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A COMPARATIVE STUDY OF NUMERICAL SOLUTION OF PANTOGRAPH EQUATIONS USING VARIOUS WAVELETS TECHNIQUES

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Abstract. The objective of the present article is to discuss a numerical method based on wavelets for finding the solution of pantograph differential equations with proportional delays. First, the pantograph differential equation is converted into system of linear algebraic equations and then unknown coefficients are induced by solving the linear system. The convergence of the approximate solution is also derived along with its error estimate. Some numerical examples are considered to demonstrate the superiority of Bernoulli wavelet over Haar, Chebyshev and Legendre wavelets etc.

Keywords: Pantograph equation, delay differential equation, Bernoulli wavelet, collocation point.

AMS Subject Classification: 65L05, 65L10, 65L20, 34K28.

1. Introduction

The pantograph equation is a special type of functional differential equations with proportional delay. It arises in different fields of pure and applied mathematics such as quantum mechanics, control system, probability theory, number theory, nonlinear dynamical systems, cell growth and electrodynamics etc. The name pantograph originated from the work of Ockendon and Tayler on the collection presented by pantograph head of an electric locomotive. Since 1990s, there has been a growing interest in the numerical treatment of pantograph equations of the retarded and advanced type. In this paper, we discuss about the numerical solution of generalized pantograph differential equation given by [[1]-[4]]

$$y^{(n)}(z) = \sum_{j=1}^{J} \sum_{k=0}^{n-1} A_{jk}(z) y^{(k)}(\alpha_j z + \beta_j) + f(z), \ z \in [0, \ T], \tag{1}$$

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with boundary conditions

$$\sum_{k=0}^{n-1} d_{jk}(z)y^{(k)}(0) = \delta_j,\ j = 0, 1, 2, ..., (n-1), \tag{2}$$

$$y(T) = \eta, \tag{3}$$

where $T > 0$, $f(z)$ and $A_{jk}(z)$ are continuous functions in the given interval and $d_{jk}, \alpha_j, \beta_j$ and δ_j are finite real or complex constants. A special feature of this equation is the existence of compactly supported solution. This phenomenon is studied in the article [5] and has a direct application to the approximation theory and wavelets [6]. These equations arise in industrial applications also.

In the recent years, several numerical methods for approximating the solution of pantograph differential equations have been established. Many authors have tried to find the solutions of pantograph differential equations using various techniques such as by Taylor polynomials [[7]-[10]], perturbation-iteration algorithms [11], Laguerre series [12], Walsh stretch matrices and functional differential equation [13], Bernstein polynomials [14], polynomial interpolation [15], Spline functions approximation [16], Adomian decomposition method [[17], [18]], Hermite interpolation [19], collocation method [[20], [21]] Chebyshev polynomials [22], Legendre polynomial approximation [23], differential transform method [24], block-pulse functions and Bernstein polynomials [25], Variational iteration method(VIM) [[26], [27]], Jacobi rational-Gauss collocation (JRC) [28], successive interpolations [29] etc.. Methods based on the wavelets are more attractive and considerable. Various wavelets techniques are applied in order to solve the equation (1) namely, Wavelets [30], Chebyshev wavelets [[31], [32]], Hermite wavelets [[33], [34]], Legendre wavelets method [[35], [36]] etc.

The rest of the discussion is summarized as follows: In section 2, we mention the short introduction about Haar, Chebyshev, Legendre and Bernoulli wavelets. The detailed methodology for solving the pantograph equations is discussed in the section 3. In the next section, we discuss the convergence analysis and error estimate for Bernoulli wavelets. Before the final section, we presented some numerical examples for comparative study considered technique along with the some available techniques in the literature. Finally, we give the concluding remarks.

2. Wavelets

Wavelets are a family of functions assembled from dilation and translation of a special function known as the mother wavelet and denoted as $\psi(z)$. When the parameters u and v vary, we obtain the family of continuous wavelets as follows [38]:

$$\psi_{uv}(z) = \frac{1}{\mid u \mid^{1/2}} \psi\left(\frac{z-v}{u}\right), u, v \in R,\ u \neq 0, \tag{4}$$

where u is the dilation parameter and v is the translation parameter. If we bound the variables u and v to be discontinuous values as $u = u_0^{-p}$, $v = lv_0u_0^{-p}$, $u_0 > 1$, $v_0 > 0$ where $p, l \in \mathbb{Z}$ are positive integers, we get the family of discrete wavelets as follows:

$$\psi_{pl}(z) = \mid u_0 \mid^{p/2} \psi(u_0^p z - lv_0), \tag{5}$$

where $\psi_{pl}(z)$ forms a wavelet basis for $L^2(\mathbb{R})$.

Now we are going to give a brief review of Haar, Chebyshev, Legendre and Bernoulli wavelets denoted by HW, CW, LW and BW, respectively which will be used in the later discussions.

2.1. **Haar Wavelets (HW).** Let us consider the interval $z \in [a, b]$, where a and b are given constants. We shall define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[a, b]$ is partitioned into $2M$ subintervals of equal length; the length of each subinterval is $\Delta z = (b-a)/(2M)$. Next two parameters are introduced: the dilatation parameter $j = 0, 1, ..., J$ and the translation parameter $k = 0, 1, \cdots, m-1$ (here the notation $m = 2^J$ is introduced). The wavelet number i is identified as $i = m + k + 1$ and the h_i Haar wavelet is defined as [[41], [42]]

$$h_i(z) = \begin{cases} 1 & for \;\; z \in [\alpha, \beta)\,, \\ -1 & for \;\; z \in [\beta, \gamma)\,, \\ 0 & elsewhere, \end{cases} \tag{6}$$

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}, \quad i = 2, 3, ..., 2M. \tag{7}$$

The case $i = 1$ corresponds to the scaling function: $h_1(z) = 1$ for $z \in [a, b]$ and $h_1(z) = 0$ elsewhere.

If we want to solve a nth order pantograph equations, we need the following integrals

$$p_{v,i}(z) = \int_a^z \int_a^z \cdots \int_a^z h_i(x) dx^v = \frac{1}{(v-1)!} \int_a^z (z-x)^{v-1} h_i(x) dx, \tag{8}$$

where $v = 1, 2, \cdots, n$, $i = 1, 2, \cdots, 2M$. The case $v = 0$ corresponds to function $h_i(z)$. Taking account of the eqn. (6) these integrals can be calculated analytically; by doing it we obtain

$$p_{t,i}(z) = \begin{cases} 0 & for \; z < \alpha, \\ \frac{1}{t!}[z-\alpha]^t & for \; z \in [\alpha, \beta], \\ \frac{1}{t!}\left\{[z-\alpha]^t - 2[z-\beta]^t\right\} & for \; z \in [\beta, \gamma], \\ \frac{1}{t!}\left\{[z-\alpha]^t - 2[z-\beta]^t + [z-\gamma]^t\right\} & for \; z > \gamma. \end{cases} \tag{9}$$

These formulas hold for $i > 1$. In case $i = 1$, we have $\alpha = a$, $\beta = \gamma = b$ and

$$p_{t,i}(z) = \frac{1}{t!}(z-a)^t. \tag{10}$$

In the present paper the collocation method for solving the pantograph equations is applied. The collocation points are $z_l = 0.5[\widetilde{z}_{l-1} + \widetilde{z}_l]$, $l = 1, 2, \cdots, 2M$ the symbol $\widetilde{z}_l$ denotes the lth grid point $\widetilde{z}_l = a + l\Delta z$, $l = 1, 2, ..., 2M$.

2.2. **Chebyshev Wavelets (CW).** Here we describe the four kinds of Chebyshev wavelets which are named accordingly the Chebyshev polynomials.

a. First Kind Chebyshev Wavelets (FSTCW): The FSTCW $\psi_{p,q} = \psi(l, p, q, z)$ have four arguments, $q = 1, 2, \ldots, 2^{l-1}$, l positive integer, $p = 0, 1, \ldots, P-1$ and p is the order for Chebyshev polynomials of the first kind and z denotes the time. They are defined on the interval $[0, 1)$ as [31]

$$\psi_{qp}(z) = \begin{cases} 2^{l/2}\widetilde{X_p}(2^l z - 2q + 1), & \frac{q-1}{2^{l-1}} \le z < \frac{q}{2^{l-1}}, \\ 0, & elsewhere, \end{cases} \tag{11}$$

where

$$\widetilde{X_p}(z) = \begin{cases} \frac{1}{\sqrt{\pi}}, & p = 0, \\ \sqrt{\frac{2}{\pi}} X_p(z), & p > 0. \end{cases} \tag{12}$$

In the eqn. (12), $X_p(z)$ is well-known Chebyshev polynomials of the first kind with degree p which are orthogonal with respect to the weight function $w(z) = (1 - z^2)^{-1/2}$ within the interval $[-1, 1]$ and fulfilled the following recursive formula:

$$X_0(z) = 1,\ X_1(z) = z,\ X_{p+1}(z) = 2zX_p(z) - X_{p-1}(z),\ p = 1, 2, 3, ... \tag{13}$$

b. Second Kind Chebyshev Wavelets (SNDCW): The SNDCW $\psi_{p,q} = \psi(l, p, q, z)$ have four arguments where $q = 1, 2, \ldots, 2^{l-1}$, l is assumed to be any positive integer, $p = 0, 1, \ldots, P - 1$ and p is the order for second kind Chebyshev polynomials and z denotes the time. They are defined on interval $[0, 1)$ by [32]

$$\psi_{pq}(z) = \begin{cases} 2^{l/2}\widetilde{Y_p}(2^l z - 2q + 1), & \frac{q-1}{2^{l-1}} \leq z < \frac{q}{2^{l-1}}, \\ 0, & elsewhere, \end{cases} \tag{14}$$

where

$$\widetilde{Y_p}(z) = \sqrt{\frac{2}{\pi}} Y_p(z),\ p \geq 0. \tag{15}$$

In the eqn. (15), the coefficients are used for the orthonormality, $Y_p(z)$ is the second kind Chebyshev polynomials of degree p with respect to weight function $w(z) = \sqrt{1 - z^2}$ in the interval $[-1, 1]$ and satisfy the following recursive formula:

$$Y_0(z) = 1,\ Y_1(z) = 2z,\ Y_{p+1}(z) = 2zY_p(z) - Y_{p-1}(z),\ p = 1, 2, 3, ... \tag{16}$$

c. Third Kind Chebyshev Wavelets (TRDCW): The TRDCW expression is given by [37]

$$\psi_{qp}(z) = \begin{cases} 2^{l/2}\widetilde{Z_p}(2^l z - 2q + 1), & \frac{q-1}{2^{l-1}} \leq z < \frac{q}{2^{l-1}}, \\ 0, & elsewhere, \end{cases} \tag{17}$$

where

$$\widetilde{Z_p}(z) = \sqrt{\frac{1}{\pi}} Z_p(z),\ p \geq 0. \tag{18}$$

The coefficient in the expression (18) is for the orthonormality and third kind Chebyshev polynomials of degree p i.e. $Z_p(z)$ are orthogonal with respect to the weight function $w(z) = \frac{\sqrt{1+z}}{\sqrt{1-z}}$ in the interval $[-1, 1]$ and satisfy the following recursive formula

$$Z_0(z) = 1,\ Z_1(z) = 2z - 1,\ Z_{p+1}(z) = 2zZ_p(z) - Z_{p-1}(z),\ p = 1, 2, 3, ... \tag{19}$$

d. Fourth Kind Chebyshev Wavelets (FTHCW) : The FTHCW is given by [37]

$$\psi_{qp}(z) = \begin{cases} 2^{l/2}\widetilde{\chi_p}(2^l z - 2q + 1), & \frac{q-1}{2^{l-1}} \leq z < \frac{q}{2^{l-1}}, \\ 0, & elsewhere, \end{cases} \tag{20}$$

where

$$\widetilde{\chi_p}(z) = \sqrt{\frac{1}{\pi}} \chi_p(z),\ p \geq 0. \tag{21}$$

The coefficient in the relation (21) is used for the orthonormality, $W_p(z)$ is fourth kind Chebyshev polynomials of degree p, which are orthogonal with respect to weight function $w(x) = \frac{\sqrt{1-z}}{\sqrt{1+z}}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$\chi_0(z) = 1,\ \chi_1(z) = 2z + 1,\ \chi_{p+1}(z) = 2z\chi_p(z) - \chi_{p-1}(z),\ p = 1, 2, 3, ... \tag{22}$$

2.3. **Legendre Wavelets (LW).** The LW $\psi_{qp}(z) = \psi(l, q, p, z)$ has four arguments $q = 1, 2, ..., 2^{l-1}$, where l is positive integer, $p = 0, 1, ..., P-1$ and $\zeta_p(z)$ are the Legendre polynomials of degree p and z denotes time. They are defined on the interval $[0, 1)$ as [[35], [36]],

$$\psi_{qp}(z) = \begin{cases} \sqrt{p+\frac{1}{2}}2^{\frac{k}{2}}\zeta_p(2^l z - 2q + 1), & \frac{q-1}{2^{l-1}} \leq z < \frac{q}{2^{l-1}}, \\ 0, & elsewhere. \end{cases} \tag{23}$$

The weight function $w(z) = 1$ on the interval the interval $[-1, 1]$ and satisfy the following recurrence formulae:

$$\zeta_0(z) = 1,\ \zeta_1(z) = z,$$
$$\zeta_{p+1}(z) = \left(\frac{2p+1}{p+1}\right) z\zeta_p(z) - \left(\frac{p}{p+1}\right)\zeta_{p-1}(z),\ p = 1, 2, 3, ...$$

2.4. **Bernoulli Wavelets (BW).** The BW $\psi_{qp}(z) = \psi(l, q, p, z)$ have four arguments, $q = 1, 2, ..., 2^{l-1}$, where l is positive integer. They are defined on the interval $[0, 1)$ as given in [38]

$$\psi_{qp}(z) = \begin{cases} 2^{\frac{l-1}{2}}\widetilde{b}_p(2^{l-1}z - q + 1) & \frac{q-1}{2^{l-1}} \leq z < \frac{q}{2^{l-1}}, \\ 0 & elsewhere, \end{cases} \tag{24}$$

with

$$\widetilde{b}_p(z) = \begin{cases} 1, & p = 0, \\ \frac{1}{\sqrt{\frac{(-1)^{p-1}(p!)^2}{(2p)!}a_{2p}}} b_p(z), & p > 0, \end{cases} \tag{25}$$

where $p = 0, 1, ..., P-1$, p is the degree of Bernoulli polynomials and z is normalized time. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{p-1}(p!)^2}{(2p)!}a_{2p}}}$ is for the orthonormality, $u = 2^{-(l-1)}$ is the dilation parameter and $v = (q-1)2^{-(l-1)}$ is the translation parameter. Here $b_p(z)$ are Bernoulli polynomials of degree p and can be determined by [[39], [40]]

$$b_p(z) = \sum_{j=0}^{p} \begin{pmatrix} p \\ j \end{pmatrix} a_{p-j} z^j, \tag{26}$$

where $a_j, j = 0, 1, 2, ..., p$. Now we mention some properties related to BW.

A. Properties of Bernoulli number: The sequence of Bernoulli numbers a_p, $p \in \mathbb{N}$ satisfies the following properties [[39], [40]]:

1. $a_{2p+1} = 0,\ a_{2p} = b_{2p}(1)$,
2. $b_p(1/2) = (2^{1-p} - 1)a_p$,
3. $a_p = -\frac{1}{p+1}\sum_{l=0}^{p-1} \begin{pmatrix} p+1 \\ l \end{pmatrix} a_l$.

B. Properties of Bernoulli's polynomial: Properties of Bernoulli polynomials are given as follows [[39], [40]]:

1. $b_p(1-z) = (-1)^p b_p(z)$, where p is positive integers,
2. $b_p^{'}(z) = p b_{p-1}(z)$, where p is positive integers,
3. $sup_{z\in[0,1]} \mid b_{2p}(z) \mid = \mid a_{2p} \mid$,
4. $\int_0^1 \mid b_p(z) \mid dz < 16\frac{p!}{(2\pi)^{p+1}},\ p \geq 0$,

5. $sup_{z\in[0,1]} \mid b_{2p+1}(z) \mid \leq \frac{2p+1}{4} \mid a_{2p} \mid$, where p is positive integers,
6. $\int_0^1 b_p(z)b_q(z) = (-1)^{p-1}\frac{p!q!}{(p+q)!}a_{p+q},\ p, q \geq 1$,
7. $\int_m^t b_p(z)dz = \frac{b_{p+1}(t)-b_{p+1}(m)}{(p+1)}$.

3. Formulation for the Solution of pantograph differential Equations

A function $y(z)$ defined over $[0, 1)$ can be expanded by CW, LW and BW as

$$y(z) = \sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z) = G^T\psi(z), \tag{27}$$

where G and ψ are $(2^{l-1}P \times 1)$ matrices given by

$$G = [g_{10}, g_{11}, \ldots, g_{1(P-1)}, g_{20}, \ldots, g_{2(P-1)}, \ldots, g_{2^{l-1}0}, \ldots, g_{2^{l-1}(P-1)}]^T,$$

$$\psi(z) = [\psi(z)_{10}, ..., \psi(z)_{1(P-1)}, \psi(z)_{20}, ..., \psi(z)_{2(P-1)}, ..., \psi(z)_{2^{l-1}0}, ..., \psi(z)_{2^{l-1}(P-1)}]^T.$$

Case 1. If the given equation is of order one then there is an initial condition, namely

$$\sum_{k=0}^{n-1} d_{jk}(z)y^{(k)}(0) = \delta_j,\ j = 0, 1, 2, ..., n-1, n = 1, \tag{28}$$

where $d_{jk} \neq 0$. Now total conditions will be reduced to $2^{l-1}P - 1$ to recover the unknown coefficients g_{qp}, which can be obtained by substituting expression (27) in the equation (1), that gives

$$\frac{d^n}{dz^n}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z) = \sum_{j=1}^{J}\sum_{k=0}^{n-1} A_{jk}(z)\frac{d^k}{dz^k}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(\alpha_j z + \beta_j) + f(z). \tag{29}$$

In the relation (29) replacing z by z_j, we get

$$\frac{d^n}{dz^n}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z_j) = \sum_{j=1}^{J}\sum_{k=0}^{n-1} A_{jk}(z_j)\frac{d^k}{dz^k}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(\alpha_j z_j + \beta_j) + f(z_j), \tag{30}$$

where $z_j's$ are collocation points given by

$$z_j = \frac{T(1 + cos\frac{(j-1)\pi}{2^{l-1}P})}{2},\ j = 2, 3, ..., 2^{l-1}P. \tag{31}$$

On combining equations (28) and (30), we get $2^{l-1}P$ linear equations from which one can get the unknown coefficients g_{qp}.

Case 2. If given equation is of order two then there is an initial condition, namely

$$\sum_{k=0}^{n-1} d_{jk}(x)y^{(k)}(0) = \delta_j,\ j = 0, 1, 2, ..., n-1, n = 2, \tag{32}$$

where $d_{jk} \neq 0$. Now total conditions will be reduced to $2^{l-1}P - 2$ to recover the unknown coefficients g_{qp}, which can be obtained by substituting equation (27) in the expression (1) as follows

$$\frac{d^n}{dz^n}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z) = \sum_{j=1}^{J}\sum_{k=0}^{n-1} A_{jk}(z)\frac{d^k}{dz^k}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(\alpha_j z + \beta_j) + f(z). \tag{33}$$

In the equation (33) replacing z by z_j, we get

$$\frac{d^n}{dz^n}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z_j) = \sum_{j=1}^{J}\sum_{k=0}^{n-1} A_{jk}(z_j)\frac{d^k}{dz^k}\sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(\alpha_j z_j+\beta_j)+f(z_j), \quad (34)$$

where $z_j's$ are collocation points defined as below

$$z_j = \frac{T(1+cos\frac{(j-1)\pi}{2^{l-1}P-1})}{2}, \; j=2,3,...,2^{l-1}P-1. \quad (35)$$

On combining equations (32) and (34), we get $2^{l-1}P$ linear equations from which we can obtain the unknown coefficients g_{qp}. Similarly, we can proceed for higher order also.

4. Convergence analysis and error estimate

In the following theorems, we derive the general formula for the upper bound of the coefficients and truncation error for Bernoulli wavelets.

Theorem 4.1. *If $y(z) \in L^2(\mathbb{R})$ be a continuous function defined on $[0,1]$ and $\mid f(z) \mid \leq M$, then the BW expansion of $y(z)$ defined in the equation (27) converges uniformly and*

$$g_{qp} < \frac{MB}{2^{\frac{l-1}{2}}}\frac{16p!}{(2\pi)^{p+1}},$$

where $B = \frac{1}{\sqrt{\frac{(-1)^{p-1}(p!)^2}{(2p)!}a_{2p}}}$.

Proof. Any function $y(z) \in L^2(\mathbb{R})$ can be expressed as the BW as

$$y(z) = \sum_{q=1}^{2^{l-1}}\sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z),$$

where the coefficient g_{qp} can be resolved as

$$g_{qp} = \langle y(z), \psi(z)\rangle.$$

Now for $p > 0$,

$$\begin{aligned} g_{qp} &= \langle y(z), \psi(z)\rangle = \int_0^1 y(z)\psi_{qp}(z)dz \\ &= 2^{\frac{l-1}{2}}B\int_{\frac{q-1}{2^{l-1}}}^{\frac{q}{2^{l-1}}} y(z)b_p(2^{l-1}z-q+1)dz. \end{aligned}$$

Changing the variable $2^{l-1}z-q+1 = x$, we get

$$g_{qp} = \frac{1}{2^{\frac{l-1}{2}}}B\int_0^1 y(\frac{x+q-1}{2^{l-1}})b_p(x)dx,$$

$$\begin{aligned} \mid g_{qp} \mid &\leq \frac{B}{2^{\frac{l-1}{2}}}\int_0^1 \mid y(\frac{x+q-1}{2^{l-1}}) \mid\mid b_p(x) \mid dx \\ &\leq \frac{B}{2^{\frac{l-1}{2}}}M\int_0^1 \mid b_p(x) \mid dx, \\ &\leq \frac{B}{2^{\frac{l-1}{2}}}M\frac{16p!}{(2\pi)^{p+1}}, \end{aligned}$$

by using the property of Bernoulli polynomials. This means that the series $\sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z)$ is absolutely convergent and hence the BW expansion is uniformly convergent [39].

□

Theorem 4.2. *Let* $y(z) = \sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} g_{qp}\psi_{qp}(z)$ *be truncated series, then the truncation error* $E_{qp}(z)$ *can be derived as*

$$\| E_{qp}(z) \|_2^2 \leq \sum_{q=2^l}^{\infty} \sum_{p=P}^{\infty} \left(\frac{B}{2^{\frac{l-1}{2}}} . \frac{16p!}{(2\pi)^{p+1}} \right)^2 .$$

Proof. Any function $y(z)^* \in L^2(\mathbb{R})$ can be expressed by the BW as

$$y(z)^* = \sum_{q=1}^{\infty} \sum_{p=0}^{\infty} g_{qp}\psi_{qp}(z).$$

If $y(z)$ be a truncated series, then the truncation error term can be determined as

$$E_{qp}(z) = y(z)^* - y(z) = \sum_{q=2^{l-1}}^{\infty} \sum_{p=P}^{\infty} g_{qp}\psi_{qp}(z).$$

Now,

$$\begin{aligned} \| E_{qp}(z) \|_2^2 &= \| \sum_{q=2^l}^{\infty} \sum_{p=P}^{\infty} g_{qp}\psi_{qp}(z) \|_2^2 \\ &\leq \int_0^1 | \sum_{q=2^l}^{\infty} \sum_{p=P}^{\infty} g_{qp}\psi_{qp}(z) |^2 \, dz \\ &\leq \sum_{q=2^l}^{\infty} \sum_{p=P}^{\infty} \left(\frac{B}{2^{\frac{l-1}{2}}} . \frac{16p!}{(2\pi)^{p+1}} \right)^2 \int_0^1 | \psi_{qp}(z) |^2 \, dz \\ &\leq \sum_{q=2^l}^{\infty} \sum_{p=P}^{\infty} \left(\frac{B}{2^{\frac{l-1}{2}}} . \frac{16p!}{(2\pi)^{p+1}} \right)^2 , \end{aligned}$$

by orthonormality ($\int_0^1 | \psi_{qp}(z) |^2 \, dz = 1$). □

5. Numerical Examples

In this section, we consider seven numerical problems for testing. The main advantage of the Bernoulli wavelets method is that more accurate results are achieved as compared to some exiting techniques. The numerical results obtained by the proposed method are also in good agreement with the exact solutions available in the literature. This scheme is easy to implement in the computer programs.

Example 5.1: Consider the following first-order pantograph equation [27]:

$$y'(z) = -y(z) + \frac{1}{2}y\left(\frac{z}{2}\right) + \frac{1}{2}y'\left(\frac{z}{2}\right); \; 0 < z \leq 1, \tag{36}$$

with initial condition

$$y(0) = 1. \tag{37}$$

The exact solution is

$$y(z) = e^{-z}. \tag{38}$$

Equation (36) came in the picture when the British Railways wanted to make the electric locomotive faster. An important construct was the pantograph, which collects current from an overhead wire which can be seen in the references [[10], [21]], [27], [44]].

Table 1 compares the absolute error for Example 5.1 along with the some existing method discussed in the article [27]. Here, TSORKM means two-stage oder-one Runge-Kutta method; OLM means one-leg $\theta-$ method; VIM means variational iteration method and BW means Bernoulli wavelets technique. Table 2 contains the absolute errors derived from Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW) techniques. The tables confirm that Bernoulli wavelets technique extracts more accurate results in most of the cases.

Table 1. Comparison of the absolute error for Example 5.1

z	TSORKM [27]	OLM [27] $\theta = 0.8$	VIM [27] N=7	VIM [27] N=8	BW l=1, P=5	BW l=1, P=6
0.1	4.55E-04	2.57E-03	7.43E-04	3.72E-04	5.21E-05	2.52E-06
0.2	8.24E-04	8.86E-03	1.42E-03	7.08E-04	5.67E-05	2.37E-06
0.3	1.12E-03	1.72E-02	2.02E-03	1.01E-03	5.04E-05	2.26E-06
0.4	1.35E-03	2.66E-02	2.58E-03	1.29E-03	4.80E-05	2.46E-06
0.5	1.52E-03	3.63E-02	3.07E-03	1.54E-03	5.07E-05	2.47E-06
0.6	1.66E-03	4.58E-02	3.52E-03	1.76E-03	5.29E-05	2.10E-06
0.7	1.75E-03	5.47E-02	3.93E-03	1.97E-03	4.86E-05	1.65E-06
0.8	1.81E-03	6.29E-02	4.30E-03	2.15E-03	3.72E-05	1.73E-06
0.9	1.84E-03	7.02E-02	4.64E-03	2.32E-03	2.86E-05	2.32E-06
1.0	1.85E-03	7.66E-02	4.94E-03	2.47E-03	4.84E-05	1.48E-06

Table 2. Comparison of the absolute error for Example 5.1 ($l = 1$, $P = 10$)

z	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW	HW(J=3)
0.1	2.06257E-12	2.06268E-12	2.06257E-12	2.06279E-12	2.06235E-12	1.15352E-13	1.268E-03
0.2	2.47191E-12	2.47202E-12	2.47180E-12	2.47213E-12	2.47158E-12	8.32667E-14	1.328E-03
0.3	2.07989E-12	2.07989E-12	2.07989E-12	2.08011E-12	2.07967E-12	2.62013E-14	1.345E-03
0.4	2.21700E-12	2.21712E-12	2.21712E-12	2.21734E-12	2.21689E-12	5.14588E-13	1.301E-03
0.5	2.41363E-12	2.41374E-12	2.41374E-12	2.41385E-12	2.41351E-12	3.23519E-13	3.708E-04
0.6	1.61127E-12	1.61149E-12	1.61138E-12	1.61171E-12	1.61127E-12	8.50986E-13	1.188E-03
0.7	1.47499E-12	1.47515E-12	1.47521E-12	1.47538E-12	1.47504E-12	1.00403E-12	1.117E-03
0.8	2.08494E-12	2.08522E-12	2.08517E-12	2.08533E-12	2.08505E-12	9.57734E-13	9.862E-04
0.9	1.59939E-12	1.59961E-12	1.59972E-12	1.59989E-12	1.59950E-12	9.19709E-13	7.929E-04
1.0	1.50774E-12	1.50796E-12	1.50813E-12	1.50829E-12	1.50796E-12	1.88233E-12	6.752E-05

Example 5.2: Consider the linear delay differential equations of first order [27]:

$$y'(z) = -y(z) + 0.1y(0.8z) + 0.5y'(0.8z) + (0.32z - 0.5)e^{-0.8z} + e^{-z}, \ z \geq 0, \tag{39}$$

with initial condition

$$y(0) = 0. \tag{40}$$

The exact solution is $y(z) = ze^{-z}$. Equation (39) arises in rather different fields of pure and applied mathematics such as electrodynamics, control systems, number theory, probability, and quantum mechanics [[10], [21]], [27], [44]].

Table 3 and 4 compare the absolute errors for Example 5.2 derived from Bernoulli wavelets (BW) technique along with the some discussed techniques in the reference [27] and Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW), respectively. The tables confirm that BW technique extracts more accurate results in most of the cases.

TABLE 3. Comparison of the absolute error for Example 5.2

z	TSORKM [27]	OLM [27] $\theta = 0.8$	VIM [27] N=5	VIM [27] N=6	BW l=1, P=5	BW l=1, P=6
0.1	8.68E-04	4.65E-03	2.62E-03	1.30E-03	1.46E-04	7.78E-06
0.2	1.49E-04	1.45E-02	4.36E-03	2.14E-04	9.40E-05	2.29E-06
0.3	1.90E-03	2.57E-02	5.40E-03	2.63E-03	1.68E-05	2.45E-07
0.4	2.16E-03	3.60E-02	5.89E-03	2.84E-03	1.51E-05	1.29E-06
0.5	2.28E-03	4.43E-02	5.96E-03	2.83E-03	1.04E-06	3.15E-06
0.6	2.31E-03	5.30E-02	5.71E-03	2.67E-03	3.40E-05	2.70E-06
0.7	2.27E-03	5.37E-02	5.23E-03	2.39E-03	4.74E-05	6.53E-07
0.8	2.17E-03	5.47E-02	4.59E-03	2.04E-03	2.54E-05	6.03E-08
0.9	2.03E-03	5.35E-02	3.84E-03	1.64E-03	4.44E-06	1.61E-06
1.0	1.86E-03	5.03E-02	3.04E-03	1.22E-03	4.92E-05	1.91E-06

Example 5.3: Consider the pantograph equations of second order [27]:

$$y''(z) = y'\left(\frac{z}{2}\right) - \frac{1}{2}zy''\left(\frac{z}{2}\right) + 2,\ 0 < z < 1, \tag{41}$$

with initial conditions

$$y(0) = 1,\ y'(0) = 0, \tag{42}$$

the exact solution is

$$y(z) = 1 + z^2. \tag{43}$$

Equation (41) has gained more interest in many application fields such a biology, physics, engineering, economy, electrodynamics [[10], [21]], [27], [44]]. On solving equation (41) by using the method defined in section 3 at $l = 1$, $P = 5$, we obtain the solution $y(z) = 1 - 2.22045 \times 10^{-16}z + z^2$, which is similar to the exact solution.

Example 5.4: Consider the pantograph equations of second order [27]:

$$y''(z) = \frac{3}{4}y(z) + y\left(\frac{z}{2}\right) + y'\left(\frac{z}{2}\right) + \frac{1}{2}y''\left(\frac{z}{2}\right) - z^2 - z + 1,\ 0 < z < 1, \tag{44}$$

with initial conditions

$$y(0) = 0,\ y'(0) = 0. \tag{45}$$

This type of pantograph functional-differential equations play an important role in the mathematical modeling of real world phenomena [[10], [21]], [27]] and Obviously, most of these equations cannot be solved exactly.

The known exact solution of the problem is $y(z) = z^2$. Table 5 and 6 include the absolute errors for Example 5.4 along with the some well discussed methods in the reference [27] and Bernoulli wavelets (BW) and Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW), respectively which concludes that Bernoulli wavelets gives comparatively better results.

Table 4. Comparison of the absolute error for Example 5.2 ($l = 1$, $P = 5$)

z	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW	HW(J=2)
0.1	1.4640395993828E-04	1.4640395993833E-04	1.4640395993833E-04	1.4640395993838E-04	1.4640395993841E-04	1.4640395993827E-04	3.879E-03
0.2	9.3954070086000E-05	9.3954070086028E-05	9.3954070086000E-05	9.3954070086055E-05	9.3954070086083E-05	9.3954070085972E-05	8.228E-03
0.3	1.6783585469693E-05	1.6783585469749E-05	1.6783585469749E-05	1.6783585469804E-05	1.6783585469860E-05	1.6783585469692E-05	1.284E-02
0.4	1.5149167490180E-05	1.5149167490070E-05	1.5149167490125E-05	1.5149167490070E-05	1.5149167489958E-05	1.5149167490180E-05	1.584E-02
0.5	1.0419012285157E-06	1.0419012286822E-06	1.0419012286267E-06	1.0419012286267E-06	1.0419012287377E-06	1.0419012285156E-06	1.371E-02
0.6	3.4036391871495E-05	3.4036391871495E-05	3.4036391871439E-05	3.4036391871439E-05	3.4036391871606E-05	3.4036391871438E-05	1.247E-02
0.7	4.7406614404832E-05	4.7406614404999E-05	4.7406614404832E-05	4.7406614404832E-05	4.7406614405055E-05	4.7406614404721E-05	1.097E-02
0.8	2.5369480115389E-05	2.5369480115500E-05	2.5369480115334E-05	2.5369480115334E-05	2.5369480115667E-05	2.5369480115278E-05	9.795E-03
0.9	4.4353577022260E-06	4.4353577019485E-06	4.4353577021705E-06	4.4353577022815E-06	4.4353577018930E-06	4.4353577021205E-06	9.255E-03
1.0	4.9191658670567E-05	4.9191658670789E-05	4.9191658670511E-05	4.9191658670456E-05	4.9191658670900E-05	4.9191658670455E-05	9.428E-03

Table 5. Comparison of the absolute error for Example 5.4

z	TSORKM [27]	OLM [27] $\theta = 0.8$	VIM [27] N=5	VIM [27] N=6	BW l=1, P=3	BW l=1, P=4
0.1	1.00E-03	6.10E-03	3.34E-04	1.67E-04	2.43E-17	1.09E-16
0.2	2.02E-03	2.58E-02	1.43E-03	7.15E-04	2.08E-17	1.04E-16
0.3	3.07E-03	6.47E-02	3.45E-03	1.73E-03	6.94E-17	1.25E-16
0.4	4.17E-03	1.37E-01	6.58E-03	3.30E-03	1.67E-16	8.33E-17
0.5	5.34E-03	2.81E-01	1.11E-02	5.55E-03	2.78E-16	5.55E-17

Table 6. Comparison of the absolute error for Example 5.4 ($l = 1, P = 6$)

z	LW	FSTCHWs	SNDCHWs	THDCHWs	FTHCHWs	BW	HW(J=2)
0.1	1.65E-16	2.15E-16	6.77E-17	9.37E-17	1.42E-16	4.86E-17	1.06E-05
0.2	1.60E-16	1.74E-16	8.33E-17	1.18E-16	1.32E-16	4.16E-17	6.51E-05
0.3	1.25E-16	1.11E-16	1.39E-16	1.80E-16	1.67E-16	5.55E-17	2.45E-04
0.4	1.67E-16	8.33E-17	1.39E-16	1.94E-16	1.39E-16	2.76E-17	7.13E-04
0.5	1.67E-16	5.55E-17	1.67E-16	2.78E-16	1.39E-16	5.55E-17	1.67E-03

Example 5.5: Consider the pantograph equations of third order [27]:

$$y'''(z) = y(z) + y'\left(\frac{z}{2}\right) + y''\left(\frac{z}{3}\right) + \frac{1}{2}y'''\left(\frac{z}{4}\right) - z^4 - \frac{z^3}{2} - \frac{4}{3}z^2 + 21z,\ 0 < z < 1, \tag{46}$$

with initial conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0, \tag{47}$$

the exact solution of the above equation is

$$y(z) = z^4. \tag{48}$$

This type of equation occurs in different fields of science, engineering and industrial applications, for more details one can see the references [[10], [27], [28]]

Table 7 and 8 contain the absolute errors for Example 5.5 obtained by considered technique along with the method presented in the article [27] and Bernoulli wavelets (BW), Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW), respectively, which conclude that Bernoulli wavelets gives comparatively better results.

Example 5.6: Consider the pantograph equation of fourth order [28]:

$$\begin{aligned} y^{(4)}(z) \;=\; & y'''\left(\frac{z}{4}\right) + zy''(2z) - y'(z) - y\left(\frac{z}{2}\right) + e^{-2z}\{e^{3z/2}\sin(z) \\ & + e^{7z/4}\left(2\cos\left(\frac{z}{2}\right) - 11\sin\left(\frac{z}{2}\right)\right) \\ & + 2e^{z}(13\cos(2z) - 4\sin(2z)) + z(3\sin(4z) + 4\cos(4z))\},\ 0 \le z \le 1, \end{aligned} \tag{49}$$

with initial conditions

$$y(0) = 0,\ y'(0) = 2,\ y''(0) = -4,\ y'''(0) = -2. \tag{50}$$

The exact solution is $y(z) = e^{-z}\sin(2z)$. This type of pantograph equations are in different fields such as quantum mechanics, probability theory, astrophysics, cell growth [[10], [28]].

Table 9 and 10 compare the absolute errors for Example 5.6 obtained by Bernoulli wavelets along with the method JRCM given in the reference [28] and Bernoulli wavelets

Table 7. Comparison of the absolute error for Example 5.5

z	TSORKM [27]	VIM [27] N=5	VIM [27] N=6	BW l=1, P=5	BW l=1, P=6
0.1	4.97E-05	3.07E-09	9.09E-12	9.39E-17	3.26E-17
0.2	4.43E-04	5.04E-08	2.98E-10	4.56E-16	1.10E-16
0.3	1.57E-03	2.62E-07	2.33E-09	8.19E-16	2.67E-16
0.4	3.85E-03	8.49E-07	1.01E-08	1.17E-15	4.27E-16
0.5	7.78E-03	2.13E-06	3.20E-08	1.53E-15	6.04E-16
0.6	1.39E-02	4.55E-06	8.24E-08	1.92E-15	7.77E-16
0.7	2.28E-03	8.69E-06	1.85E-07	2.44E-15	1.14E-15
0.8	3.53E-02	1.53E-05	3.76E-07	2.72E-15	1.22E-15
0.9	5.19E-02	2.54E-05	7.09E-07	3.11E-15	1.33E-15
1.0	7.34E-02	4.01E-05	1.26E-06	3.55E-15	1.67E-15

Table 8. Comparison of the absolute error for Example 5.5

z	LW l=1, P=10	FSTCHW l=1, P=10	SNDCHW l=1, P=10	THDCHW l=1, P=10	FTHCHW l=1, P=10	BW l=1, P=5	HW J=3
0.1	1.757717224809E-11	1.757744407113E-11	9.73964E-10	1.75773E-11	1.757744407E-11	9.39E-17	7.526E-03
0.2	1.165313788273E-07	1.165313796014E-07	1.22182E-07	1.16531E-07	1.165313789E-07	5.52E-16	4.156E-02
0.3	2.070899618996E-06	2.070899619575E-06	1.22182E-06	2.07090E-06	2.070899619E-06	4.57E-16	1.208E-01
0.4	1.560169093249E-05	1.560169093284E-05	1.56369E-05	1.56017E-05	1.560169093E-05	1.67E-15	2.650E-01
0.5	7.449772886716E-05	7.449772886722E-05	7.45617E-05	7.44977E-05	7.449772887E-05	1.53E-15	4.882E-01
0.6	2.670494454790E-04	2.670494454788E-04	2.67152E-04	2.67050E-04	2.670494455E-04	1.92E-15	8.040E-01
0.7	7.857019672545E-04	7.857019672539E-04	7.85855E-04	7.85702E-04	7.857019673E-04	2.44E-15	1.223E-00
0.8	2.000658007823E-03	2.000658007821E-03	2.00088E-03	2.00066E-03	2.000658008E-03	2.72E-15	1.753E-00
0.9	4.562120117422E-03	4.562120117420E-03	4.56242E-03	4.56212E-03	4.562120117E-03	3.11E-15	2.401E-00
1.0	9.535780138820E-03	9.535780138818E-03	9.53618E-03	9.53578E-03	9.535780139E-03	3.55E-15	3.170E-00

(BW), Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW), respectively which conclude Bernoulli wavelets gives comparatively better results.

Table 9. Comparison of the absolute error for Example 5.6

z	JRCM [28] N=12	BW l=1, P=9	BW l=1, P=10	BW l=1, P=11	BW l=1, P=12
0.0	4.440E-16	0.000E-00	0.000E-00	0.000E-00	0.000E-00
0.1	1.259E-06	1.049E-08	1.843E-08	1.065E-09	3.389E-10
0.2	1.030E-05	1.070E-07	1.592E-07	8.155E-09	2.237E-09
0.3	3.703E-05	3.781E-07	4.799E-07	2.474E-08	7.094E-09
0.4	8.804E-05	9.053E-07	1.068E-06	5.786E-08	1.799E-08
0.5	1.760E-04	1.736E-06	2.133E-06	1.146E-07	3.468E-08
0.6	3.181E-04	2.997E-06	3.942E-06	1.795E-07	3.962E-08
0.7	5.247E-04	5.990E-06	7.013E-06	1.072E-07	8.196E-08
0.8	8.045E-04	1.673E-05	1.323E-05	7.469E-07	8.317E-07
0.9	1.117E-03	5.361E-05	2.847E-05	4.414E-06	3.794E-06
1.0	1.693E-03	1.609E-04	6.718E-05	1.595E-05	1.291E-05

Example 5.7: Consider the pantograph equation of third order [28]:

$$y'''(z) = -y(z) - y(z - 0.3) + e^{-z+0.3},\ 0 \leq z \leq 1, \tag{51}$$

with initial conditions

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 1. \tag{52}$$

TABLE 10. Comparison of the absolute error for Example 5.6 ($l = 1, P = 10$)

z	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW	HW(J=3)
0.0	0.00000000E-00	0.00000000E-00	0.00000000E-00	0.00000000E-00	0.00000000E-00	0.0000E-00	0.000E-00
0.1	1.84315804E-08	1.84315808E-08	1.843158045E-08	1.84315796E-08	1.84315808E-08	1.2300E-10	4.312E-06
0.2	1.59233276E-07	1.59233277E-07	1.59233277E-07	1.59233276E-07	1.59233277E-07	4.5900E-10	2.724E-05
0.3	4.79928420E-07	4.79928421E-07	4.79928420E-07	4.79928419E-07	4.79928421E-07	6.3955E-08	5.604E-05
0.4	1.06758756E-06	1.06758756E-06	1.06758755E-06	1.06758756E-06	1.06758756E-06	1.3196E-08	7.603E-04
0.5	2.13252927E-06	2.13252927E-06	2.13252926E-06	2.13252927E-06	2.13252927E-06	2.1804E-07	3.228E-03
0.6	3.94149103E-06	3.94149102E-06	3.94149103E-06	3.94149103E-06	3.94149102E-06	1.1822E-07	9.498E-03
0.7	7.01329406E-06	7.01329405E-06	7.01329405E-06	7.01329406E-06	7.01329405E-06	7.3653E-06	2.290E-02
0.8	1.32305595E-05	1.32305594E-05	1.32305595E-05	1.32305594E-05	1.32305594E-05	1.323E-06	4.876E-02
0.9	2.84742761E-05	2.84742762E-05	2.84742761E-05	2.84742762E-05	2.84742761E-05	3.2260E-05	9.546E-02
1.0	6.71792414E-04	6.71792414E-05	6.71792414E-05	6.71792414E-05	6.71792414E-05	1.0842E-05	1.762E-01

The exact solution is $y(z) = e^{-z}$. This example plays an important role in explaining various problems in engineering and sciences such as biology, economy, control and electrodynamics [43].

Table 11 and 12 include the absolute errors derived by considered techniques along with Taylor series method (TSM), Chebyshev method (CM), HC method (HCM) and Bernoulli wavelets (BW) and Legendre wavelets (LW), Chebyshev wavelets (CHW) (all four kinds), Bernoulli wavelets (BW) and Haar wavelets (HW), respectively which conclude that Bernoulli wavelets gives comparatively better results.

An additional effective approach to compare the efficiency of methods is CPU time used in the implementation of program. At this point, the CPU time has been computed by means of the command "TimeUsed [ ]" in "Mathematica 9". The CPU time depends on the specification of computer. The computer characteristic is Microsoft Windows 10 Intel(R) Core(TM) i3 CPU M 380@ 2.53 GHz with 3.00 GB of RAM, 64-bit operating system throughout this study. The mean CPU time is calculated by considering the mean of 30 performances of the program. Table 13 contains CPU time for Exa. 1 to Exa. 7 utilized by Legendre , Chebyshev (four kinds) and Bernoulli wavelets (BW). Its clear from the table that Bernoulli wavelets results are more significant.

TABLE 11. Comparison of the absolute error for Example 5.7

z	TSM [28]	CM [28]	HCM [28]	JRCM [28]	BW	BW	BW
				N=26,$\alpha, \beta = 0.5$	l=1, P=12	l=1, P=13	l=1, P=14
0.0	0.00E-00	0.00E-00	0.000E-00	0.000E-00	0.000E-00	0.000E-00	0.000E-00
0.2	8.54E-08	3.70E-07	6.200E-09	3.605E-08	2.294E-12	7.761E-14	2.331E-15
0.4	5.36E-06	2.38E-06	5.760E-08	9.299E-09	1.079E-11	3.577E-13	1.055E-14
0.6	5.95E-05	5.97E-06	1.796E-07	3.503E-10	2.556E-11	8.418E-13	2.465E-14
0.8	3.26E-04	3.48E-05	3.735E-07	8.345E-09	4.644E-11	1.524E-12	4.452E-14
1.0	1.21E-03	2.03E-04	6.368E-07	1.161E-08	7.304E-11	2.392E-12	6.950E-14

TABLE 12. Comparison of the absolute error for Example 5.7 ($l = 1, P = 10$)

z	LW	FSTCHW	SNDCHW	THDCHW	FTHCHW	BW	HW(J=3)
0.1	4.0151216E-12	4.0150105E-12	4.0153436E-12	4.0147885E-12	4.0150105E-12	4.92939E-14	1.70783E-06
0.2	1.3591239E-11	1.3591128E-11	1.3591461E-11	1.3590906E-11	1.3591017E-11	1.55631E-12	7.25176E-06
0.3	2.8645864E-11	2.8645863E-11	2.8646197E-11	2.8645530E-11	2.8645752E-11	7.01328E-12	1.66174E-05
0.4	5.4234728E-11	5.4234617E-11	5.4235061E-11	5.4234395E-11	5.4234506E-11	1.02137E-11	2.96974E-05
0.5	8.6948670E-11	8.6948448E-11	8.6948782E-11	8.6948226E-11	8.6948448E-11	6.95699E-12	4.64011E-05
0.6	1.2281254E-10	1.2281243E-10	1.2281276E-10	1.2281221E-10	1.2281221E-10	6.19760E-12	6.65013E-05
0.7	1.6551188E-10	1.6551172E-10	1.6551222E-10	1.6551149E-10	1.6551149E-10	1.75381E-11	8.99132E-05
0.8	2.1815166E-10	2.1815161E-10	2.1815183E-10	2.1815133E-10	2.1815122E-10	3.12795E-11	1.16405E-04
0.9	2.7573505E-10	2.7573488E-10	2.757353E-10	2.7573466E-10	2.7573471E-10	3.23493E-11	1.45763E-04
1.0	3.3819120E-10	3.3819103E-10	3.3819131E-10	3.3819075E-10	3.3819081E-10	3.74238E-11	1.77759E-04

Table 13. CPU Time of Legendre, Chebyshev and Bernoulli wavelets

	CPU Time for LW	CPU Time for FSTCHW	CPU Time for SNDCW	CPU Time for THDCHW	CPU Time for FTHCHW	CPU Time for BW
Example 1 (l=1, p=10)	0.421	0.484	0.359	0.406	0.421	0.344
Example 2 (l=1, p=5)	0.359	0.360	0.359	0.375	0.343	0.327
Example 4 (l=1, p=6)	0.484	0.438	0.392	0.501	0.548	0.313
Example 5 (l=1, p=10)	0.501	0.422	0.438	0.454	0.470	0.391(l=1,p=5)
Example 6 (l=1, p=10)	0.578	0.531	0.442	0.516	0.515	0.423
Example 7 (l=1, p=10)	0.438	0.453	0.484	0.516	0.548	0.406

6. Conclusions

In this paper, we have demonstrated the feasibility of the Bernoulli wavelets method for solving pantograph equation. The considered examples reveal that the results obtained by Bernoulli wavelets method are in excellent agreement as compared to generated by Haar, Legendre, Chebyshev wavelets (four kinds) etc. The effective of the considered technique is also justified by calculating utilized CPU time of the proposed method along with Legendre and Four kinds of Chebyshev wavelets.

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References

[1] Ajella, W. G., Freedman, H. I. and Wu, J., (1992), A model of stage structured population growth with density depended time delay, SIAM J. Appl. Math., (52), pp. 855-869.

[2] Muroya, Y., Ishiwata, E. and Brunner H., (2003), On the attainable order of collocation method for pantograph integro-differential equations, J. Comput. Appl. Math., (152), pp. 347-366.

[3] Reutskiy, S. Y., (2015), A new collocation method for approximate solution of the pantograph functional equations with proportional delay, Appl. Math. Comput., (266), pp. 642-655.

[4] Liu, M. Z. and Li, D. (2004), Properties of analytic solution and numerical solution of multi-pantograph equation, Appl. Math. Comput., (155), pp. 853-871.

[5] Derfel, G., (1980), On compactly supported solutions of a class of functional-differential equations, in: Modern Problems of Function Theory and Functional Analysis, Karaganda University Press (in Russian).

[6] Derfel, G., Dyn, N. and Levin, D., (1995), Generalized refinement equation and subdivision process, J. Approx. Theory, (80), pp. 272-297.

[7] Gulsu, M. and Sezer, M., (2005), The approximate solution of high-order linear difference equations with variable coefficients in terms of Taylor polynomials, Appl. Math. Comput., (168), pp. 76-88.

[8] Sezer, M., Yalcinbas, S. and Sahin, N., (2008), Approximate solution of multi-pantograph equation with variable coefficients, J. comput. Appl. Math., (214), pp. 406-416.

[9] Yuzbasi, S. and Ismailov, N., (2018), A Taylor operation method for solutions of generalized pantograph type delay differential equations, Turk. J. Math., (42), pp. 395-406.

[10] Sezer, M. and Akyuz-Dascroglu, A., (2007), A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, J. Comput. Appl. Math., (200), pp. 217-225.

[11] Bahsi, M. M., Cevik, M., (2015), Numerical solution of pantograph-type delay differential equations using perturbation-iteration algorithms, J. Appl. Math., (2015), Art. ID 139821, 10 pages.

[12] Hwang, C. and Shih, Y. P., (1982), Laguerre series solution of a functional differential equation, Int. J. Syst. Sci., (13), pp. 783-788.

[13] Rao, G. P. and Palanisamy, K. R., (1982), Walsh stretch matrices and functional differential equation, IEEE Trans. Autom. Control, (27), pp. 272-276.

[14] Yalcinbas, S., Sorkun, H. H., Sezer, M., (2015), A numerical method for solutions of pantograph type differential equations with variable coefficients using Bernstein polynomials, New Trends Math. Sci., (3), pp. 179-195.

[15] Isik, O. R., Guney, Z., and Sezer, M., (2012), Bernstein series solutions of pantograph equations using polynomial interpolation, J. Differ. Equ. Appl., (18), pp. 357-374.

[16] El-Safty, A., Salim, M. S. and El-Khatib, M. A., (2003), Convergence of the spline function for delay dynamic system, Int. J. Comput. Math., (80), pp. 509-518.

[17] Ogunfiditimi, F. O., (2015), Numerical solution of delay differential equations using the Adomian decomposition method (ADM), Int. J. Engg. Sci., (4), pp. 18-23.

[18] Evans, D. J., Raslan, K. R., (2005), The Adomian decomposition method for solving delay differential equation, Int. J. Comp. Math., (82), pp. 49-54.

[19] Oberle, H. J., Pesch, H. J., (1981), Numerical treatment of delay differential equations by Hermite interpolation, Numer. Math., (37), pp. 235-255.

[20] Davaeifar, S., Rashidinia, J., (2017), Solution of a system of delay differential equations of multi pantograph type, J. Taibah University Sci., (11), pp. 1141-1157.

[21] Reutskiy, S. Y., (2015), A new collocation method for approximate solution of the pantograph functional differential equations with proportional delay, Appl. Math. Comp., (266), pp. 1642-1655.

[22] Sedaghat, S., Ordokhani, Y. and Dehghan, M., (2012), Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials, Commu. Nonl. Sci. Numer. Simul., (17), pp. 4815-4830.

[23] Yuzbasi, S., Sezer, M., (2015), Shifted Legendre approximation with the residual correction to solve pantograph-delay type differential equations, Appl. Math. Model., (39), pp. 6529-6542.

[24] Karakoc, F., Bereketoglu, H., (2009), Solutions of delay differential equations by using differential transform method, Int. J. Comp. Math., (86), pp. 914-923.

[25] Behroozifar, M., Yousefi, S. A., (2013), Numerical solution of delay differential equations via operational matrices of hybrid of block-pulse functions and Bernstein polynomials, Comput. Meth. Differ. Equ., (1), pp. 78-95.

[26] Yu, Z. H., (2008), Variational iteration method for solving the multi-pantograph delay equation, Physics Letters A, (372), pp. 6475-6479.

[27] Chen, X. and Wang, L., (2010), The variational iteration method for solving a neutral functional-differential equation with proportional delays, Comput. Math. Appl., (59), pp. 2696-2702.

[28] Doha, E. H., Bhrawy, A. H., Baleanu, D., Hafez, R. M., (2014), A new Jacobi rational-Gauss collocation method for numerical solution of generalized pantograph equations, Appl. Numer. Math., (77), pp. 43-54.

[29] Bica, A. M., (2011), The numerical method of successive interpolations for two-point boundary value problems with deviating argument, Appl. Math. Comput., (62), pp. 3829-3843.

[30] Shiralashetti, S. C., Kumbinarasaiah, S., Mundewadi, R. A., Hoogar, B. S., (2016), Series solution of pantograph equations using wavelets, Open J. Appl. Theo. Math., (2), pp. 505-518.

[31] Ghasemi, M., Kajani, M. T., (2011), Numerical solution of time-varying delay systems by Chebyshev wavelets, Appl. Math. Model., (35), pp. 5235-5244.

[32] Ali, A., Iqbal, M. A., Mohyud-Din, S. T., (2013), Chebyshev wavelts method for delay differential equations, Int. J. Mod. Math. Sci., (8), pp. 102-110.

[33] Shiralashetti, S. C., Hoogar, B. S., Kumbinarasaiah, S., (2017), Hermite wavelets based method for numerical solution of linear and nonlinear delay differential equations, Int. J. Engg. Sci. Math., (6), pp. 71-79.

[34] Saeed, U. and Rehman, M. R., (2014), Hermite wavelet method for fractional delay differential equations, J. Differ. Equ., (2014), Article ID 359093, 8 pages.

[35] Hafshejani, M. S., Vanani, S. K., Hafshejani, J. S., (2011), Numerical solution of delay differential equations using Legendre wavelet method, World Appl. Sci. J., (13), pp. 27-33.

[36] Vanani, S. K., Hafshejani, J. S., Soleymani, F., Khan M., (2011), Numerical solution of functional differential equations using Legendre wavelet method, World Appl. Sci. J., (13), pp. 2522-2525.

[37] Bahmanpour, M., Araghi, M. A. F., (2013), A method for solving fredholm integral equations of the first kind based on chebyshev wavelets, Anal. Theo. Appl., (29), pp. 197-207.

[38] Keshavarz, E., Ordokhani, Y., Razzaghi, M., (2014), Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, Appl. Math. Mod., (38), pp. 6038-6051.

[39] Sahu, P. K., Ray, S. S., (2017), A new Bernoulli wavelet method for accurate solutions of nonlinear fuzzy Hammerstein–Volterra delay integral equations, Fuzzy Sets Syst., (309), pp. 131-144.

[40] Kouba, O., (2013), Lecture notes, Bernoulli polynomials and applications, arXiv:1309.7560v1 [mah.CA].

[41] Lepik, U., (2008), Haar wavelet method for solving higher order differential equations, Int. J. Math. Comput., (1), pp. 84-94.

[42] Chang, P., Piau, P., (2008), Haar wavelet matrices designation in numerical solution of ordinary differential equations, IAENG Int. J. Appl. Math., (38), pp. 1-5.

[43] Xie, L.J., Zhou, C. L., and Xu, S., (2018), A new computational approach for the solutions of generalized pantograph-delay differential equations, Comp. Appl. Math., (37), pp. 1756-1783.

[44] Ozturk, Y., Gulsu, M., (2016), Approximate solution of generalized pantograph equations with variable coefficients by operational method, J. Opt. Theory App., (7), pp. 66-74.

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