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AN APPROXIMATION TECHNIQUE FOR FIRST PAINLEVÉ EQUATION

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Abstract. In this study, we introduce a new approximative algorithm to get numerical solutions of the nonlinear first Painlevé equation. Indeed, to obtain an approximate solution, a combination of exponential matrix method based on collocation points and quasilinearization technique is used. The quasilinearization method is used to transform the original non-linear problem to a sequence of linear equations while the exponential collocation method is employed to solve the resulting linear equations iteratively. Furthermore, since the exact solution of the model problem is not known, an error estimation based on the residual functions is presented to check the accuracy of the proposed method. Finally, the benefits of the method are illustrated with the aid of numerical calculations. Comparisons with other well-known schemes show that the combined technique is easy to implement while capable of giving results of very high accuracy with a relatively low number of exponential functions.

Keywords: Collocation points, Exponential functions, Painlevé equation, Quasilinearization method.

AMS Subject Classification: 65M70, 33B10, 34A34, 65L05.

1. Introduction

The six Painlevé equations were first discovered by Painlevé and his co-workers in their study of nonlinear second-order irreducible ordinary differential equations [24]. In this paper, a novel efficient approximative algorithm based on exponential basis functions and quasilinearization technique is introduced to solve the first Painlevé equation with initial conditions given as

$$\begin{cases} x''(t) = 6\,x^2(t) + t\,\mu, \quad 0 \leqslant t \leqslant 1, \\ x(0) = 0, \quad x'(0) = 1, \end{cases} \tag{1}$$

where μ is an arbitrary real constant. In fact, Painlevé equations satisfy the so-called Painlevé properties (the general solutions are free from movable branch points) and their solutions are known as Painlevé transcendents. Painlevé equations appear in many important physical applications. Among others, we emphasize as a model for describing the

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electric field in a semiconductor [18], quantum gravity [12], random matrix theory [32], modelling the viscous shocks in Hele-Shaw flow and also Stokes phenomena [21] as well as the existence of tronquée and hyperasymptotics solutions [5]. Moreover, the exact solutions to many nonlinear partial differential equations such as Korteweg-de Vries (KdV), cylindrical KdV and Boussinesq equations can be written in terms of Painlevé transcendent [1, 31]. The mathematical theory of the classical Painlevé differential equations along with some applications are considered in [4, 13, 19].

The properties of Painlevé differential equations have been studied from both analytical and numerical point of views in many publications. The most considerable analytical schemes include Adomian's decomposition (ADM) and homotopy perturbation methods (HPM) [7], variational iteration method (VIM) and HPM [15], homotopy analysis method [10, 14], and optimal homotopy asymptotic method [22]. On the other hand, numerical techniques such as Chebyshev series [6, 16], computational intelligence technique based on neural networks and particle swarm optimization [28, 29, 30], and reproducing kernel Hilbert space algorithms [2] have been developed in the past to solve the nonlinear equation (1).

Exponential polynomials or exponential functions have found important applications in numerous branches of science and engineering. Physical systems which can be characterized by linear differential equations with constant coefficients naturally admit exponential solutions. The function $1/x$ can be well approximated by sums of exponentials on finite or infinite intervals [9]. The use of orthonormal exponentials as basis functions provide efficient representation of a large class of signals arising in physical processes [20]. An old bibliography for approximation with exponential sums can be found in [17].

The main subject of this paper is to approximate the solution of (1) on the interval $[0, 1]$ as a linear combination of real exponential functions. Rather than exploiting the basis functions with negative exponents $\{1, e^{-t}, e^{-2t}, \ldots\}$ as considered in [33, 34, 35, 36], we are going to find an accurate approximate solution of (1) of the form

$$x(t) \cong x_N(t) = \sum_{n=0}^{N} a_n\, e^{nt}, \quad 0 \leqslant t \leqslant 1, \tag{2}$$

where $a_0, a_1, \ldots, a_N$ are the coefficients of the exponential polynomial to be determined through a collocation procedure. The main idea of the proposed technique based on using these exponential functions along with collocation points is that it transforms the model problem (2) to an algebraic form, thus greatly reducing the computational effort. However, due to the nonlinearity of the problem under consideration, the resulting system of equations obtained via exponential-collocation is nonlinear and its solving may be inefficient for a large value of N. Alternatively, to obviate this difficulty we first apply the quasilinearization method (QLM) to (1) and then utilize the exponential-collocation scheme to the resulting linear equation.

The rest of the paper is organized as follows: in Section 2, a concise introduction to the technique of quasilinearization is expressed. In Section 3, the methodology of collocation for the corresponding quasi-linear equations is explained. Results and discussions of the combined proposed method are reported in Section 4. Finally, a conclusion is provided.

2. Quasilinearization Method

As previously mentioned, solving a nonlinear system of equations obtained via direct exponential-collocation scheme using nonlinear solvers such as Newton's methods may be inefficient when the number of basis functions is getting large. To overcome this difficulty, we may first convert the original equation (2) into a sequence of linear equations and then apply the aforementioned exponential collocation scheme to them. To this end, we describe briefly the quasi-linearization method (QLM) as a generalized Newton-Raphson scheme for functional equations, see [3, 23, 26, 25, 27].

Let us consider the general form of nonlinear differential equation (2),

$$x''(t) = f(x(t), t), \tag{3}$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$. Here f is a function of $x(t)$. To start computation, we need to choose an initial approximation of the function $x(t)$. Assuming that $x_0(t) = 0$ as an initial guess, the QLM iteration for (3) is determined as follows

$$x''_{r+1}(t) = f(x_r(t), t) + \big(x_{r+1}(t) - x_r(t)\big)\, f_x(x_r(t), t), \tag{4}$$

with the same boundary conditions $x_{r+1}(0) = 0$, $x'_{r+1}(0) = 1$, $r = 0, 1, \ldots$, and the function $f_x = \partial f/\partial x$ denotes the functional derivative of $f(x(t), t)$. By applying the QLM technique on the first Painlevé equation (1) we get

$$x''_{r+1}(t) = t\mu - 6x_r^2(t) + 12\, x_r(t)\, x_{r+1}(t), \quad 0 \leqslant t \leqslant 1, \tag{5}$$

with the corresponding initial conditions

$$x_{r+1}(0) = 0, \qquad x'_{r+1}(0) = 1. \tag{6}$$

Therefore, instead of applying the exponential collocation scheme directly to Painlevé equation we solve a sequence of linear equations (5) by the collocation method, which is referred to as the exponential-QLM.

3. Exponential-QLM

Our goal is to solve the model problem (1) approximately such that the desired solutions expressed in terms of the truncated exponential series form (2). This task is instead accomplished for the corresponding approximated quasi-linear model problem (5). To this end, assuming that we have already an approximation solution $x_{N,r}(t)$ in the iteration $r = 0, 1, \ldots$, we take the solution in the next iteration as

$$x_{N,r+1}(t) = \sum_{n=0}^{N} a_n^{(r)}\, e^{nt}, \quad 0 \leqslant t \leqslant 1, \tag{7}$$

where the unknown coefficients $a_n^{(r)}$, $n = 0, 1, \ldots, N$ to be sought. We can rewrite the finite series (7) in a matrix form compactly as

$$x_{N,r+1}(t) = E_N(t)\, A^{(r)}, \tag{8}$$

where two known and unknown vectors $E_N(t)$ and $A^{(r)}$ are defined as

$$E_N(t) = \begin{bmatrix} 1 & e^t & e^{2t} & \ldots & e^{Nt} \end{bmatrix}, \quad A^{(r)} = \begin{bmatrix} a_0^{(r)} & a_1^{(r)} & \ldots & a_N^{(r)} \end{bmatrix}^T.$$

Here, a superscript T denotes the matrix transpose operation. The next aim is to find a relationship between $x_{N,r+1}(t)$ and its second-order derivative, which appears in (5). It can easily check that the derivatives of $E_N(t)$ satisfy

$$E_N^{(\ell)}(t) = \begin{bmatrix} 0 & e^t & 2^\ell\, e^{2t} & \ldots & N^\ell\, e^{Nt} \end{bmatrix}, \quad \ell = 1, 2 \ldots.$$

Now, we are able to relate $E_N(t)$ and its derivatives through the following matrix representation for $\ell = 1, 2, \ldots$

$$E_N^{(\ell)}(t) = E_N(t)\, D^\ell, \tag{9}$$

and where the diagonal matrix D^ℓ has the form

$$D^\ell = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & 2^\ell & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & N^\ell \end{bmatrix}_{(N+1)\times(N+1)}.$$

By the aid of (9) we can express all derivatives involved in the given equation. In particular for the second derivative in (1) we have

$$x''_{N,r+1}(t) = E_N(t)\, D^2\, A^{(r)}. \tag{10}$$

To obtain a solution in the form (2) or (7) of the problem (1) on the interval $0 \leqslant t \leqslant 1$, we will use the collocation points defined by

$$t_k = \frac{k}{N}, \quad k = 0, 1, \dots, N. \tag{11}$$

Proceeding by inserting the collocation points (11) into the relations (8) and (10) we get

$$X_{r+1} = E\, A^{(r)}, \qquad X_{r+1} = \begin{bmatrix} x_{N,r+1}(t_0) \\ x_{N,r+1}(t_1) \\ \vdots \\ x_{N,r+1}(t_N) \end{bmatrix}, \quad E = \begin{bmatrix} E_N(t_0) \\ E_N(t_1) \\ \vdots \\ E_N(t_N) \end{bmatrix}, \tag{12}$$

and

$$\ddot{X}_{r+1} = E\, D^2\, A, \qquad \ddot{X}_{r+1} = \begin{bmatrix} x''_{N,r+1}(t_0) \\ x''_{N,r+1}(t_1) \\ \vdots \\ x''_{N,r+1}(t_N) \end{bmatrix}. \tag{13}$$

Now, we are able to compute the exponential solutions of (5). The collocation procedure is based on calculating these exponential coefficients by means of collocation points (11). To continue, inserting the collocation points into the first Painlevé differential equation to get the system

$$x''_{r+1}(t_k) - 12\, x_r(t_k)\, x_{r+1}(t_k) = \mu\, t_k - 6\, x_r^2(t_k), \quad k = 0, 1, \dots, N.$$

Following matrix notation we may write the above equations as compactly as

$$\ddot{X}_{r+1} + P_r\, X_{r+1} = F_r, \tag{14}$$

where the constant coefficient matrix P_r and the right-hand side vector F_r have the following representations

$$P_r = \begin{bmatrix} -12\, x_r(t_0) & 0 & \dots & 0 \\ 0 & -12\, x_r(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -12\, x_r(t_N) \end{bmatrix}_{(N+1)\times(N+1)},$$

$$F_r = \mu \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_N \end{bmatrix} - 6 \begin{bmatrix} x^2_{N,r}(t_0) \\ x^2_{N,r}(t_1) \\ \vdots \\ x^2_{N,r}(t_N) \end{bmatrix}.$$

Let us place the relations (12) and (13) into (14). This gives us the fundamental matrix equation

$$W_r\, A^{(r)} = F_r, \tag{15}$$

where

$$W_r := E\, D^2 + P_r\, E.$$

Evidently, the fundamental matrix equation (15) is a set of $(N+1)$ linear equations in terms of $(N+1)$ unknown coefficients $a_0^{(r)}, a_1^{(r)}, \ldots, a_N^{(r)}$ to be found.

In order to take into account the initial conditions, we must also convert them into matrix form. The relations (8) and (9) with $\ell = 1$ will be used to show the initial conditions $x_{r+1}(0) = 0$, $x'_{r+1}(0) = 1$ in the matrix notation. To this end, at $t = 0$ we get

$$\begin{aligned} \overline{W}_0\, A^{(r)} &= 0, \qquad \overline{W}_0 := E_N(0) = [1 \quad 1 \quad 1 \quad \ldots \quad 1]^T, \\ \overline{W}_1\, A^{(r)} &= 1, \qquad \overline{W}_1 := E_N(0)\, D^1 = [0 \quad 1 \quad 2 \quad \ldots \quad N]^T. \end{aligned}$$

Now, by replacing the first and last rows of the augmented matrix $[W_r; F_r]$ by the row matrices $[\overline{W}_0; 0]$ and $[\overline{W}_1; 1]$, we arrive at the linear algebraic system of equations

$$\overline{W}_r\, A = \bar{F}_r. \tag{16}$$

Thus, the unknown exponential coefficients in (2) will be calculated via solving this linear system of equations. Note that if rank($\overline{W}_r$)=rank($[\overline{W}_r; \bar{F}_r]$) $= N+1$, then the vector of unknown $A^{(r)}$ is uniquely determined through computing the inverse $(\overline{W}_r)^{-1}$ multiplied by $\bar{F}_r$. This means that the initial-value problem (5) has a unique solution obtained via (7). Otherwise, one may find no solution or find a particular solution [35].

3.1. **Error estimation based on residual functions.** Since the exact solution of the first Painlevé differential equation is not known yet, we need some tools to measure the accuracy of the proposed collocation scheme. In this section, the error estimation based on the residual function is introduced for the method. For this purpose, let $\mathcal{E}_{N,r+1}(t)$ denote the residual error function, which obtains by putting the truncated exponential series solution (5) into (1). This implies that the error functions $\mathcal{E}_{N,r+1} : [0,1] \to \mathbb{R}$ can be defined by [33]-[36]

$$\mathcal{E}_{N,r+1}(t) = x''_{N,r+1}(t) - 6\, x^2_{N,r+1}(t) - t\mu, \quad t \in [0,1]. \tag{17}$$

Due to the fact that the truncated exponential series (7) is the approximate solution of (5) and consequently (1), we expect that the residual obtained by inserting the computed approximated solutions $x_{N,r+1}(t)$ into the differential equation becomes approximately small. As the error functions are clearly zero at the collocation points (11), we expect that when $\mathcal{E}_{N,r+1}(t)$ tend to zero as N increase. In other words, the smallness of the residual error function means that the approximate solutions are close to the exact solution.

4. Experimental Results

In this section we illustrate the accuracy and effectiveness of the proposed Exponential-QLM collocation method numerically when applied to the first Painlevé equation. Comparisons with existing well-established numerical schemes are also made to justify our results. All numerical computations have been done by using MATLAB R2017a.

To start computation, we take $\mu_1 = 1$ in (1) and set the number of basis function $N = 10$. The number of iteration utilized in QLM is $r = 5$, which is sufficient to get the desired solution. The approximate solution $x_{10,6}(t)$ of this model problem using exponential basis functions (5) in the interval $0 \leqslant t \leqslant 1$ is obtained as follows

$$\begin{aligned} x_{10,6}(t) &= 0.0053514797698255\, e^{10t} - 0.0974685921933804\, e^{9t} + 0.79484640484\, e^{8t} \\ &- 3.81700030192051\, e^{7t} + 11.9322976008606\, e^{6t} - 25.2690718754434\, e^{5t} \\ &+ 36.3985238406147\, e^{4t} - 34.4059616985225\, e^{3t} + 19.2007019388484\, e^{2t} \\ &- 3.84210696576906\, e^{t} - 0.900111831084693. \end{aligned}$$

To show that there is no significant difference between the direct and quasi-linearization approaches in terms of accuracy, we also report the corresponding approximation by means of direct exponential-collocations scheme. In this case we get

$$\begin{aligned} x_{10}(t) &= 0.00535147976577493\, e^{10t} - 0.0974685921340833\, e^{9t} + 0.79484640444\, e^{8t} \\ &- 3.8170003003456\, e^{7t} + 11.9322975967392\, e^{6t} - 25.2690718680475\, e^{5t} \\ &+ 36.3985238314116\, e^{4t} - 34.4059616906897\, e^{3t} + 19.2007019344877\, e^{2t} \\ &- 3.84210696433558\, e^{t} - 0.900111831296. \end{aligned}$$

Note, however, that the required CPU-time for the direct collocation procedure is about 8-10 times larger compared to the exponential-QLM, for which it takes about 6-7 seconds.

Figure 1 visualizes the two above approximations. To see the impact of using different numbers of iterations, we also depict the numerical solution after $r = 25$ iterations. Looking at Fig.1 shows that no significant gain is achieved if one exploits a higher number of iterations in exponential-QLM. The influence on results by changing the number of

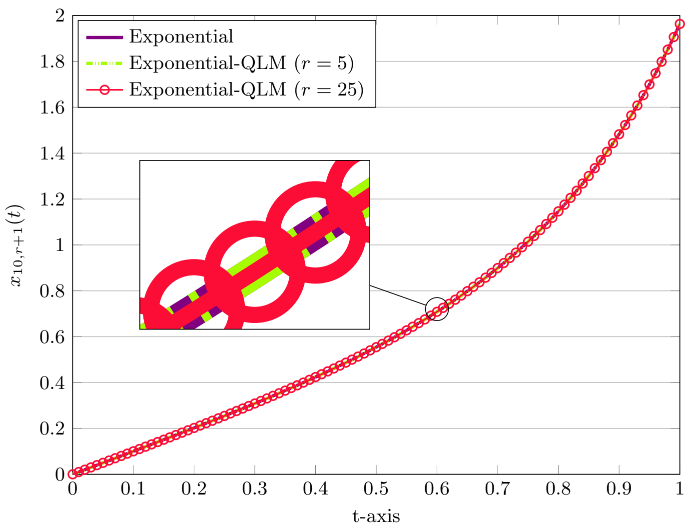

FIGURE 1. The approximated exponential (dashdotdotted) and exponential-QLM ($r = 5, 25$) series solutions $x_{10,r+1}(t)$ using $\mu = 1$.

basis functions used in the approximation form (7) are analyzed in the next experiment. We use different numbers of basis functions $N = 10, 15$, and $N = 20$. A comparison between numerical solutions obtained via direct exponential-collocation and its variant exponential-QLM is reported in Table 1.

The accuracy and convergence of the exponential-QLM are analyzed further by calculating the estimated errors via (17). The corresponding accuracies related to the results shown in Table 1, i.e., for $N = 10, 15, 20$ as well as $N = 5$ are shown in Fig. 2. Results

	Exponential	Exponential-QLM		
t	$N = 10$	$N = 10$	$N = 15$	$N = 20$
0.0	**5.72458747**$\times 10^{-16}$	**0.000000000000000**	0.000000000000000	0.000000000000000
0.1	**0.100208773704**597	**0.100208773704**608	0.100216728639073	0.100216747706758
0.2	**0.20212094690833**1	**0.202120946908**391	0.202139411269529	0.202139452778939
0.3	**0.308601374057**235	**0.308601374057**432	0.308630684225899	0.308630749264295
0.4	**0.423944902448**233	**0.423944902448**789	0.423986198578745	0.423986289623842
0.5	**0.554284564949**3518	**0.55428456494**5013	0.554339997390958	0.554340119178205
0.6	**0.708388623252**529	**0.70838862325**6570	0.708461927484248	0.708462088285766
0.7	**0.899152049**132979	**0.899152049**144271	0.899249724572649	0.899249938322129
0.8	**1.146399272**299335	**1.146399272**332506	1.146531437041188	1.146531726857257
0.9	**1.482331498**884031	**1.482331498**987892	1.482524018900693	1.482524431395324
1.0	**1.963159451**563367	**1.963159451**914160	1.963118370987331	1.963128716139353

TABLE 1. Comparison of numerical solutions in exponential and exponential-QLM methods for $N = 10, 15, 20$.

are plotted on semi-logarithmic scale in order to see the differences elaborately. It can be clearly seen from Fig. 2 that the errors are exponentially decreased while N is increased. In fact, using $N = 15$ the error function $\mathcal{E}_{15,6}(t)$ has the form

$$\begin{aligned}\mathcal{E}_{15,6}(t) = &-2.291779 \times 10^{-6}\, e^{30t} + 1.260489 \times 10^{-4}\, e^{29t} - 0.00334939474952182\, e^{28t}\\ &+ 0.0572703212328443\, e^{27t} - 0.70808241479334\, e^{26t} + 6.74407593990554\, e^{25t}\\ &- 51.4750588875573\, e^{24t} + 323.382275604955\, e^{23t} - 1704.33827885901\, e^{22t}\\ &+ 7642.11041899027\, e^{21t} - 29462.5590324569\, e^{20t} + 98445.457843787\, e^{19t}\\ &- 286813.415939263\, e^{18t} + 731812.465393484\, e^{17t} - 1640320.39584424\, e^{16t}\\ &+ 3235913.1642476\, e^{15t} - 5622456.95956756\, e^{14t} + 8601078.32479796\, e^{13t}\\ &- 11566496.7869567\, e^{12t} + 13634033.504754\, e^{11t} - 14024612.0140905\, e^{10t}\\ &+ 12508945.7012303\, e^{9t} - 9587975.369255\, e^{8t} + 6237942.34911554\, e^{7t}\\ &- 3386582.34819993\, e^{6t} + 1498302.33048981\, e^{5t} - 522349.847436787\, e^{4t}\\ &+ 136555.704864704\, e^{3t} - 24747.1595899454\, e^{2t} + 2700.48555324498\, e^{t}\\ &- 1.0\, t - 128.401784724185.\end{aligned}$$

In the next experiments, we fix $N = 15$ and use $r = 5$ iterations and see the impact of utilizing different parameter μ on numerical solutions. We employ $\mu = 0, 1, \ldots, 5$ except $\mu = 1$ that previously considered in Table 1. Table 2 demonstrates the numerical solutions obtained via exponential-QLM at some points $t \in [0, 1]$.

Comparisons are provided in Table 3 to show the validity and accuracy of the proposed exponential-QLM. For this purpose, we report the numerical solutions obtained by the state-of-the-art analytical and computational procedures. These methods include the reproducing kernel algorithm (RKA) [2], the predictor-corrector PECE method of Adams-Bashforth-Moulton type described in [8] using step size $h = 1/2000$, and the rationalized Haar wavelets method (RHWM) [11]. We run exponential-QLM when $N = 17$, $\mu = 1$, and

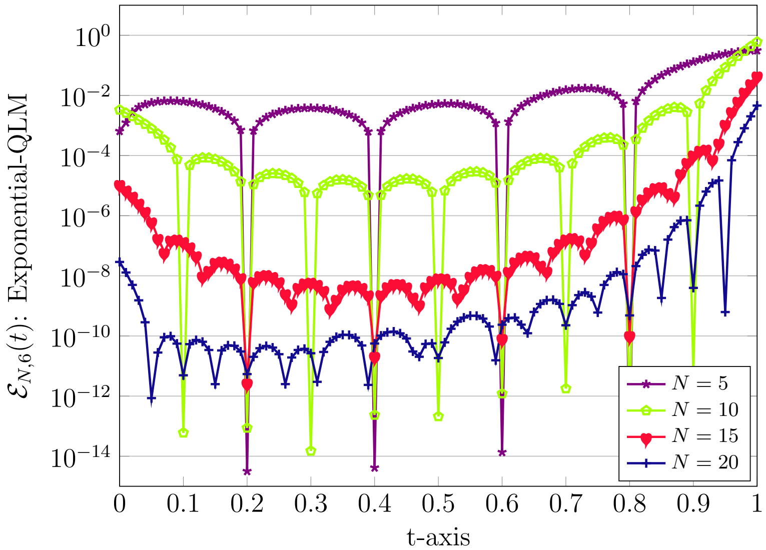

FIGURE 2. Comparison of the error functions obtained in exponential-QLM with $r = 5$ and for various $N = 5, 10, 15$, and 20.

t	$\mu = 0$	$\mu = 2$	$\mu = 3$	$\mu = 4$	$\mu = 5$
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1000500063	0.1003834504	0.1005501716	0.1007168922	0.1008836123
0.2	0.2008018149	0.2034770216	0.2048146459	0.2061522841	0.2074899363
0.3	0.3040814279	0.3131803345	0.3177303788	0.3222808171	0.3268316494
0.4	0.4130378202	0.4349386089	0.4458950526	0.4568555314	0.4678200468
0.5	0.5324019325	0.5763029198	0.5982907301	0.6203034587	0.6423411359
0.6	0.6690262707	0.7480101005	0.7876711366	0.8274453841	0.8673331923
0.7	0.8329100132	0.9660056560	1.0331806389	1.1007775270	1.1687991966
0.8	1.0391044229	1.2553048998	1.3654431987	1.4769650134	1.5898893214
0.9	1.3113237258	1.6577236396	1.8370251653	2.0205342522	2.2083597438
1.0	1.6891104843	2.2484822759	2.5457250670	2.8553980162	3.1780826167

TABLE 2. Comparison of numerical solutions in exponential-QLM with $N = 15$, $r = 5$, and different $\mu = 0, 2, 3, 4, 5$.

$r = 5$. It can be seen from Table 3 that the solutions of initial value problem (7) match between six to eight decimal places with the RKA and PECE.

To further show the advantage of the exponential-QLM proposed in this paper and validate our results, we now present comparison experiments for the first Painlevé equation at different t in $[0, 1]$ in terms of the magnitude of errors. For comparison, the following numerical methods are used, see also [2]. They are the RKA [2], the varational iteration method (VIM) [15], the homotopy perturbation method (HPM) [15], the homotopy analysis method (HAM) [15], the particle swarm optimization algorithm (PSOA) [28], and the neural networks algorithm (NNA) [29]. The numerical errors achieved by (17) while using $N = 10$ and $N = 20$ are shown in Table 4. We emphasize that the values of errors at the endpoints $t = 0$ and $t = 1$ may be pessimistic; this fact can be concluded from the results shown in Tables 1 and 3, which are in good agreement with other well-established

t	EXP-QLM	RHWM	RKA	PECE
0.0	0.000000000000000	–	0.0000000000000000	0.00000000000000
0.1	**0.10021674**7516847	**0.10021**1	**0.10021674**67681712	**0.10021674**893349
0.2	**0.2021394**52369758	**0.2021**25	**0.2021394**491589076	**0.2021394**5784765
0.3	**0.30863074**8625048	**0.3086**71	**0.30863074**10263387	**0.3086307**6126022
0.4	**0.4239862**88730206	**0.42**4303	**0.4239862**750679019	**0.423986**31278537
0.5	**0.554340**117984486	**0.5**17173	**0.554340**0973632109	**0.554340**16011750
0.6	**0.7084620**86708691	–	**0.7084620**572155395	**0.708462**15823036
0.7	**0.899249**936218413	**0.89**4911	**0.899249**8909209865	**0.8992**5005749290
0.8	**1.146531**723985065	–	**1.146531**6432231625	**1.146531**93362504
0.9	**1.482524**426463903	**1.4**77958	**1.482524**2507589982	**1.482524**80271840
1.0	**1.963127**374898169	–	**1.963127**6465421460	**1.96312**892826707

TABLE 3. Comparison of various numerical results with exponential-QLM for $N = 17$ and $\mu = 1$.

schemes. Finally, the logarithmic graph of the absolute coefficients of exponential functions obtained via the exponential-QLM using $N = 20$ and $N = 50$ are plotted in Fig. 3. These graphs indicate that the new proposed scheme has an appropriate convergence rate.

t	$N = 10$	$N = 20$	RKA	VIM	HPM	HAM	PSOA	NNA
0.0	3.31_{-03}	2.89_{-08}	0	0	0	0	0	0
0.1	6.03_{-14}	4.92_{-12}	1.32_{-10}	1.35_{-08}	7.96_{-10}	8.00_{-10}	1.05_{-03}	6.15_{-06}
0.2	8.72_{-14}	5.39_{-12}	4.74_{-09}	1.85_{-06}	4.88_{-09}	1.19_{-09}	8.05_{-04}	2.58_{-06}
0.3	1.50_{-14}	2.66_{-11}	1.38_{-08}	3.20_{-05}	2.22_{-07}	5.62_{-09}	6.71_{-04}	2.00_{-06}
0.4	2.35_{-13}	5.61_{-11}	2.48_{-08}	2.45_{-04}	3.94_{-06}	1.12_{-08}	6.39_{-04}	2.21_{-06}
0.5	2.11_{-13}	1.85_{-11}	4.42_{-08}	1.20_{-03}	3.79_{-05}	5.31_{-08}	6.79_{-04}	1.17_{-06}
0.6	1.21_{-12}	2.36_{-10}	7.41_{-08}	4.50_{-03}	2.45_{-04}	6.38_{-07}	7.72_{-04}	4.55_{-06}
0.7	1.78_{-12}	2.27_{-10}	1.22_{-07}	1.40_{-03}	1.21_{-03}	7.55_{-06}	9.10_{-04}	4.05_{-06}
0.8	2.88_{-13}	4.83_{-10}	2.06_{-07}	3.84_{-02}	4.97_{-03}	6.89_{-05}	1.07_{-03}	8.42_{-06}
0.9	6.30_{-12}	3.97_{-09}	3.84_{-07}	9.63_{-02}	1.78_{-02}	5.02_{-04}	1.29_{-03}	8.85_{-06}
1.0	6.05_{-01}	4.60_{-03}	9.14_{-07}	2.27_{-01}	5.74_{-02}	3.07_{-03}	1.98_{-03}	4.13_{-05}

TABLE 4. Comparison of error functions in exponential-QLM for $N = 10, 20$ and absolute errors in various numerical schemes when $\mu = 1$.

5. CONCLUSIONS

In this article, a new numerical matrix technique in terms of exponential functions, which is based on collocation points is developed for the approximate solution of first Painlevé differential equation. The direct application of this method termed exponential-collocation method is transformed the model under consideration into a nonlinear matrix equation, which may be solved inefficiently when a large number of exponential functions is used. To get ride of the nonlinearity, a variant of this algorithm based on

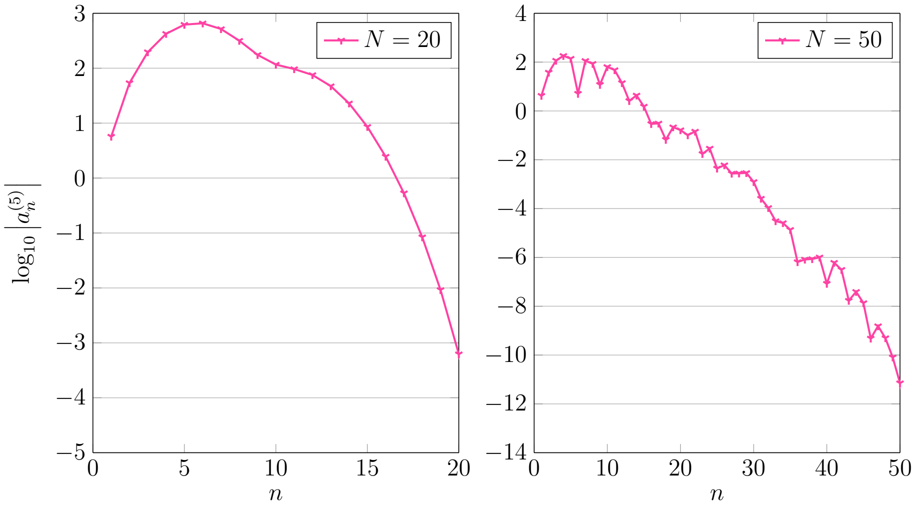

FIGURE 3. Logarithmic graph of absolute coefficients $|a_n^{(5)}|$ in exponential-QLM using $N = 20$ (left) and $N = 50$ (right) with $r = 5, \mu = 1$.

quasi-linearization methodology, i.e., the exponential-QLM is then presented to solve the Painlevé initial-value problem efficiently. Numerical examples are included to demonstrate the validity and applicability of the combined collocation and quasi-linearization techniques and comparisons are made with existing well-known results. Moreover, since the exact solution of this problem is not yet known, an error analysis technique based on residual function is developed.

On the basis of the simulations and their comparative studies provided in the last section, it can be concluded that the first Painlevé equation can be solved effectively by the exponential-QLM. Indeed, six to eight digit agreement was found between solutions obtained by the exponential-QLM and the analytical and computational techniques of RKA and PECE. A main benefit of the proposed scheme is that the solutions are obtained very easily by means of today's modern mathematical softwares such as MATLAB, MAPLE. and MATHEMATICA.

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