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THE RE-NND SOLUTIONS OF THE MATRIX EQUATION $AXB = C$ WITH REFERENCE TO INDEFINITE INNER PRODUCT

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Abstract. In this paper, we first consider the matrix equation $AXA^{[*]} = C$, where $A \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{n\times n}$ and establish necessary and sufficient conditions for the existence of Re-nnd solutions. Further, we determine the necessary and sufficient conditions for the existence of Re-nnd solutions of the equation AXB=C.

Keywords: Indefinite inner product, Re-nnd solutions, matrix equation.

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1. Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0$, $\forall y \in \mathbb{C}^n_J$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] = \langle x, Jy\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. We also make an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors [3, 20]. This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [20]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where J_m and J_n are weights.

Many properties of this product are similar to that of the usual matrix product [20]. Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results with reference to Indefinite Inner Product in a manner analogous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar also established in

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[20] that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate that of the usual matrix product. Range symmetric (EP) matrices with respect to indefinite inner product have been intensively studied in [10, 12, 13, 14, 15, 16, 26].

Recall that the Moore-Penrose inverse exists if and only if $rank(AA^*) = rank(A^*A) = rank(A)$. If we take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $AA^{[*]}$ and $A^{[*]}A$ are both the zero matrix and so $rank(AA^{[*]}) < rank(A)$, thereby proving that the Moore-Penrose inverse does not exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$. Thus, the Moore-Penrose J-inverse with real or complex entries exists over an indefinite inner product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinent to extend the study of generalized inverses to the setting of indefinite inner product.

The Hermitian part of X is defined as $H(X) = \frac{1}{2}(X + X^*)$. We say that X is Re-nnd if $H(X) \geq 0$ and X is Re-pd if $H(X) > 0$. The Hermitian part of X is defined as $H(X) = \frac{1}{2}(JXJ + X^{[*]})$ with reference to indefinite inner product. We will say that X is Re-nnd if $H(X) \geq 0$ and X is Re-pd if $H(X) > 0$.

Many authors have studied the well-known equation $AXB = C$ with the unknown matrix X, such that X belongs to some special class of matrices. For example, in [4] and [19] the existence of reflexive and anti-reflexive, with respect to a generalized reflection matrix P, solutions of the matrix equation $AXB = C$ was considered, while in [5, 11, 17, 18] necessary and sufficient conditions for the existence of symmetric and antisymmetric solutions of the equation $AXB = C$ were investigated.

The Hermitian nonnegative definite solutions for the equation $AXA^* = B$ were investigated by Khatri and Mitra [11], Baksalary [1], Dai and Lancaster [6], Groß [9], Zhang and Cheng [25] and Zhang [24].

Wu [22] studied Re-pd solutions of the equation $AX = C$, and Wu and Cain [23] found the set of all complex Re-nnd matrices X such that $XB = C$ and presented a criterion for Re-nndness. Groß [8] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [23]. Some results from [23] were extended in the paper of Wang and Yang [21], in which the authors presented criteria for 2×2 and 3×3 partitioned matrices to be Re-nnd, found necessary and sufficient conditions for the existence of Re-nnd solutions of the equation $AXB = C$ and derived an expression for these solutions. In addition to these papers many other papers have dealt with the problem of finding the Re-nnd and Re-pd solutions of some other forms of equations.

2. PRELIMINARIES

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. [20] *Let $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, $B \in \mathbb{C}^{n\times k}_{J_n,J_k}$. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_nB$.*

Definition 2.2. [20] *For $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, $A^{[*]} = J_nA^*J_m$ is the adjoint of A relative to J_n and J_m.*

Definition 2.3. [20] *A matrix $A \in \mathbb{C}^{n\times n}_{J_n}$ is said to be J-invertible if there exists $X \in \mathbb{C}^{n\times n}_{J_n}$, such that $A \circ X = X \circ A = J_n$. Such an X is denoted by $A^{[-1]} = JA^{-1}J$.*

Remark 2.1. *For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\} = A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if J_nXJ_m is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X : J_nXJ_m$ is a generalized inverse of $A\}$.*

Definition 2.4. [10] *For $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, and $X \in \mathbb{C}^{n\times m}_{J_n,J_m}$ is called the Moore - Penrose J-inverse of A if it satisfies the following equations:*

$(i)\ \ A \circ X \circ A = A$ $\qquad (\{1\}\,inverse)$

$(ii)\ \ X \circ A \circ X = X$ $\qquad (\{2\}\,inverse)$

$(iii)(A \circ X)^{[*]} = A \circ X$ $\qquad (\{3\}\,inverse)$

$(iv)\ (X \circ A)^{[*]} = X \circ A.$ $\qquad (\{4\}\,inverse)$

such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^{\dagger}J_m$.

Definition 2.5. [15] *The range space $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$. The null space of $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.*

Property 2.1. [15] *Let $A \in \mathbb{C}^{m\times n}_{J_n}$. Then*

$(i)\ (A^{[*]})^{[*]} = A.$

$(ii)\ (A^{[\dagger]})^{[\dagger]} = A.$

$(iii)\ (AB)^{[*]} = B^{[*]}A^{[*]}.$

$(iv)\ Ra(A^{[*]}) = Ra(A^{[\dagger]}).$

$(v)\ Ra(A \circ A^{[*]}) = Ra(A),\ Ra(A^{[*]} \circ A) = Ra(A^{[*]}).$

$(vi)\ Nu(A \circ A^{[*]}) = Nu(A^{[*]}),\ Nu(A^{[*]} \circ A) = Nu(A).$

Definition 2.6. [15] *A is range symmetric in $\mathbb{C}^{n\times n}_J$ if and only if $Ra(A) = Ra(A^{[*]})$ (or) equivalently $Nu(A) = Nu(A^{[*]})$.*

Remark 2.2. *In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.*

Theorem 2.1. [15] *For $A \in C^{n\times n}_{J_n}$, the following are equivalent:*
(i) A is range symmetric in $\mathbb{C}^{n\times n}_J$.
(ii) AJ is EP.
(iii) JA is EP.
(iv) $Nu(A) = Nu(A^{[*]})$.
(v) $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.
(vi) $(A^{\dagger}A)^{[*]} = JA^{\dagger}AJ = AA^{\dagger}$.
(vii) A is J-EP.

3. The Re-nnd solutions of the Matrix equation $AXB = C$

Lemma 3.1. [13] *Let $A, B \in \mathbb{C}^{m\times n}$, then $N(A^*) \subseteq N(B^*)$ if and only if $Nu(A^{[*]}) \subseteq Nu(B^{[*]})$.*

Theorem 3.1. [12] *Let $M \in \mathbb{C}^{(n+m)\times(n+m)}$ be an J-symmetric matrix given by $M = \begin{pmatrix} A & B \\ B^{[*]} & D \end{pmatrix}$, where $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$. Then $M \geq 0$ if and only if $A \geq 0,\ AA^{[\dagger]}B = B,\ D - B^{[*]}A^{[\dagger]}B \geq 0$.*

Next, we give necessary and sufficient conditions for the matrix equation $AX = B$ to have a Re-nnd solution X, where A and B are given matrices of suitable size and presents a possible explicit expression for X with reference to indefinite inner product.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{n\times m}$. Then there exists a Re-nnd matrix $X \in \mathbb{C}^{m\times m}$ satisfying $AX = B$ if and only if $AA^{[\dagger]}B = B$ and $AB^{[*]}$ is Re-nnd.*

Proof. Suppose $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{n\times m}$. Assume that there exists a Re-nnd matrix $X \in \mathbb{C}^{m\times m}$ satisfying $AX = B$ implies that $X = A^{[\dagger]}B$. Therefore $AA^{[\dagger]}B = B$.
Next we show that $AB^{[*]}$ is Re-nnd.
For if $AB^{[*]} = A(AX)^{[*]} = AX^{[*]}A^{[*]} = (A^{[*]})^{[*]}X^{[*]}A^{[*]} = (AXA^{[*]})^{[*]} \geq 0$.
Hence $AB^{[*]}$ is Re-nnd.
Conversely, let us assume that $AA^{[\dagger]}B = B$ and $AB^{[*]}$ is Re-nnd.
It suffices to show that $AX = B$ for any Re-nnd matrix $X \in \mathbb{C}^{m\times m}$.

$$
\begin{aligned}
AX &= A(X_0 + (I - A^{[\dagger]}A)Y(I - A^{[\dagger]}A)), \\
&\quad \text{where } X_0 \text{ is one of the Re-nnd solutions of } AX = B \\
&= AX_0 + (AY - AA^{[\dagger]}AY)(I - A^{[\dagger]}A) \\
&= AX_0 + (AY - AY)(I - A^{[\dagger]}A) \\
&= AX_0 \\
&= B.
\end{aligned}
$$

□

Our main aim is to generalize these results to the equation $AXB = C$ and to present a general form of Re-nnd solutions of it. First we will consider the equation

$$AXA^{[*]} = C \tag{1}$$

and find necessary and sufficient conditions for the existence of Re-nnd solutions. The next auxiliary result presents a general form of a solution X of (1) which satisfies $H(X) = 0$.

Lemma 3.2. *If $A \in \mathbb{C}^{n\times m}$, then $X \in \mathbb{C}^{m\times m}$ is a solution of the equation*

$$AXA^{[*]} = 0 \tag{2}$$

which satisfies $H(X) = 0$ if and only if

$$X = W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]} \tag{3}$$

for some $W \in \mathbb{C}^{m\times m}$.

Proof. Denote $r = rank(A)$. Let us suppose that X is a solution of the equation $AXA^{[*]} = 0$ and $H(X) = 0$. Using a singular value decomposition of $A = U^{[*]}(D \oplus 0)V$, where $U \in \mathbb{C}^{n\times n}, V \in \mathbb{C}^{m\times m}$ are unitary and $D \in \mathbb{C}^{r\times r}$ is an invertible matrix, we have that $A^{[\dagger]} = V^{[*]}(D^{[-1]} \oplus 0)U$ and $X = V^{[*]}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}V$, for some $X_1 \in \mathbb{C}^{r\times r}$ and $X_4 \in \mathbb{C}^{(m-r)\times(m-r)}$.
From $AXA^{[*]} = 0$ we obtain that $X_1 = 0$ and, by $H(X) = 0$ that $X_3 = -X_2^{[*]}$ and $H(X_4) = 0$. Hence $X = V^{[*]}\begin{pmatrix} 0 & X_2 \\ -X_2^{[*]} & X_4 \end{pmatrix}V$.
Taking into account that $H(X_4) = 0$, for $W = V^{[*]}\begin{pmatrix} I & X_2 \\ 0 & \frac{1}{2}X_4 \end{pmatrix}V$, we have that $X = W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]}$.

In the other direction we have to check that for arbitrary $W \in \mathbb{C}^{m\times m}$, X defined by $X = W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]}$ is a solution of the equation $AXA^{[*]} = 0$. That is

$$\begin{aligned}
AXA^{[*]} &= A(W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]})A^{[*]} \\
&= A(W - WA^{[\dagger]}A - W^{[*]} + A^{[\dagger]}AW^{[*]})A^{[*]} \\
&= AWA^{[*]} - AWA^{[\dagger]}AA^{[*]} - AW^{[*]}A^{[*]} + AA^{[\dagger]}AW^{[*]}A^{[*]} \\
&= AWA^{[*]} - AWA^{[\dagger]}AA^{[*]} - AW^{[*]}A^{[*]} + AW^{[*]}A^{[*]} \ (By\ Definition\ 2.4) \\
&= AWA^{[*]} - AWA^{[*]} - AW^{[*]}A^{[*]} + AW^{[*]}A^{[*]} \ (By\ Theorem\ 3.2) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
H(X) &= \frac{1}{2}(JXJ + X^{[*]}) \\
&= \frac{1}{2}(J(W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]})J + (W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]})^{[*]}) \\
&= \frac{1}{2}(J(W - WA^{[\dagger]}A) - (W^{[*]} - A^{[\dagger]}AW^{[*]})J + (W - WA^{[\dagger]}A - W^{[*]} + A^{[\dagger]}AW^{[*]})^{[*]}) \\
&= \frac{1}{2}(J(W - W) - (W^{[*]} - W^{[*]})J + (W - W - W^{[*]} + W^{[*]})^{[*]}) \ (By\ Theorem\ 3.2) \\
&= 0.
\end{aligned}$$

□

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{n\times n}$ be given matrices such that the equation (1) is consistent and let $r = rank\ H(C)$. Then there exists a Re-nnd solution of the equation (1) if and only if C is Re-nnd. In this case the general Re-nnd solution is given by*

$$X = (A^{=}CA^{=})^{[*]} + (I - A^{[\dagger]}A)UU^{[*]}(I - A^{[\dagger]}A)^{[*]} + W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]} \quad (4)$$

with

$$A^{=} = A^{[\dagger]} + (I - A^{[\dagger]}A)Z((H(C))^{\frac{1}{2}})^{[\dagger]}, \quad (5)$$

where $A^{[\dagger]}$ and $(H(C)^{\frac{1}{2}})^{[\dagger]}$ are arbitrary but fixed Moore-Penrose inverses of A and $(H(C)^{\frac{1}{2}})$, respectively, and $Z \in \mathbb{C}^{m\times n}$, $U \in \mathbb{C}^{m\times(m-r)}$, $W \in \mathbb{C}^{m\times m}$ are arbitrary matrices.

Proof. If X is a Re-nnd solution of the equation (1), then $AH(X)A^{[*]} = H(C) \geq 0$.
In the other direction, if C is Re-nnd, then $X_0 = A^{[\dagger]}C(A^{[\dagger]})^{[*]}$ is a Re-nnd solution of the equation (1).
Let us prove that a representation of the general Re-nnd solution is given by (4). If X is defined by (4), then X is Re-nnd and $AXA^{[*]} = AA^{[\dagger]}C(AA^{[\dagger]})^{[*]} = C$.
If X is an arbitrary Re-nnd solution of (1), then $H(X)$ is a J-symmetric non-negative definite solution of the equation $AZA^{[*]} = H(C)$,
$H(X) = A^{=}H(C)(A^{=})^{[*]} + (I - A^{[\dagger]}A)UU^{[*]}(I - A^{[\dagger]}A)^{[*]}$, where $A^{=}$ is given by (5), for some $Z \in \mathbb{C}^{m\times n}$ and $U \in \mathbb{C}^{m\times(m-r)}$.
Note that, $H(X) = H(A^{=}C(A^{=})^{[*]} + (I - A^{[\dagger]}A)UU^{[*]}(I - A^{[\dagger]}A)^{[*]})$,
implying $X = A^{=}C(A^{=})^{[*]} + (I - A^{[\dagger]}A)UU^{[*]}(I - A^{[\dagger]}A)^{[*]} + Z$, where $H(Z) = 0$ and $AZA^{[*]} = 0$.
Using Lemma 3.2, we have that $Z = W(I - A^{[\dagger]}A) - (I - A^{[\dagger]}A)W^{[*]}$, for some $W \in \mathbb{C}^{m\times n}$.
Hence, we obtain that X has a representation as in (1). □

Now, let us consider the equation

$$AXB = C, \quad (6)$$

where $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times n}$ are given matrices and find necessary and sufficient conditions for the existence of a Re-nnd solution.

This follows from the fact that whenever $AXB = C$ is solvable then X is a solution of that equation if and only if X is a solution of the equation $A^{[*]}AXBB^{[*]} = A^{[*]}CB^{[*]}$. Hence from now on, we assume that A and B are non-negative definite matrices from the space $\mathbb{C}^{n\times n}$. when $m = n$ is particular.

The following theorem presents necessary and sufficient conditions for the equation $AXB = C$ to have a Re-nnd solution.

Theorem 3.4. *Let $A, B, C \in \mathbb{C}^{n\times n}$ be given matrices such that A and B are non negative definite and the equation (6) is consistent. Then there exists a Re-nnd solution of (6) if and only if $T = B(A+B)^{[\dagger]}C(A+B)^{[\dagger]}A$ is Re-nnd, where $(A+B)^{[\dagger]}$ is a M-P inverse of $(A+B)$. In this case a general Re-nnd solution is given by*

$$X = \begin{cases} (A+B)^{=}(C+Y+Z+W)((A+B)^{=})^{[*]} \\ +(I-(A+B)^{[\dagger]}(A+B))UU^{[*]}(I-(A+B)^{[\dagger]}(A+B))^{[*]} \\ +Q(I-(A+B)^{[\dagger]}(A+B)) - (I-(A+B)^{[\dagger]}(A+B))Q^{[*]}, \end{cases} \tag{7}$$

where Y, Z, W are arbitrary solutions of the equations

$$\begin{cases} Y(A+B)^{[\dagger]}B = C(A+B)^{[\dagger]}A, \\ A(A+B)^{[\dagger]}Z = B(A+B)^{[\dagger]}C, \\ A(A+B)^{[\dagger]}W(A+B)^{[\dagger]}B = T, \end{cases} \tag{8}$$

such that $C+Y+Z+W$ is Re-nnd, $(A+B)^{=}$ is defined by

$$(A+B)^{=} = (A+B)^{[\dagger]} + (I-(A+B)^{[\dagger]}(A+B))P((H(C+Y+Z+W))^{\frac{1}{2}}),$$

where $U \in \mathbb{C}^{n\times(n-r)}, Q \in \mathbb{C}^{n\times n}, P \in \mathbb{C}^{n\times n}$ are arbitrary, $r = rank(C+Y+Z+W)$.

Proof. Denote by $E = (A+B)^{[\dagger]}B, F = C(A+B)^{[\dagger]}A$,
$K = A(A+B)^{[\dagger]}, L = B(A+B)^{[\dagger]}C$.
Now equations (8) are equivalent to

$$YE = F, KZ = L, KWE = T. \tag{9}$$

Using Definition 2.4 and the fact that E is invertible with reference to indefinite inner product and $E^{[\dagger]} = B^{[\dagger]}(A+B)$, we have that

$$\begin{aligned} FE^{[\dagger]}E &= C(A+B)^{[\dagger]}AB^{[\dagger]}(A+B)(A+B)^{[\dagger]}B \\ &= C(A+B)^{[\dagger]}AB^{[\dagger]}B \\ &= C(A+B)^{[\dagger]}A \\ &= F, \end{aligned}$$

which implies that the equation $YE = F$ is consistent. In a similar way, we can prove that the other two equations from (9) are consistent. Furthermore, $T^{[*]} = F^{[*]}E = KL^{[*]}$ is Re-nnd which implies by Theorem 3.2, that the first two equations from (9) have Re-nnd solutions.

Now, suppose that the equation (6) has a Re-nnd solution X. Then

$$\begin{aligned} H(T) &= H(B(A+B)^{[\dagger]}AXB(A+B)^{[\dagger]}A) \\ &= (B(A+B)^{[\dagger]}A)H(X)(B(A+B)^{[\dagger]}A)^{[*]} \geq 0. \end{aligned}$$

Conversely, let T be Re-nnd. We can check that

$$X_0 = (A+B)^{[\dagger]}(C+Y+Z+W)(A+B)^{[\dagger]}, \tag{10}$$

is a solution of the equation (1), where Y, Z, W are arbitrary solutions of the equation (6). This follows from

$$\begin{aligned} AX_0B &= (A+B)(A+B)^{[\dagger]}C(A+B)^{[\dagger]}(A+B) \\ &= (A+B)(A+B)^{[\dagger]}AA^{[\dagger]}CB^{[\dagger]}B(A+B)^{[\dagger]}(A+B) \\ &= AA^{[\dagger]}CB^{[\dagger]}B \\ &= C. \end{aligned}$$

Now, we have to prove that for some choice of Y, Z, W, the matrix $C+Y+Z+W$ is Re-nnd which would imply that X_0 is Re-nnd.

Let $Y = FE^{[\dagger]} - (FE^{[\dagger]})^{[*]} + (E^{[\dagger]})^{[*]}F^{[*]}EE^{[\dagger]} + (I - EE^{[\dagger]})^{[*]}(I - EE^{[\dagger]})$,
$Z = K^{[\dagger]}L - (K^{[\dagger]}L)^{[*]} + K^{[\dagger]}KL^{[*]}(K^{[\dagger]})^{[*]} + (I - K^{[\dagger]}K)Q(I - K^{[\dagger]}K)^{[*]}$,
$W = K^{[\dagger]}TE^{[\dagger]} - (I - K^{[\dagger]}K)S - S(I - EE^{[\dagger]})$,
where $Q = (C^{[*]} - K^{[\dagger]}T^{[*]}E^{[\dagger]})(C^{[*]} - K^{[\dagger]}T^{[*]}E^{[\dagger]})^{[*]}$ and $S = K^{[\dagger]}KC^{[*]} + C^{[*]}EE^{[\dagger]}$.

Obviously, Y, Z, W are solutions of the equation (1) and

$$\begin{aligned} H(Y) &= (E^{[\dagger]})^{[*]}H(T)E^{[\dagger]} + (I - EE^{[\dagger]})^{[*]}(I - EE^{[\dagger]}) \\ H(Z) &= K^{[\dagger]}H(T)(K^{[\dagger]})^{[*]} + (I - K^{[\dagger]}K)H(Q)(I - K^{[\dagger]}K)^{[*]} \\ H(W) &= K^{[\dagger]}TE^{[\dagger]} + (E^{[\dagger]})^{[*]}T^{[*]}(K^{[\dagger]})^{[*]} - H(C^{[*]}EE^{[\dagger]} + K^{[\dagger]}KC^{[*]} - 2K^{[\dagger]}T^{[*]}E^{[\dagger]}). \end{aligned}$$

Using

$$\begin{aligned} &K^{[\dagger]}KK^{[\dagger]}T^{[*]}E^{[\dagger]} = K^{[\dagger]}KK^{[\dagger]}KL^{[*]}E^{[\dagger]} = K^{[\dagger]}KL^{[*]}E^{[\dagger]} = K^{[\dagger]}T^{[*]}E^{[\dagger]}, \\ &K^{[\dagger]}T^{[*]}E^{[\dagger]}EE^{[\dagger]} = K^{[\dagger]}F^{[*]}EE^{[\dagger]}EE^{[\dagger]} = K^{[\dagger]}F^{[*]}EE^{[\dagger]} = K^{[\dagger]}T^{[*]}E^{[\dagger]}, \\ &KC^{[*]}E = KL^{[*]} = T^{[*]}, \end{aligned}$$

we compute,

$$\begin{aligned} H(C+Y+Z+W) &= ((E^{[\dagger]})^{[*]} + K^{[\dagger]})H(T)((E^{[\dagger]})^{[*]} + K^{[\dagger]})^{[*]} \\ &\quad + [(I - EE^{[\dagger]})^{[*]}(I - K^{[\dagger]}K)]D\begin{pmatrix} I - EE^{[\dagger]} \\ (I - K^{[\dagger]}K)^{[*]} \end{pmatrix}, \\ &\quad \text{where } D = \begin{pmatrix} I & C - (E^{[\dagger]})^{[*]}T(K^{[\dagger]})^{[*]} \\ C^{[*]} - K^{[\dagger]}T^{[*]}E^{[\dagger]} & H(Q) \end{pmatrix}. \end{aligned}$$

By Theorem 3.1, it follows that D is nonnegative definite, so $H(C+Y+Z+W) \geq 0$. Hence, with such a choice of Y, Z, W, it can be seen that X_0 defined by (10) is Re-nnd solutions of (6). So, we proved the sufficient part of the theorem.

Let X be an arbitrary Re-nnd solutions of (6). It is evident that $Y = AXA, Z = BXB$ and $W = BXA$ are solutions of (9), and that $(A+B)X(A+B) = C+Y+Z+W$ is Re-nnd. Now, using Theorem 3.3, we get that X has the representation (7).

Note that if the equation $AX = C$ is consistent then X is a solution of it if and only if $A^{[*]}AX = A^{[*]}C$. By Theorem 3.4, we get that there exists a Re-nnd solution of the equation AX = C if and only if $T = (A^{[*]}A + I)^{[-1]}A^{[*]}C(A^{[*]}A + I)^{[-1]}A^{[*]}A$ is Re-nnd. Note that in this case $(I + A^{[*]}A)$ is invertible matrix.

Let us prove that T is Re-nnd if and only if $CA^{[*]}$ is Re-nnd. By $(A^{[*]}A + I)^{[-1]}A^{[*]}A = A^{[*]}A(A^{[*]}A + I)^{[-1]}$,
we have that $T = (A^{[*]}A + I)^{[-1]}A^{[*]}(CA^{[*]})((A^{[*]}A + I)^{[-1]}A^{[*]})^{[*]}$, That is $H(T) = ((A^{[*]}A + I)^{[-1]}A^{[*]})H(CA^{[*]})((A^{[*]}A + I)^{[-1]}A^{[*]})^{[*]}$. From the last equality, $H(CA^{[*]}) \geq 0$ implies that $H(T) \geq 0$.

Now, suppose that $H(T) \geq 0$, then, by the consistence of the equation $AX = C$, it follows that $AA^{[\dagger]}C = C$ which implies that $(A^{[\dagger]})^{[*]}(A^{[*]}A + I)T((A^{[\dagger]})^{[*]}(A^{[*]}A + I))^{[*]} = (A^{[\dagger]})^{[*]}A^{[*]}CA^{[*]}AA^{[\dagger]} = AA^{[\dagger]}CA^{[*]} = CA^{[*]}$.

That is $H(CA^{[*]}) = ((A^{[\dagger]})^{[*]}(A^{[*]}A + I))H(T)((A^{[\dagger]})^{[*]}(A^{[*]}A + I))^{[*]} \geq 0$. $\square$

Example:

Let us consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and $J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$A^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $A^*A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $trace(A^*A) = 2$, $A^\dagger = \frac{A^*}{trace(A^*A)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$,

$(A+B)^{[\dagger]} = J(A+B)^\dagger J = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $B(A+B)^{[\dagger]}C(A+B)^{[\dagger]}A$ is Re-nnd.

4. Conclusion

In this paper we consider some special cases and give a complete characterization of the set of Re-nnd solution of $AXA^{[*]} = C$. The necessary and sufficient conditions for the existence of Re-nnd solutions of the equation $AXB = C$ with reference to indefinite inner product is determined.

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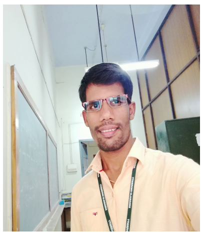

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