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# FEKETE-SZEGÖ INEQUALITY FOR ANALYTIC AND BI-UNIVALENT FUNCTIONS SUBORDINATE TO $(p,q)$ –LUCAS POLYNOMIALS

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ABSTRACT. In this paper, a subclass of analytic and bi-univalent functions by means of $(p,q)-$ Lucas polynomials is introduced. Certain coefficients bounds for functions belonging to this subclass are obtained. Furthermore, the Fekete-Szegö problem for this subclass is solved.

Keywords: Lucas polynomials, bi-univalent functions, analytic functions, Fekete-Szegö problem.

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## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions $f$ defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [7]; see also some of the recent investigations [2, 3, 4, 14, 19]).

Two of the important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha$ in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}, \tag{2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : \ f \in \mathcal{S} \ \text{ and } \ \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}. \tag{3}$$

It is clear from the definitions (2) and (3) that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$. Also we have

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$$f(z) \in \mathcal{K}(\alpha) \text{ iff } zf'(z) \in \mathcal{S}^*(\alpha),$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \text{ iff } \int_0^z \frac{f(t)}{t}dt = F(z) \in \mathcal{K}(\alpha).$$

It is well-known that, if $f(z)$ is an univalent analytic function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g\left(f(z)\right) = z, \quad (z \in \mathbb{D}_1),$$

is an analytic and univalent mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$. Moreover, by the familiar Koebe one-quarter theorem (for details, see [7]), we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.

According to this, every function $f \in \mathcal{S}$ has an inverse map $f^{-1}$ that satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Examples of functions in the class $\Sigma$ are

$$\frac{z}{1-z}, \ -\log(1-z), \ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \cdots .$$

It is worth noting that the familiar Koebe function is not a member of $\Sigma$, since it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. Thus, clearly, the image of the domain does not contain the unit disk $\mathbb{U}$. For a brief history and some intriguing examples of functions and characterization of the class $\Sigma$, see Srivastava et al. [12] and Yousef et al. [15, 16, 17].

In 1967, Lewin [9] investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [11] showed that $\max\limits_{f\in\Sigma} |a_2| = \frac{4}{3}$. The best known estimate for functions in $\Sigma$ has been obtained in 1984 by Tan [13], that is, $|a_2| < 1.485$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1, 2\})$ for each $f \in \Sigma$ given by (1) is presumably still an open problem.

For the polynomials $p(x)$ and $q(x)$ with real coefficients, the $(p, q)$ −Lucas polynomials $L_{p,q,k}(x)$ are defined by the following recurrence relation (see [8]):

$$L_{p,q,k}(x) = p(x)L_{p,q,k-1}(x) + q(x)L_{p,q,k-2}(x), \quad (k \geq 2),$$

with

$$L_{p,q,0}(x) = 2, \ L_{p,q,1}(x) = p(x), \text{ and } L_{p,q,2}(x) = p^2(x) + 2q(x). \tag{5}$$

The generating function of the $(p, q)$ −Lucas Polynomials $L_{p,q,k}(x)$ (see [10]) is given by

$$A_{\{L_{p,q,k}(x)\}}(z) = \sum_{k=2}^{\infty} L_{p,q,k}(x)z^k = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}.$$

The concept of $(p,q)$ −Lucas polynomials was introduced by Altınkaya and Yalçın [1]. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

## 2. The class $\mathfrak{B}_{\Sigma}^{\mu}(\alpha,\lambda,\delta)$

Firstly, we consider a comprehensive class of analytic bi-univalent functions introduced and studied by Yousef et al. [18] defined as follows:

**Definition 2.1.** *(See* [18]*) For $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 0$ and $0 \leq \alpha < 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathfrak{B}_{\Sigma}^{\mu}(\alpha,\lambda,\delta)$ if the following conditions hold for all $z, w \in \mathbb{U}$:*

$$Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta z f''(z)\right)>\alpha \tag{6}$$

*and*

$$Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\xi\delta w g''(w)\right)>\alpha, \tag{7}$$

*where the function $g(w)=f^{-1}(w)$ is defined by (4) and $\xi=\frac{2\lambda+\mu}{2\lambda+1}$.*

**Remark 2.1.** *In the following special cases of Definition 2.1; we show how the class of analytic bi-univalent functions $\mathfrak{B}_{\Sigma}^{\mu}(\alpha,\lambda,\delta)$ for suitable choices of $\lambda$, $\mu$ and $\delta$ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.*

*(i) For $\delta = 0$, we obtain the bi-univalent function class $\mathfrak{B}_{\Sigma}^{\mu}(\alpha,\lambda,0) := \mathfrak{B}_{\Sigma}^{\mu}(\alpha,\lambda)$ introduced by Çağlar et al.* [6].

*(iii) For $\delta = 0$, $\mu = 1$, and $\lambda = 1$, we obtain the bi-univalent function class $\mathfrak{B}_{\Sigma}^{1}(\alpha,1,0) := \mathfrak{B}_{\Sigma}(\alpha)$ introduced by Srivastava et al.* [12].

*(iv) For $\delta = 0$, $\mu = 0$, and $\lambda = 1$, we obtain the well-known class $\mathfrak{B}_{\Sigma}^{0}(\alpha,1,0) := \mathcal{S}_{\Sigma}^{*}(\alpha)$ of bi-starlike functions of order $\alpha$.*

*(iv) For $\mu = 1$, we obtain the well-known class $\mathfrak{B}_{\Sigma}^{1}(\alpha,\lambda,\delta) := \mathfrak{B}_{\Sigma}(\alpha,\lambda,\delta)$ of bi-univalent functions.*

## 3. Main Results

We begin this section by defining the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$ as follows:

**Definition 3.1.** *For $\lambda \geq 1$, $\mu \geq 0$ and $\delta \geq 0$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$ if the following subordinations are satisfied:*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta z f''(z) \prec A_{\{L_{p,q,k}(x)\}}(z)-1$$

*and*

$$(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu}+\lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1}+\xi\delta z\left(f^{-1}(w)\right)''\prec A_{\{L_{p,q,k}(x)\}}(w)-1,$$

*where* $f^{-1}$*is given by (4).*

**Theorem 3.1.** *For* $\lambda\geq 1$, $\mu\geq 0$ *and* $\delta\geq 0$, *let* $f\in\mathcal{A}$ *belongs to the class* $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$. *Then*

$$|a_2|\leq\frac{2\,|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]p^2(x)-2\left(\mu+\lambda+2\xi\delta\right)^2\left(p^2(x)+2q(x)\right)\right|}},$$

*and*

$$|a_3|\leq\frac{p^2(x)}{\left(\mu+\lambda+2\xi\delta\right)^2}+\frac{|p(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

*Proof.* Let $f\in\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$. From Definition 3.1, for some analytic functions $\phi$, $\psi$ such that $\phi(0)=\psi(0)=0$ and $|\phi(z)|<1$, $|\psi(w)|<1$ for all $z,w\in\mathbb{U}$, then we can write

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta zf''(z)=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\phi(z)+L_{p,q,2}(x)\phi^2(z)+\cdots \tag{8}$$

and

$$(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu}+\lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1}+\xi\delta z\left(f^{-1}(w)\right)''=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\psi(w)+L_{p,q,2}(x)\psi^2(w)+\cdots. \tag{9}$$

From the equalities (8) and (9), we obtain that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta zf''(z)=1+L_{p,q,1}(x)r_1z+\left[L_{p,q,1}(x)r_2+L_{p,q,2}(x)r_1^2\right]z^2+\cdots \tag{10}$$

and

$$(1-\lambda)\left(\frac{f^{-1}(w)}{w}\right)^{\mu}+\lambda\left(f^{-1}(w)\right)'\left(\frac{f^{-1}(w)}{w}\right)^{\mu-1}+\xi\delta z\left(f^{-1}(w)\right)''=1+L_{p,q,1}(x)s_1w+\left[L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2\right]w^2+\cdots. \tag{11}$$

It is fairly well known that if

$$|\phi(z)|=\left|r_1z+r_2z^2+r_3z^3+\cdots\right|<1,\quad(z\in\mathbb{U})$$

and

$$|\psi(w)|=\left|s_1w+s_2w^2+s_3w^3+\cdots\right|<1,\quad(w\in\mathbb{U}),$$

then

$$|r_k|<1\text{ and }|s_k|<1\text{ for }k\in\mathbb{N}. \tag{12}$$

Thus, upon comparing the corresponding coefficients in (10) and (11), we have

$$\left(\mu+\lambda+2\xi\delta\right)a_2=L_{p,q,1}(x)r_1, \tag{13}$$

$$(\mu+2\lambda)\left[\left(\frac{\mu-1}{2}\right)a_2^2+\left(1+\frac{6\delta}{2\lambda+1}\right)a_3\right]=L_{p,q,1}(x)r_2+L_{p,q,2}(x)r_1^2, \tag{14}$$

$$-\left(\mu+\lambda+2\xi\delta\right)a_2 = L_{p,q,1}(x)s_1, \tag{15}$$

and

$$(\mu+2\lambda)\left[\left(\frac{\mu+3}{2}+\frac{12\delta}{2\lambda+1}\right)a_2^2-\left(1+\frac{6\delta}{2\lambda+1}\right)a_3\right] = L_{p,q,1}(x)s_2 + L_{p,q,2}(x)s_1^2. \tag{16}$$

It follows from (13) and (15) that

$$r_1 = -s_1 \tag{17}$$

and

$$2\left(\mu+\lambda+2\xi\delta\right)^2 a_2^2 = L_{p,q,1}^2(x)\left(r_1^2+s_1^2\right). \tag{18}$$

If we add (14) and (16), we get

$$(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]a_2^2 = L_{p,q,1}(x)\left(r_2+s_2\right)+L_{p,q,2}(x)\left(r_1^2+s_1^2\right). \tag{19}$$

Substituting the value of $\left(r_1^2+s_1^2\right)$ from (18) in the right hand side of (19), we deduce that

$$\left[(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]L_{p,q,1}^2(x)-2\left(\mu+\lambda+2\xi\delta\right)^2 L_{p,q,2}(x)\right]a_2^2 = L_{p,q,1}^3(x)\left(r_2+s_2\right). \tag{20}$$

Moreover computations using (5), (12) and (20), we find that

$$|a_2| \leq \frac{2\left|p(x)\right|\sqrt{|p(x)|}}{\sqrt{\left|(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]p^2(x)-2\left(\mu+\lambda+2\xi\delta\right)^2\left(p^2(x)+2q(x)\right)\right|}}.$$

Moreover, if we subtract (16) from (14), we obtain

$$2\left(\mu+2\lambda\right)\left(1+\frac{6\delta}{2\lambda+1}\right)\left(a_3-a_2^2\right) = L_{p,q,1}(x)\left(r_2-s_2\right)+L_{p,q,2}(x)\left(r_1^2-s_1^2\right). \tag{21}$$

Then, in view of (17) and (18), Eq. (21) becomes

$$a_3 = \frac{L_{p,q,1}^2(x)}{2\left(\mu+\lambda+2\xi\delta\right)^2}\left(r_1^2+s_1^2\right)+\frac{L_{p,q,1}}{2\left(\mu+2\lambda\right)\left(1+\frac{6\delta}{2\lambda+1}\right)}\left(r_2-s_2\right).$$

Thus applying (5), we conclude that

$$|a_3| \leq \frac{p^2(x)}{\left(\mu+\lambda+2\xi\delta\right)^2}+\frac{|p(x)|}{\left(\mu+2\lambda\right)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

□

By setting $\mu=\delta=0$ and $\lambda=1$ in Theorem 3.1, we obtain the following consequence.

**Corollary 3.1.** *If $f$ belongs to the class $\mathfrak{B}_\Sigma(1)= \mathcal{S}_\Sigma^*$ of bi-starlike functions, then*

$$|a_2| \leq \frac{2\left|p(x)\right|\sqrt{|p(x)|}}{\sqrt{\left|2p^2(x)-2\left(p^2(x)+2q(x)\right)\right|}},$$

*and*

$$|a_3| \leq p^2(x)+\frac{|p(x)|}{2}.$$

## 4. Fekete–Szegö problem for the function class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$

In this section, we aim to provide Fekete–Szegö inequalities for functions in the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$. These inequalities are given in the following theorem.

**Theorem 4.1.** *For $\lambda \geq 1$, $\mu \geq 0$ and $\delta \geq 0$, let $f \in \mathcal{A}$ belongs to the class $\mathfrak{B}_{\Sigma}^{\mu}(\lambda,\delta)$. Then*

$$\left|a_3 - \upsilon a_2^2\right| \leq \begin{cases} \frac{|p(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}, & |\upsilon - 1| \leq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)} \times |\Upsilon(x)| \\ \frac{2|p(x)|^3|1-\upsilon|}{|p(x)\Upsilon(x)|}, & |\upsilon - 1| \geq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)} \times |\Upsilon(x)|, \end{cases}$$

*where* $\Upsilon(x) = (\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]p(x) - 2(\mu+\lambda+2\xi\delta)^2 \frac{p^2(x)+2q(x)}{p(x)}$.

*Proof.* From (20) and (21)

$$\begin{aligned} a_3 - \upsilon a_2^2 &= (1-\upsilon)\frac{L_{p,q,1}^3(x)\,(r_2+s_2)}{\left[(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]L_{p,q,1}^2(x) - 2(\mu+\lambda+2\xi\delta)^2 L_{p,q,2}(x)\right]} \\ &\quad + \frac{L_{p,q,1}}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}(r_2 - s_2) \\ &= L_{p,q,1}\left[\left[\varphi(\upsilon,x) + \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\right]r_2 + \left[\varphi(\upsilon,x) - \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}\right]s_2\right], \end{aligned}$$

where

$$\varphi(\upsilon,x) = \frac{L_{p,q,1}^2(x)\,(1-\upsilon)}{\left[(\mu+2\lambda)\left[1+\mu+\frac{12\delta}{2\lambda+1}\right]L_{p,q,1}^2(x) - 2(\mu+\lambda+2\xi\delta)^2 L_{p,q,2}(x)\right]},$$

Then, in view of (5), we conclude that

$$\left|a_3 - \upsilon a_2^2\right| \leq \begin{cases} \frac{|p(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)} & 0 \leq |\varphi(\upsilon,x)| \leq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}, \\ 2|p(x)|\,|\varphi(\upsilon,x)| & |\varphi(\upsilon,x)| \geq \frac{1}{2(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}. \end{cases}$$

Which completes the proof of Theorem 4.1. □

Putting $\mu = \delta = 0$ and $\lambda = 1$ in Theorem 4.1, we conclude the following result:

**Corollary 4.1.** *If $f$ belongs to the class $\mathcal{S}_{\Sigma}^{*}$, then*

$$\left|a_3 - \upsilon a_2^2\right| \leq \begin{cases} \frac{|p(x)|}{2}, & |\upsilon - 1| \leq \left|\frac{q(x)}{p(x)}\right| \\ \frac{2|p(x)|^3|1-\upsilon|}{4|q(x)|}, & |\upsilon - 1| \geq \left|\frac{q(x)}{p(x)}\right| \end{cases}.$$

Putting $\upsilon = 1$ in Theorem 4.1, we conclude the following result:

**Corollary 4.2.** *If $f$ belongs to the class $\mathcal{S}_{\Sigma}^{*}$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|p(x)|}{(\mu+2\lambda)\left(1+\frac{6\delta}{2\lambda+1}\right)}.$$

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