aligned}$$

Then, we obtain

$$\left|\frac{\psi_2(z)-1}{\psi_2(z)+1}\right| \leq \frac{(1+\lambda_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n| - (\lambda_{k+1}-1)\sum\limits_{n=k+1}^{\infty}|a_n|} \leq 1$$

if and only if

$$\sum_{n=2}^{k}|a_n| + \lambda_{k+1}\sum_{n=k+1}^{\infty}|a_n| \leq 1. \tag{18}$$

Finally, we can state that the left-hand side of (18) is bounded above by $\sum\limits_{n=2}^{\infty}\lambda_n|a_n|$, and therefore we obtain (15). This completes the proof. □

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