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OPTIMALITY CONDITIONS FOR APPROXIMATE SOLUTIONS OF SET-VALUED OPTIMIZATION PROBLEMS IN REAL LINEAR SPACES

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Abstract. In this paper, we deal with optimization problems without assuming any topology. We study approximate efficiency and Q- Henig proper efficiency for the set-valued vector optimization problems, where Q is not necessarily convex. We use scalarization approaches based on nonconvex separation function to present some necessary and sufficient conditions for approximate (proper and weak) efficient solutions.

Keywords: Vector optimization, Set-valued maps, Approximate weak efficiency, Approximate proper efficiency, Vector closure.

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1. Introduction

Recently some researchers have turned their study to vector optimization of set-valued maps [7, 8, 9, 29]. This kind of optimization problems have many applications in stochastic programming, control theory and economic theory. Various concepts of proper efficiency have been introduced [4, 5, 10, 16, 19, 27, 30], and many researches efforts focus on concepts of approximate solutions or ϵ- efficient solutions [12, 15, 20, 21, 28].
The concept of ϵ-efficiency is practically useful regarding decision-making problems. The approximate solutions may be produced in applied optimization and numerical algorithms. Some of the applications of approximate efficiency in radiotherapy treatment planning have been addressed by Shao and Ehrgott [28].
There are many publications on optimization problems refrain from using topological concepts. Adan and Novo [1, 2, 3] defined concepts of proper efficiency in the sense of Hurwicz

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and Benson, and utilized some algebraic concepts such as algebraic interior, and vector closure instead of topological interior, and closure to obtain optimality conditions. The relation between approximate solutions and weakly efficient solutions of vector optimization problems with set-valued maps was discussed in [25]. Optimality conditions for ϵ-global properly efficient elements of set-valued optimization problems were established under the assumption of generalized cone subconvexlikeness in [31]. These set-valued problems are based on the algebraic interior in real linear spaces.
However, all these papers focus on convex problems, as far as we know, nonconvex vector optimization has not reached attention of reserchers.
The nonconvex separation function is studied in a real linear space not necessarily endowed with a topology in [13, 18, 24]. Specifically, the main properties of this function are extended to the linear setting. By using the nonconvex separation function, authors characterized the weak efficient solutions of vector equilibrium problems defined through algebraic solid ordering sets in [13].
In this paper, using algebraic notions, we study the concept of approximate proper efficiency. Moreover, by means of Gerstewitz's function generated by a convex cone, we provide the necessary and sufficient optimality conditions for K-weak efficient solutions and K-Henig proper efficient points. Furthermore, we define Q-Henig proper efficiency for the set-valued vector optimization problems, where Q is not necessarily convex. Moreover, nonconvex separation function generated by a nonconvex set is used to obtain optimality conditions for Q-efficient solutions.
The results of this paper extend those given in the literatures (e.g., [13, 18, 24]) and provide some new applications for the notions introduced in [1, 2, 3].
The outline of this paper is as follows: main definitions and notations are given in Section 2. In Section 3, we will define the concept of approximate (weak and proper) efficient solutions. Scalarization tool, including Gerstewitz's function are utilized to provide some necessary and sufficient conditions are given in Section 4. Finally, in Section 5, we use nonconvex separation function to obtain optimality conditions for Q-efficient solutions, where Q is not a convex set.

2. Preliminaries

Throughout this paper X,Y , and Z are three (real) linear spaces; A is a subset of X, and $K \subseteq Y$ is a pointed convex proper cone which introduces a partial order on Y by the equivalence $y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in K$. K is called pointed if $K \cap (-K) = \{0\}$ and the cone generated by A is denoted by cone(A). Moreover, a nonempty set $F \subset Y$ is said to be free disposal with respect to a convex cone $K \subseteq Y$ if $F + K = F$. The algebraic interior of A and the vectorial closure of A are denoted by $cor(A)$ and $vcl(A)$, respectively and these are defined as follows:

$$cor(A) = \{x \in A : \ \forall x' \in X \ , \ \exists \lambda' > 0, \ s.t. \ x + \lambda x' \in A, \ \forall \lambda \in [0, \lambda']\},$$
$$vcl(A) = \{b \in X : \exists x \in X \ ; \ \forall \lambda' > 0, \ \exists \lambda \in [0, \lambda'], \quad s.t. \quad b + \lambda x \in A\}.$$

We say that A is solid if $cor(A) \neq \emptyset$, and A is deemed vectorially closed if $A = vcl(A)$. It is known that, if $cor(K) \neq \emptyset$, then $cor(K) \cup \{0\}$ is a convex cone, in addition $cor(K) + K = cor(K)$ and $cor(cor(K)) = cor(K)$ for solid nontrivial convex cone K. Moreover, $cor(K \times M) = cor(K) \times cor(M)$ and $cor(K - M) = cor(K) - cor(M)$ for solid nontrivial convex cones K and M[2]. For each $q \in Y$, q-vector closure of A is denoted by $vcl_q(A)$ and defined as follows:

$$vcl_q(A) = \{x \in X : \ \forall \lambda' > 0, \quad \exists \lambda \in [0, \lambda'] \ s.t. \ \ x + \lambda q \in A\}.$$

In fact it can be shown that

$$vcl_q(A) = \{x \in X : \ \exists \lambda_n \geq 0, \, \lambda_n \to 0 \ \ s.t. \ \ x + \lambda_n q \in A, \ \ \forall n \in N\}.$$

Obviously,

$$A \subseteq vcl_q(A) \subseteq \cup_{q \in Y} vcl_q(A) = vcl(A).$$

The algebraic dual of Y is denoted by $Y^{'}$, and also the positive dual and the strict positive dual of K are defined by

$$K^{+} = \{l \in Y^{'} : \langle l, a \rangle \geq 0, \forall a \in K\};$$
$$K^{+s} = \{l \in Y^{'} : \langle l, a \rangle > 0\}, \forall a \in K \backslash \{0\}\}.$$

The following propositions will be used in sequel. The proof of proposition 2.1 was given in [24, Proposition 2.5] and proposition 2.2 shows that $vcl_e(Q) + (0, \infty)e = cor(Q)$ for $e \in cor(K)$ and it was proved in [13, Proposition 18].

Proposition 2.1. [24] *Suppose $K \subset Y$ be a convex cone, $e \in cor(K)$ and $vint_e(K) = K + (0, \infty)e$. Then $vint_e(K) = cor(K)$.*

Proposition 2.2. [13] *Suppose Q is free disposal with respect to an algebraic solid convex cone K. Then*

$$vcl_e(Q) + cor(K) = vcl_e(Q) + (0, +\infty)e = cor(Q),$$

where $e \in cor(K)$.

3. Approximate proper efficiency

Let F be a set-valued map from X to a real space Y and G be a set-valued map from X to a linear space Z. We define $\langle F(x), y^* \rangle := \{\langle y, y^* \rangle ; \ y \in F(x)\}$ and $\langle F(A), y^* \rangle = U_{x \in A} \langle F(x), y^* \rangle$. Now, consider the following set-valued vector optimization problems

$$(UP) \ Min\{F(x) : x \in X\},$$
$$(CP) \ Min\{F(x) : x \in X, \ G(x) \cap (-M) \neq 0\},$$

and the following vector optimization problem:

$$(P) \ Min\{F(x) : \ x \in S\}.$$

where the feasible set S can be either

$$S = X \text{ or } \ S = \{x \in X, \ G(x) \cap (-M) \neq \emptyset\},$$

ans $M \subseteq Z$ is a pointed convex proper cone which introduces a partial order on Z by the equivalence $z_1 \leq z_2 \Leftrightarrow z_2 - z_1 \in M$.

In what follows, we discuss efficiency for set-valued optimization problems. Comparing the approximate weak efficiency defined here with the standard definition shows that here $int(K)$ has been replaced by $cor(K)$, because as no topology is used. Proper efficiency is one of the most important concepts in vector optimization. This concept has been studied by several authors [1, 2, 3, 18, 30, 31]. One can find the approximate proper efficient solutions for single-valued problems in [18].

Definition 3.1. *Let $\epsilon \in K \setminus \{0\}$, $x_0 \in S$ and $y_0 \in F(x_0)$. A solution (x_0, y_0) is called a K-efficient solution of (P), denoted by $(x_0, y_0) \in EP_\epsilon(F, S, K)$, if*

$$(F(x) - y_0 + \epsilon) \cap (-K \setminus \{0\}) = \emptyset, \quad x \in S.$$

Furthermore, if K is solid, (x_0, y_0) is called a K-weak efficient solution of (P), denoted by $(x_0, y_0) \in WEP_\epsilon(F, S, K)$, if

$$(F(x) - y_0 + \epsilon) \cap (-cor(K)) = \emptyset, \quad x \in S.$$

It is clear that $WEP_\epsilon(F, S, K) = EP_\epsilon(F, S, cor(K))$, where $cor(K) \neq \emptyset$.

Definition 3.2. *Consider $\epsilon \in K \setminus \{0\}$, $x_0 \in S$ and $y_0 \in F(x_0)$. If*

$$(F(x) - y_0 + \epsilon) \cap (-cor(C)) = \emptyset, \quad x \in S,$$

where $0 \neq C \neq Y$ is an ordering convex cone such that $K \setminus \{0\} \subset cor(C)$, then (x_0, y_0) is called a K-Henig proper efficient solution of (P). The set of K-Henig proper efficiient solutions of problem (P) will be denoted by $HP_\epsilon(F, S, K)$. Hereafter, whenever we talk about the K- weak efficiency, it is assumed that K is solid, i.e., $cor(K) \neq \emptyset$. Also, notice that $0 \notin cor(K)$ because $K \neq Y$. In fact, it can be shown that if $0 \notin cor(K)$, then $K = Y$. By definition, it is clear that if $(x_0, y_0) \in HP_\epsilon(F, S, K)$, then $(x_0, y_0) \in EP_\epsilon(F, S, K)$ and $(x_0, y_0) \in WEP_\epsilon(F, S, K)$.

4. Scalarization

One of the most important tools to solve optimization problems is scalarization. This concept plays a vital role in sketching the numerical algorithms and duality results. Gerstewitz's function is well known and widely used in optimization problems [17, 22]. When underlying space Y is a topological vector space and $K \subset Y$ is a closed convex (solid) cone, Gerstewitz's function and its properties have been studied in [6, 11, 31]. This function introduced in [13, 22, 23, 24, 26] has different names such as Gerstewitz's function, smallest strictly monotonic function, shortage function, nonlinear scalarization function. Gerstewitz's function was generated by a general convex cone in a linear space in [24].
In this section, we present necessary and sufficient conditions to characterize approximate (weak/proper) efficient solutions of set-valued vector optimization problems in a linear space without any topology. Here, we also consider Gerstewitz's function ξ_e generated by the convex cone K. Now, let $e \in cor(K)$. The function $\xi_e(y) : Y \to \mathbb{R}$ can be defined as

$$\xi_e(y) = inf\{t \in \mathbb{R} \,:\, y \in te - K\}, \tag{1}$$

which is generated by K and e. Notice that since the set $\{t \in \mathbb{R} \,:\, y \in te - K\}$ is nonempty, closed, and bounded from below. Also, ξ_e is finite [32, Lemma 2.2], and

$$\xi_e(y) = sup\{h(y) \,:\, h \in K^+, \, \xi(e) = 1\} \quad \forall y \in Y,$$

where, K^+ is the positive polar cone of K [18]. One can find the proof of Proposition 4.2 in [24]. The Proposition 4.1 states that ξ_e is sub-additive and positively homogeneous [24, Lemma 2.8].

Proposition 4.1. [24] *(i) For any $y_1, y_2 \in Y$, $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ except that the right side becomes $\infty - \infty$*
(ii) For any $y \in Y$ and $\alpha \geq 0$ we have $\xi_e(\alpha y) = \alpha \xi_e(y)$.

Proposition 4.2. [24] *Let $y \in Y$ and $r \in \mathbb{R}$. Then we have*
(i) $S(\xi_e, r, <) = re - vint_e(K)$.
(ii) $S(\xi_e, r, \leq) = re - vcl_e(K)$.
(iii) $S(\xi_e, r, =) = re - (vcl_e(K) \setminus vint_e(K))$.
(iv) $S(\xi_e, r, \geq) = Y \setminus (re - vint_e(K))$.
(v) $S(\xi_e, r, >) = Y \setminus (re - vcl_e(K))$.

Definition 4.1. *Consider the scalar problem*

$$Min\{\xi_e(F(x)) \,:\, x \in S\}.$$

A point (x_0, y_0), where $x_0 \in S$ and $y_0 \in F(x_0)$, is said to be an ϵ-minimum solution of the scalar problem if

$$\xi_e(F(x) - y_0 + \epsilon) \geq 0, \quad \forall x \in S,$$

and if

$$\xi_e(F(x) - y_0 + \epsilon) > 0, \quad \forall x \in S,$$

then (x_0, y_0) *is a strict* ϵ*-minimum solution.*

Theorem 4.1. *Let* $\epsilon \in K$, $e \in cor(K)$, $x_0 \in S$ *and* $y_0 \in F(x_0)$. *Then* (x_0, y_0) *is an* ϵ*-minimum solution to problem*

$$Min\{\xi_e(F(x)) : x \in S\}, \tag{2}$$

if and only if $(x_0, y_0) \in WEP_\epsilon(F, S, K)$.

Proof. Since (x_0, y_0) is an ϵ-minimum solution of (2), we have

$$\xi_e(y - y_0 + \epsilon) \geq 0 \qquad \forall (y \in F(x),\, x \in S).$$

By Proposition 4.2, we have

$$y - y_0 + \epsilon \notin (-vint_e(K)) \qquad \forall (y \in F(x), \quad x \in S)$$

Since $e \in cor(K)$, by Proposition 2.1

$$y - y_0 + \epsilon \notin (-cor(K)) \quad \forall (y \in F(x), \quad x \in S)$$

Thus,

$$(F(x) - y_0 + \epsilon) \cap (-cor(K)) = \emptyset \quad \forall x \in S$$

Therefore, by definition 3.1, $(x_0, y_0) \in WEP_\epsilon(F, S, K)$. Conversely, suppose that (x_0, y_0) is not an ϵ-minimum of scalar problem (2). Then there exist $x \in S$ and $y \in F(x)$ such that

$$\xi_e(y - y_0 + \epsilon) < 0.$$

From Propositions 4.2 and 2.1, it follows that

$$y - y_0 + \epsilon \in -vint_e(K) = -cor(K).$$

On the other hand,

$$y - y_0 + \epsilon \in F(x) - y_0 + \epsilon.$$

Therefore,

$$y - y_0 + \epsilon \in (F(x) - y_0 + \epsilon) \cap (-cor(K)),$$

which is a contradiction, since $(x_0, y_0) \in WEP_\epsilon(F, S, K)$. □

The following result provides necessary and sufficient conditions for K-Henig proper efficiency.

Theorem 4.2. *Let* K *be a convex cone,* $\epsilon \in K$, $x_0 \in S$ *and* $y_0 \in F(x_0)$. *If* $e \in cor(K)$ *and* (x_0, y_0) *is a strict* ϵ*-minimum solution to the problem*

$$Min\,\{\xi_e(F(x)) : x \in S\}, \tag{3}$$

then $(x_0, y_0) \in HP_\epsilon(F, S, K)$. *Conversely, if there exists a convex cone* C *such that* $K \setminus \{0\} \subseteq cor(C)$ *and* $(x_0, y_0) \in HP_\epsilon(F, S, K)$, *then* (x_0, y_0) *is an* ϵ*-minimum solution to the problem* (3).

Proof. Since (x_0, y_0) is a strict ϵ-minimum solution of (3), we have

$$\xi_e(y_0 - y - \epsilon) < 0 \qquad \forall (x \in S, y \in F(x)).$$

By Propositions 4.2 and 2.1, we have $y_0 - y - \epsilon \in (-vint_e(K)) = (-cor(K))$. Therefore,

$$\langle y_0 - y - \epsilon, l\rangle < 0 \qquad \forall (x \in S,\ y \in F(x),\ l \in K^{+s}) \tag{4}$$

Now, consider $l \in K^{+s}$, we have

$$\langle k, l\rangle > 0 \quad \forall k \in K \setminus \{0\}. \tag{5}$$

By setting $V = \{k \in K;\ \langle k, l\rangle = 1\}$ and $U = \{y \in Y;\ \mid \langle y, l\rangle \mid < 1/2\}$, one has $V + U \subset \{y \in Y; \langle y, l\rangle \geq 1/2\}$. Let $C = cone(V + U)$. If $y \in C$, then $\langle y, l\rangle \geq 0$. We will show $K \setminus \{0\} \subseteq cor(C)$. On the contrary, suppose that there exists $k \in K \setminus \{0\}$ such that $k \notin cor(C)$. Therefore,

$$\exists y \in Y, \quad \forall \lambda > 0, \quad \exists \lambda \in [0, \lambda^{'}]; \quad k + \lambda y \notin C.$$

Thus,

$$\langle k + \lambda y, l\rangle = \langle k, l\rangle + \lambda \langle y, l\rangle < 0.$$

If $\lambda^{'} \longrightarrow 0$, then $\lambda \longrightarrow 0$ and $\langle k, l\rangle \leq 0$, which contradicts to (5). Hence, $K \setminus \{0\} \subseteq cor(C)$ and then we have

$$(F(x) - y_0 + \epsilon) \cap (-C \setminus \{0\}) = \emptyset \qquad x \in S.$$

Therefore, $(x_0, y_0) \in HP_\epsilon(F, S, K)$. Conversely, since $(x_0, y_0) \in HP_\epsilon(F, S, K)$, then $(x_0, y_0) \in WEP_\epsilon(F, S, K)$. From Theorem 4.1 we conclude that (x_0, y_0) is ϵ-minimum for the problem (3). □

5. A Nonconvex Separation Function

The main idea of this section obtained from [13, 24], where some preliminary properties of Gerstewitz's function such as subadditivity and positively homogeneous are proved in the setting of a real linear space. Gerstewitz's function can also be generated by any set, as opposed to a convex set and recalls the nonconvex separation function in [13]. In this section, we also consider the nonconvex separation function φ^e_Q generated by a set Q to obtain optimality conditions for Q-efficient (Q-weak efficient) solutions and Q-Henig proper efficient solutions. Let $\emptyset \neq Q \subset Y$, $\varphi^e_Q : Y \to \mathbb{R} \cup \{\pm\infty\}$ is definedd as follows

$$\varphi^e_Q = inf\{t \in \mathbb{R} :\ y \in te - Q\} \tag{6}$$

In the following theorem, some properties of the nonconvex separation function are given.

Theorem 5.1. [13] *Consider $0 \neq e \in Y$ and $\emptyset \neq Q \subset Y$. We have the following properties of φ^e_Q*
i) $S(\varphi^e_Q, 0, \leq) = (-\infty, 0]e - vcl_eQ$,
ii) $S(\varphi^e_Q, 0, <) = (-\infty, 0)e - vcl_eQ$,
iii) $S(\varphi^e_Q, 0, =) = (-vcl_e(Q)) \setminus ((-\infty, 0)e - vcl_e(Q))$,
iv) $S(\varphi^e_Q, 0, \geq) = Y \setminus ((-\infty, 0]e - vcl_e(Q))$.

Definition 5.1. *Consider the scalar problem*

$$Min\{\varphi^e_Q(F(x)) : \quad x \in S\}$$

A point (x_0, y_0), where $x_0 \in S$ and $y_0 \in F(x_0)$, is said to be an ϵ-minimum solution of the scalar problem if

$$\varphi^e_Q(F(x) - y_0 + \epsilon) \geq 0 \qquad \forall x \in S,$$

and if

$$\varphi^e_Q(F(x) - y_0 + \epsilon) > 0 \qquad \forall x \in S,$$

then (x_0, y_0) *is a strict* ϵ*-minimum solution.*

The notion of Q-minimal solution of vector optimization problems via topological concepts presented in [14], where Q is some nonempty open (not necessarily convex) cone. Moreover, these Q-minimal points were characterized by the Hiriart-Urruty function. Here, we define the concept of Q-efficient (Q-weak efficient) solutions and Q-Henig proper efficient solutions in vector optimization.

Definition 5.2. *Let* $x_0 \in S$*,* $y_0 \in F(x_0)$ *and* $0 \neq Q \subset Y$*. A solution* (x_0, y_0) *is called a* Q*-efficient solution of (P), denoted by* $EP_\epsilon(F, S, Q)$*, if*

$$(F(x) - y_0 + \epsilon) \cap (-Q \setminus \{0\}) = \emptyset, \quad x \in S.$$

Furthermore, if $cor(Q) \neq \emptyset$*, then* (x_0, y_0) *is called a* Q*-weak efficient solution of (P), denoted by* $WEP_\epsilon(F, S, Q)$*, if*

$$(F(x) - y_0 + \epsilon) \cap (-cor(Q)) = \emptyset, \quad x \in S.$$

Definition 5.3. *Consider* $x_0 \in S$ *and* $y_0 \in F(x_0)$*. If there exists* $0 \neq Q \subset Y$ *such that* $K \setminus \{0\} \subseteq cor(Q)$ *and*

$$(F(x) - y_0 + \epsilon) \ (-Q \setminus \{0\}) = \emptyset \quad x \in S,$$

then (x_0, y_0) *is called a* Q*-Henig proper efficient solution of (P). The set of* Q*-Henig proper efficient solutions of problem (P) will be denoted by* $HP_\epsilon(F, S, Q)$*.*

Proposition 5.1. *Let* $Q \subset Y$ *and consider the following set*

$$\bar{H} := \{Q \subset Y : \ K \setminus \{0\} \subseteq cor((0, \infty)e + vcl_e(Q))\}$$

where $e \in cor(K)$*. Then we have* $\bar{H} \neq \emptyset$*.*

Proof. Assume $Q \subset Y$ and $q \in cor(Q)$. There exists $\lambda > 0$ such that $q - [0, \lambda]e \in Q$. Thus, $q \in [0, \lambda]e + Q \subseteq [0, \lambda]e + vcl_e(Q)$, Therefore,

$$cor(Q) \subseteq (0, +\infty)e + vcl_e(Q).$$

Now, consider $K \setminus \{0\} \subseteq cor(cor(Q))$. Since $cor(Q) \subseteq (0, \infty)e + vcl_e(Q)$, we have

$$cor(cor(Q)) \subseteq cor((0, +\infty)e + vcl_e(Q)),$$

then $Q \in \bar{H}$. □

Theorem 5.2. *Let* $x_0 \in S$*,* $y \in F(x_0)$ *and* $e \in cor(K)$*.*
(i) If (x_0, y_0) *is a strict* ϵ*-minimum solution to the problem*

$$Min \{\varphi_Q^e(F(x)) : \quad x \in S\}, \tag{7}$$

then $(x_0, y_0) \in EP_\epsilon(F, S, Q)$*.*
(ii) If (x_0, y_0) *is an* ϵ*-minimum solution to the problem* (7)*, then* $(x_0, y_0) \in WEP_\epsilon(F, S, Q)$*.*

Proof. (i) Suppose (x_0, y_0) is not a Q-efficient solution for (P). Thus, there exists $x \in S$ such that

$$(F(x) - y_0 + \epsilon) \cap (-Q \setminus \{0\}) \neq \emptyset$$

This implies that there exists $y \in F(x)$ such that

$$y - y_0 + \epsilon \in (F(x) - y_0 + \epsilon) \cap (-Q \setminus \{0\}).$$

Since (x_0, y_0) is a strict ϵ-minimum solution of the problem (7), we have

$$\varphi_Q^e(y - y_0 + \epsilon) > 0. \tag{8}$$

On the other hand, $y - y_0 + \epsilon \in (-Q \setminus \{0\})$. Hence,

$$y - y_0 + \epsilon \in (-vcl_e(Q)) \subset (-\infty, 0]e - vcl_e(Q),$$

by Theorem 5.1, one has

$$\varphi_Q^e(y - y_0 + \epsilon) \leq 0,$$

which is a contradiction to (8).
(ii) Since (x_0, y_0) is an ϵ-minimum solution of the problem (7), we have

$$\varphi_Q^e(y - y_0 + \epsilon) \geq 0 \qquad \forall (y \in F(x),\ x \in S). \tag{9}$$

By Theorem 5.1, one has

$$(y - y_0 + \epsilon) \notin (-\infty, 0]e - vcl_e(Q).$$

Since $(-cor(Q)) \subseteq (-\infty, 0]e - vcl_e(Q)$, then $(y - y_0 + \epsilon) \notin (-cor(Q))$. Therefore,

$$(F(x) - y_0 + \epsilon) \cap (-cor(Q)) \neq \emptyset \qquad \forall x \in S.$$

□

Theorem 5.3. *Let $x_0 \in S$, $y_0 \in F(x_0)$, and Q is free disposal with respect to the convex cone $K \subseteq Y$. If $(x_0, y_0) \in WEP_\epsilon(F, S, Q)$, then (x_0, y_0) is an ϵ-minimum for the problem*

$$Min\{\varphi_Q^e(F(x)), \quad \forall x \in S\}. \tag{10}$$

Proof. Suppose (x_0, y_0) is not an ϵ-minimum for the problem (10). Then there exists $x \in S$ and $y \in F(x)$ such that

$$\varphi_Q^e(y - y_0 + \epsilon) < 0,$$

and by Theorem 5.1 we obtain

$$y - y_0 + \epsilon \in (-\infty, 0)e - vcl_e(Q).$$

Now, since Q is free disposal with respect to the convex cone $K \subseteq Y$, then by Proposition 2.2 one has $(-\infty, 0)e - vcl_e(Q) = -cor(Q)$. Hence,

$$y - y_0 + \epsilon \in (-cor(Q)).$$

On the other hand, $y - y_0 + \epsilon \in F(S) - y_0 + \epsilon$. Consequently,

$$y - y_0 + \epsilon \in (F(S) - y_0 + \epsilon) \cap (-cor(Q)),$$

which contradicts to the assumption. Therefore, (x_0, y_0) is an ϵ-minimum of the scalar problem (10). □

It can be seen that Theorem 5.3 is also valid for $(x_0, y_0) \in EP_\epsilon(F, S, Q)$. From Theorems 5.2 and 5.3 we conclude the following corollary.

Corollary 5.1. *Let $e \in cor(K)$ and $\emptyset \neq Q \subset Y$ be free disposal with respect to the convex cone K. A point (x_0, y_0) is a Q-weak efficient solution for (P) if and only if (x_0, y_0) is an ϵ-minimum of the scalar problem*

$$Min\{\varphi_Q^e(F(x)) : \ x \in S\}.$$

Theorem 5.4. *Let $x_0 \in S, y_0 \in F(x_0)$ and $Q \in \bar{H}$. If $e \in cor(K)$ and (x_0, y_0) is an ϵ-minimum solution to the problem*

$$Min\{\varphi_Q^e(F(x)) : \ x \in S\}, \tag{11}$$

then there exists $\emptyset \neq Q' \subseteq Y$ such that $(x_0, y_0) \in HP_\epsilon(F, S, Q')$.

Proof. Since (x_0, y_0) is an ϵ-minimum solution to the problem (11), then we have

$$\varphi_Q^e(y - y_0 + \epsilon) \geq 0, \quad \forall(y \in F(x),\ x \in S). \tag{12}$$

By Theorem 5.1, one has

$$y - y_0 + \epsilon \in Y \setminus (-\infty, 0)e - vcl_e(Q), \qquad \forall(y \in F(x), x \in S).$$

Put $Q^{'} = (0, \infty)e + vcl_e(Q)$, then it is clear that

$$(F(x) - y_0 + \epsilon) \cap (-Q^{'} \setminus \{0\}) = \emptyset \qquad \forall x \in S$$

Moreover, since $cor(cor(Q)) \subset cor(Q^{'})$ and $Q \in \bar{H}$, one has

$$K \setminus \{0\} \subseteq cor(Q^{'}).$$

Hence, $(x_0, y_0) \in HP_\epsilon(F, S, Q^{'})$. □

Theorem 5.5. *Let $x_0 \in S$ and $y_0 \in F(x_0)$. If there exists Q subset of Y which is free disposal with respect to the convex cone $K \subset Y$, $K \setminus \{0\} \subseteq cor(Q)$, and*

$$(F(x) - y_0 + \epsilon)\ (-Q \setminus \{0\}) = \emptyset \qquad \forall x \in S$$

then (x_0, y_0) is an ϵ-minimum for the problem

$$Min\,\{\varphi_Q^e(F(x)), \quad \forall \in S\}.$$

Proof. This theorem can be proved similar to Theorem 5.3. □

Here, we use the results of previous theorems to obtain optimality conditions for constrained problems without convexity assumption. To obtain optimality conditions, we need a constraint qualification.

Definition 5.4. *We say that the Slater constraint qualification for constrained problems holds if there exists $x \in S$ such that $G(x) \subseteq (-cor(M))$.*

Hereafter, the set of all linear operators from Z into Y is denoted by $O(Z, Y)$, and Γ is denoted by

$$\Gamma = \{T \in O(Z, Y) : \quad T(M) \subseteq K\}$$

where M and K are as above. The following theorem will be used in sequel.

Theorem 5.6. *Consider $0 \neq e \in Y$ and suppose that $\emptyset \neq Q \subseteq Y$ be closed under addition and Q is finite. Then*

$$\varphi_Q^e(y_1 + y_2) \leq \varphi_Q^e(y_1) + \varphi_Q^e(y_2),$$

for all $y_1, y_2 \in Y$, except that these make the right side into an indeterminate form $\infty - \infty$.

Proof. From definition of φ_Q^e and Lemma 3 in [15], we have

$$y_i \in \varphi_Q^e(y_i)e - vcl_e(Q) \qquad i = 1, 2.$$

Now, we can use $\varphi_{vcl_e(Q)}^e = \varphi_Q^e$ to obtain

$$y_i \in \varphi_{vcl_e(Q)}^e(y_i)e - vcl_e(Q), \qquad i = 1, 2.$$

Obviously,

$$y_1 + y_2 \in (\varphi_{vcl_e(Q)}^e(y_1) + \varphi_{vcl_e(Q)}^e(y_2))e - vcl_e(Q),$$

which implies

$$\varphi_{vcl_e(Q)}^e(y_1 + y_2) \leq \varphi_{vcl_e(Q)}^e(y_1) + \varphi_{vcl_e}^e(Q)(y_2),$$

and hence we have

$$\varphi^e_Q(y_1 + y_2) \leq \varphi^e_Q(y_1) + \varphi^e_Q(y_2).$$

□

Constrained problems are studied in [2,20]. In particular, the relation between "Hurwicz victorial proper efficient solutions" and "Benson victorial proper efficient solutions" in unconstrained and constrained problems were studied in [2, Theorem 4.3]. Moreover, in [20, Theorem 4.12] authors discussed the relation between "Benson victorial proper efficient solutions" in unconstrained and constrained problems. In the following theorem we obtain optimality condition for Q-efficient solutions in the constrained problems.

Theorem 5.7. *Consider the constrained vector optimization problem (CP), given in section 3. Suppose convex cones K and M are pointed and $e \in cor(K)$. Let $\emptyset \neq Q \subseteq Y$ be free disposal with respect to the algebraic solid convex cone K, $x_0 \in S$, and $y_0 \in F(x_0)$.*
(i) Assume that $T \in \Gamma$, $0 \in Q$, and Q is closed under addition. If we have

$$\varphi^e_Q(F(x) - y_0 + T(z) + \epsilon) > 0, \qquad \forall(x \in S,\, z \in G(x)), \tag{13}$$

then $(x_0, y_0) \in EP_\epsilon(F, S, Q)$ for the CP.
(ii) If for any $x \in S$ we have

$$\varphi^e_Q(F(x) - y_0 + T(z) + \epsilon) \geq 0, \qquad \forall z \in G(x), \tag{14}$$

then $(x_0, y_0) \in WEP_\epsilon(F, S, Q)$ for the CP.

Proof. (i) Suppose that $(x_0, y_0) \neq EP_\epsilon(F, S, Q)$ for the CP. Then by definition, there exists $x \in S$ such that

$$(F(x) - y_0 + \epsilon) \cap (-Q\backslash\{0\}) \neq \emptyset,$$

which implies that there exists $y \in F(x)$ such that

$$y - y_0 + \epsilon \in (-Q\backslash\{0\}).$$

Now, by Theorem 5.1, this yields

$$\varphi^e_Q(y - y_0 + \epsilon) \leq 0. \tag{15}$$

On the other hand, by Theorem 5.6, for $z \in G(x)$ we have

$$\varphi^e_Q(F(x) - y_0 + \epsilon) + \varphi^e_Q(T(z)) \geq \varphi^e_Q(F(x) - y_0 + T(z) + \epsilon),$$

and by assumption

$$\varphi^e_Q(F(x) - y_0 + T(z) + \epsilon) > 0 \tag{16}$$

Now, equations (15) and (16) yield

$$\varphi^e_Q(T(z)) > 0, \qquad \forall z \in G(x). \tag{17}$$

Moreover, since $0 \in Q$ and $T(G(x)) \subseteq -K$, we have

$$T(G(x)) + 0 \subset -K - Q$$

and Q is free disposal with respect to K, so one has

$$T(G(x)) \subset -K - Q = -Q \subset (-\infty, 0]e - vcl_e(Q).$$

By Theorem 5.1, we deduce

$$\varphi_Q^e(T(z)) \leq 0 \qquad \forall (z \in G(x),\, x \in X) \tag{18}$$

This is a contradiction by (17). Therefore, $(x_0, y_0) \in EP_\epsilon(F, S, Q)$ for the CP.
(ii) Suppose that there exist $x \in S$ and $y \in F(x)$ such that $y - y_0 + \epsilon \in (-cor(Q))$. Thus,

$$y - y_0 + T(z) + \epsilon \in (-cor(Q)) - K = -cor(Q + K) = -cor(Q),$$

where $z \in G(x)$. Since $-cor(Q) \subseteq (\infty, 0)e - vcl_e(Q)$, one has

$$y - y_0 + T(z) + \epsilon \in (-\infty, 0)e - vcl_e(Q)$$

Therefore,

$$\varphi_Q^e(F(x) - y_0 + T(z) + \epsilon) < 0, \qquad \forall (x \in S,\, z \in G(x)),$$

which is a contradiction by (14). So, $(x_0, y_0) \in WEP_\epsilon(F, S, Q)$ for the CP.

□

Remark: It is well-know that the Lagrangian mapping $L : X \times \Gamma \longrightarrow Y$, corresponding to the constrained vector optimization problem, is defined by $L(x, T) = F(x) + T(G(x))$, where $T \in \Gamma$. Using the map L, one can convert the CP to an unconstrained vector optimization problem:

$$Min\{F(x) + T(G(x)) : \quad x \in X\}. \tag{19}$$

Theorem 5.8. *In a constrained vector optimization problem, suppose $Q \in \bar{H}$ be closed under addition and free disposal with respect to the algebraic solid convex cone K. Let $e \in cor(K), 0 \in Q$ and the Slater constraint qualification holds. Moreover, assume that $T(z) = 0$ for $T \in \Gamma$, $z \in G(x_0)$, and $x_0 \in S$. If $(x_0, y_0) \in HP_\epsilon(F, S, Q)$ for problem given in (19), then $(x_0, y_0) \in HP_\epsilon(F, S, Q)$ for the CP.*

Proof. As mentioned in Theorem 5.3, since $(x_0, y_0) \in HP_\epsilon(F, S, Q)$ for problem (19), then (x_0, y_0) is an ϵ- minimum for the following problem

$$Min\,\{\varphi_Q^e(F(x) + T(G(x)) : \quad x \in X\}.$$

Hence, for $y \in F(x), z \in G(x)$ and $z_0 \in G(x_0)$, one has

$$\varphi_Q^e(y + T(z) - y_0 - T(z_0) + \epsilon) \geq 0.$$

Now, by Theorem 5.6, we have

$$\varphi_Q^e(y - y_0 + \epsilon) + \varphi_Q^e(T(z) - T(z_0)) \geq \varphi_Q^e(y + T(z) - y_0 - T(z_0) + \epsilon) \geq 0.$$

Since $T(z_0) = 0$ for $z_0 \in G(x_0)$, then

$$\varphi_Q^e(y - y_0 + \epsilon) + \varphi_Q^e(T(z)) \geq 0. \tag{20}$$

On the other hand, since $0 \in Q$ and Q is free disposal with respect to K, we have

$$T(G(x)) \subseteq -K - Q = -Q \subseteq (-\infty, 0]e - vcl_e(Q), \qquad \forall x \in S$$

and by Theorem 5.1, we obtain

$$\varphi_Q^e(T(z)) \leq 0, \qquad \forall (z \in G(x), \quad x \in S). \tag{21}$$

Now, (20) and (21) yield

$$\varphi_Q^e(y - y_0 + \epsilon) \geq 0$$

and then, by Theorem 5.4, there exists $Q \subseteq Y$ such that $(x_0, y_0) \in HP_\epsilon(F, S, Q)$ for the CP. □

6. Conclusions

In this paper we introduced a new version of efficient solutions in vector optimization problems with set-valued mapping. First, optimality conditions for approximate (Henig proper/weak) efficiency in set-valued optimization are provided by using Gerstewitz's function generated by a convex cone. Moreover, Q-proper efficient solutions of a vector optimization problem have been generalized to a linear space not necessarily endowed with a topology by using the algebraic concepts of interior and closure. Furthermore, we used the scalarization technique including the nonconvex separation function generated by a nonconvex set to characterize Q-weak efficient solutions and Q-Henig proper efficient solutions.

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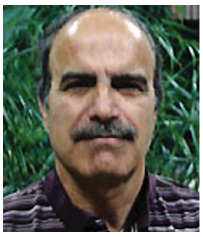

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