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AN EFFICIENT METHOD FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH A WEAKLY SINGULAR KERNEL

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Abstract. In this work, a class of volterra integro-differential equation with a weakly singular kernel is discussed. The shifted Legendre Tau method is introduced for finding the unknown function. The proposed method is based on expanding the approximate solution as the elements of a shifted Legendre polynomials. We reduce the problem to a set of algebraic equations by using operational matrices. Also the convergence analysis and error estimation have been discussed and approved with the exact solution. Finally, several numerical examples are given to demonstrate the high accuracy of the method.

Keywords: Shifted Legendre Tau method; Weakly singular kernel; Integro-differential equation.

AMS Subject Classification: 65R20, 45D05.

1. Introduction

In recent years there has been a high level of interest in the field of integro-differential equations, which are the combination of differential and Fredholm-Volterra integral equations. Many phenomena in various fields of engineering, mechanics, economics, potential theory, astronomy, chemistry, physics, electrostatics, etc, are modeled by integro-differential equation [2, 3, 17].

In 1981, Ortiz and Samara [12] proposed an operational technique for the numerical solution of nonlinear ordinary differential equations. During recent years considerable works have been done for solving integral equations using Tau method [4,5,14,16]. We can see the progress of this method for the numerical solution of partial differential equations

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and their related eigenvalue problems, iterated solutions of linear operator equations [1, 8–11, 13].

Let us consider the general form of a class of weakly singular kernel of integro-differential equation.

$$Dy(t) = f(t) + \lambda_1 \int_0^t \frac{y^{(k)}(s)}{(t-s)^{\alpha}} ds + \lambda_2 \int_0^t k(t,s) y^{(l)}(s) ds, \quad t \geqslant 0, \quad s \in [0,b], \tag{1.1}$$

where D is a linear differential operator of order $m-1$ with polynomial coefficient $p_i(t)$,

$$D = \sum_{i=0}^{J} p_i(t) \frac{d^i}{dt^i}. \tag{1.2}$$

under the conditions

$$\sum_{k=0}^{m-1} a_{jk} y^{(k)}(0) = d_j, \quad m = max\{J, l, k\}, \quad j = 0, 1, \ldots, m-1, \tag{1.3}$$

Also, $y^{(0)}(t) = y(t)$ is an unknown function, the known functions $p_i(t)$, $f(t)$ are defined on interval $0 \leqslant t \leqslant b$, and a_{jk}, d_j, λ_1, λ_2 are real or complex constants.
For $\lambda_2 = 0$, Equation (1.1) becomes Volterra integro-differential equation with a weakly singular kernel. Especially if $k = 0$, $\alpha = \frac{1}{2}$ and all $p_i = 0$ Equation (1.1) reduces to the Abel's integral equation

$$f(t) = -\lambda_1 \int_0^t \frac{y(s)}{\sqrt{t-s}} ds,$$

which occurs in many branches of science such as microscopy, seismology, radio astronomy, atomic scattering. Some authors have used Tau method for solving integro-differential equations [6, 7, 12, 15]. The Organization of the rest of this article is as follows. In the next section, we describe the basic formulation of shifted Legendre polynomials. In section 3, we construct the operational matrices of Legendre polynomials. Section 4, by using Tau spectral method we construct and develop an algorithm for the solution of weakly singular Volterra integro-differential equation with boundary conditions. We discuss the convergence analysis and error estimation for the proposed function approximation in sections 5, 6. In section 7, some illustrative numerical experiments are given. The paper ends with some conclusions in section 8.

2. Properties of shifted Legendre polynomials

The classical Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae

$$L_0(x) = 1, \ L_1(x) = x,$$

$$L_{i+1}(x) = \frac{2i+1}{i+1} x \ L_i(x) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \ldots .$$

For $t \in [0, l]$, let $L_{l,i}(t) = L_i(\frac{2t-l}{l})$, $\quad i = 0, 1, 2, \ldots$. Then the shifted Legendre polynomials $\{L_{l,i}(t)\}$ are defined by

$$L_{l,0}(t) = 1, \quad L_{l,1}(t) = \frac{2t-l}{l},$$

$$L_{l,i+1}(t) = \frac{(2i+1)(2t-l)}{(i+1)l} L_{l,i}(t) - \frac{i}{i+1} L_{l,i-1}(t), \quad i = 1, 2, \ldots ..$$

If $\Phi_{l,m}(t)$ be a vector function of shifted Legendre polynomials on the interval $[0, l]$, as

$$\Phi_{l,m}(t) = [L_{l,0}, L_{l,1}, \ldots, L_{l,m}]^T, \tag{2.1}$$

then the set of $L_{l,i}(t)$ is a complete $L^2(0, l)$-orthogonal system, namely

$$\int_0^l L_{l,i}(t)L_{l,j}(t)dt = \begin{cases} \dfrac{l}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases}$$

So we define $\Pi_m = span\{L_{l,0}, L_{l,1}, \ldots, L_{l,m}\}$. For any $y(t) \in L^2(0, l)$, we write $y(t) = \sum_{j=0}^{\infty} c_j L_{l,j}(t)$, where the coefficients c_j are given by

$$c_j = \frac{2j+1}{l} \int_0^l y(t)L_{l,j}(t)dt, \quad j = 0, 1, 2, \ldots. \tag{2.2}$$

In practice, only the first $(m+1)-$ terms of shifted Legendre polynomials are considered. Hence we can write

$$y_m(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T \Phi_{l,m}(t) = C^T V X_t, \tag{2.3}$$

where $C^T = [c_0, c_1, \ldots, c_m]$ and V is a non-singular matrix given by $\Phi_{l,m}(t) = V X_t$ with a standard basic vector , $X_t = \left[1, t, t^2, \ldots, t^m\right]^T$, $(.)^T$ stands for the transpose.

Similarly a function of two independent variables $k(t, s)$ may be expressed in terms of the double shifted Legendre polynomials as

$$k(t, s) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m} k_{i,j} L_{l,i}(t)L_{l,j}(s) = \Phi_{l,m}^T(t) K \Phi_{l,m}(s), \tag{2.4}$$

where K is a $(m+1) \times (m+1)$ matrix and

$$k_{i,j} = \left(\frac{2i+1}{l}\right)\left(\frac{2j+1}{l}\right) \int_0^l \int_0^l k(t, s)L_{l,i}(t)L_{l,j}(s)dtds, \qquad i, j = 0, 1, \ldots, m. \tag{2.5}$$

Also, $k(t, s)$ can be expressed as $k(t, s) \simeq \Phi_{l,m}^T(t) K \Phi_{l,m}(s) = X_t^T V^T K V X_s$, where $V = [v_{i,j}]_{i,j=0,1,\ldots,m}$ is a non-singular matrix given by $\Phi_{l,m}(t) = V X_t$ with a standard basic vector , $X_t = \left[1, t, t^2, \ldots, t^m\right]^T$. If we take $\overline{K} = V^T K V$ then we can write $k(t, s) = X_t^T \overline{K}\, X_s$.

3. Operational matrices of shifted Legendre polynomials

3.1. **Matrix representation of differential part.** As a consequence of the previous section, and aid of following lemma and theorems we derive formulas for numerical solvability of weakly singular Volterra integro-differential equation (1.1) based on shifted Legendre polynomial of the operational Tau method.

Also, we convert the operational approach to the Tau method proposed by Ortiz and Samara [12] is based on following matrices

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \\ & & & & \ddots \end{bmatrix} \text{ and } \eta = \begin{bmatrix} 0 & & & \dots \\ 1 & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & \\ & & & \ddots \end{bmatrix}.$$

Lemma 3.1. *Let $y_m(t) \simeq C^T V X_t$ be a polynomial where $C^T = [c_0, c_1, \dots, c_m, 0, \dots]$, $X_t = [1, t, \dots]^T$ then we have*

$$\frac{d^k}{dt^k} y_m(t) = C^T V \eta^k X_t, \quad t^k y_m(t) = C^T V \mu^k X_t, \quad k = 0, 1, 2, \dots$$

Theorem 3.2. *For any linear differential operator D defined by (1.2) and any series $y(t) \cong C^T \Phi_{l,m}(t) = C^T V X_t$, we have*

$$Dy(t) \cong C^T V \Pi V^{-1} \Phi_{l,m}(t), \tag{3.1}$$

where, $\Pi = \sum_{i=0}^{m} \eta^i p_i(\mu)$.

Proof. See [12]. □

3.2. **Matrix representation of integral parts.**

Theorem 3.3. *Let $\Phi_{l,m}(t) = V X_t$ be the shifted Legendre vector then*

$$\int_0^t \frac{y^{(k)}(s)}{(t-s)^\alpha} ds \simeq C^T V \eta^k \Gamma A\, V\, X_t, \tag{3.2}$$

where Γ is a diagonal matrix with elements $\Gamma_{i,i} = \frac{\Gamma(1-\alpha)\Gamma(i+1)}{\Gamma(i-\alpha+2)}$, $i = 0, 1, 2, \dots, m,$ and

$$A = \Big[B_0, B_1, \dots, B_m\Big]^T, \qquad B_j = \Big[t_{j,0}, t_{j,1}, \dots, t_{j,m}\Big]$$

which $t_{j,i}$, $i, j = 0, 1, \dots, m$ is the coefficients of $L_{l,i}$,$i = 0, 1, \dots, m$ in expansion of $t^{j-\alpha+1}$.

Proof.

$$\int_0^t \frac{y^{(k)}(s)}{(t-s)^\alpha} \mathrm{d}s \simeq \int_0^t \frac{C^T V \eta^k X_s}{(t-s)^\alpha} \mathrm{d}s = C^T V \eta^k \int_0^t \frac{[1, s, \dots, s^m]^T}{(t-s)^\alpha} \mathrm{d}s$$
$$= C^T V \eta^k \left[\int_0^t \frac{1}{(t-s)^\alpha} \mathrm{d}s, \int_0^t \frac{s}{(t-s)^\alpha} \mathrm{d}s, \dots, \int_0^t \frac{s^m}{(t-s)^\alpha} \mathrm{d}s \right]^T,$$

by using the relation

$$\int_0^t \frac{s^m}{(t-s)^\alpha} \mathrm{d}s = \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)} t^{m-\alpha+1}, \qquad m = 0, 1, 2, \dots.$$

we can write

$$\int_0^t \frac{y^{(k)}(s)}{(t-s)^\alpha} \mathrm{d}s \simeq C^T V \eta^k \left[\frac{\Gamma(1-\alpha)\Gamma(1)}{\Gamma(-\alpha+2)} t^{-\alpha+1}, \frac{\Gamma(1-\alpha)\Gamma(2)}{\Gamma(-\alpha+3)} t^{-\alpha+2}, \dots, \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)} t^{m-\alpha+1} \right]^T$$
$$= C^T V \eta^k \Gamma \Pi, \tag{3.3}$$

where $\Pi = \Big[t^{-\alpha+1}, t^{-\alpha+2}, \dots, t^{m-\alpha+1}\Big]^T$. By approximating $t^{j-\alpha+1}$, $j = 0, 1, \dots, m$, we get

$t^{j-\alpha+1} = \sum_{i=0}^{m} t_{j,i} L_{l,i}(t) = B_j \Phi_{l,m}(t), \quad B_j = [t_{j,0}, t_{j,1}, \dots, t_{j,m}]$,
then we obtain

$$\Pi = [B_0 V X_t, B_1 V X_t, \dots, B_m V X_t]^T = A\Phi_{l,m}(t), \qquad A = [B_0, B_1, \dots, B_m]^T. \tag{3.4}$$

By substituting (3.4) into (3.3) we obtain

$$\int_0^t \frac{y^{(k)}(s)}{(t-s)^\alpha}\,\mathrm{d}s \simeq C^T V \eta^k \Gamma A V X_t. \tag{3.5}$$

□

Theorem 3.4. *Let the analytic function $y(t)$ and $k(t,s)$ be expressed as* (2.3) *and* (2.5) *then we have*

$$\int_0^t k(t,s) y^{(l)}(s) ds \simeq C^T V \eta^l M X_t, \tag{3.6}$$

where

$$M = \begin{bmatrix} 0 & \overline{k}_{0,0} & \overline{k}_{0,1} + \frac{1}{2}\overline{k}_{1,0} & \overline{k}_{0,2} + \frac{1}{2}\overline{k}_{1,1} + \frac{1}{3}\overline{k}_{2,0} & \cdots \\ 0 & 0 & \frac{1}{2}\overline{k}_{0,0} & \frac{1}{2}\overline{k}_{0,1} + \frac{1}{3}\overline{k}_{1,0} & \cdots \\ 0 & 0 & 0 & \frac{1}{3}\overline{k}_{1,0} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \frac{1}{m}\overline{k}_{0,0} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. Using lemma (3.1) we have $y^{(l)}(s) \simeq C^T V \eta^l X_s$,

$$k(t,s) y^{(l)}(s) \simeq C^T V \eta^l \Big[k(t,s), sk(t,s), s^2 k(t,s), \dots, s^m k(t,s)\Big]^T,$$

$$k(t,s) s^n = \sum_{i=0}^{m} \sum_{j=0}^{m} \overline{K}_{i,j} t^j s^{n+i},$$

so the desired integration term can be written as

$$\int_0^t k(t,s) y^{(l)}(s) ds \simeq C^T V \eta^l \Big[\sum_{i=0}^{m} \sum_{j=0}^{m} \overline{K}_{i,j} t^j \frac{t^{n+i+1}}{n+i+1}\Big]_{n=0}^{m},$$

on the other hand we can show

$$\sum_{i=0}^{m} \sum_{j=0}^{m} \overline{K}_{i,j} \frac{t^{n+j+i+1}}{n+i+1} = \Big[\frac{1}{n+i+1}\Big]_{i=0}^{m} \overline{K}(n) X_t,$$

such that $\overline{K}(n)$ is a matrix having the following entries

$$\overline{k}_{(i,j)}(n) = \begin{cases} \overline{k}_{(i,j-i-1-n)}, & j > n+i, \\ 0, & j \le n+i. \end{cases}$$

Therefore, we can write

$$\begin{aligned} \int_0^t k(t,s) y^{(l)}(s) ds &\simeq C^T V \eta^l \Big(\Big[\Big[\frac{1}{n+i+1}\Big]_{i=0}^{m} \overline{k}(n) X_t\Big]_{n=0}^{m}\Big)^T \\ &= C^T V \eta^l \Big[\Big[\frac{1}{n+i+1}\Big]_{i=0}^{m} \overline{k}(n) X_t\Big]_{n=0}^{m} X_t \\ &= C^T V \eta^l M X_t. \end{aligned}$$

□

4. Matrix representation for the supplementary conditions

Let $y(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T V X_t$ on the left hand side of (1.3), it can be written as

$$\sum_{k=0}^{m-1} a_{jk} y^{(k)}(0) = d_j, \quad m = max(J,l,k), \quad j = 0,1,\dots,m-1,$$

$$C^T V \sum_{k=0}^{m-1} \Big[a_{jk} \eta^k X_0 \Big] = d_j.$$

Let $H_j = \sum_{k=0}^{m-1} a_{jk} \eta^k X_0$ where $X_0 = [1,0,0,\dots,0]^T$ thus the (jth) condition number of (1.3) is converted to

$$C^T V H_j = d_j \quad j = 0,1,\dots,m-1.$$

Now by setting H as the matrix with columns H_j, $j = 0,1,\dots,m-1$ and by setting $d = [d_0, d_1, \dots, d_{m-1}]$, as the vector that contains right-hand side of supplementary conditions, they take the form

$$C^T V H = d. \tag{4.1}$$

Now, Let us start our algorithm to solve (1.1), (1.3).

We approximate $f(t)$ by the shifted Legendre polynomials as

$$f(t) \simeq \sum_{j=0}^{m} f_j L_{l,j}(t) = F V X_t, \tag{4.2}$$

where $F = [f_0, f_1, \dots, f_m]$ and f_j are given in (2.2).

Using (3.1), (4.2), (3.2), (3.6) and substituting in equation (1.1), it is easy to obtain that

$$C^T V \Pi X_t = F V X_t + \lambda_1 C^T V \eta^k \Gamma A V X_t + \lambda_2 C^T V \eta^l M X_t,$$

thus, the matrix vector multiplication representation for the (1.1) is as follows

$$C^T \Pi_v \Phi_{l,m}(t) = F \Phi_{l,m}(t) + \lambda_1 C^T K_1 \Phi_{l,m}(t) + \lambda_2 C^T M_1 \Phi_{l,m}(t),$$

where $\Pi_v = V \Pi V^{-1}$, $K_1 = V \eta^k \Gamma A$ and $M_1 = V \eta^l M V^{-1}$. As we pointed out in section 2, the orthogonality of $\{L_{l,i}(t)\}_{i=0}^{m-1}$, so we have

$$C^T \Pi_v = F + \lambda_1 C^T K_1 + \lambda_2 C^T M_1,$$

also from equation(4.1) we have following system

$$\begin{cases} C^T [\Pi_v - \lambda_1 K_1 - \lambda_2 M_1] = F, \\ C^T V H = d. \end{cases} \tag{4.3}$$

Now setting

$$\Delta = \Pi_v - \lambda_1 K_1 + \lambda_2 M_1, \quad \overline{H} = V H$$

and

$$G = [\overline{H_1}, \overline{H_2}, \dots, \overline{H_J}, \Delta_1, \Delta_2, \dots, \Delta_{m+1-J}], \quad g = [d_1, d_2, \dots, d_J, F_0, F_1, \dots, F_{m-J}],$$

where $\overline{H_i}$ denotes the (ith) column of $\overline{H}$, system of (4.3) can be written as $C^T G = g$, which must be solved for the unknown coefficients $c_0, c_1, \dots, c_m$.

5. Convergence analysis

In this section we present the shifted Legendre expansion of a function $y(t)$ with bounded second derivative, converges uniformly to $y(t)$. Also, we state the estimated error for the proposed method.

Theorem 5.1. *(convergence theorem) If a continuous function $y(t)$, defined on $[0, l]$, has bounded the second derivative $\frac{d^2y}{dt^2}$, then the shifted Legendre expansion of the function as $\sum_{i=0}^{\infty} c_i L_{l,i}(t)$ converges uniformly to the $y(t)$.*

Proof. Let $y(t)$ be a function defined on $[0, l]$ such that $\left|\frac{d^2y(t)}{dt^2}\right| \leq \alpha$, where α is a positive constant and

$$c_i = \left(\frac{2i+1}{l}\right) \int_0^l y(t) L_{l,i}(t) dt, \quad i = 0, 1, \dots, m.$$

By partial integration and using following equation

$$L'_{l,i+1} - L'_{l,i-1} = \frac{2}{l}(2i+1)L_{l,i}(t)$$

we have

$$\begin{aligned} c_i &= \frac{1}{2}\Big(y(t)\Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\Big|_0^l - \int_0^l \Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\frac{\partial y}{\partial t} dt \\ &= -\frac{1}{2}\int_0^l \frac{l}{2(2i+3)}\Big(L'_{l,i+2}(t) - L'_{l,i}(t)\Big)\frac{\partial y}{\partial t} dt + \frac{l}{2}\int_0^l \frac{l}{2(2i-1)}\Big(L'_{l,i}(t) - L'_{l,i-2}(t)\Big)\frac{\partial y}{\partial t} dt \\ &= \frac{l}{4}\int_0^l \frac{\partial^2 y(t)}{\partial t^2}\Big(\frac{L_{l,i+2}(t) - L_{l,i}(t)}{2i+3}\Big)dt - \frac{l}{4}\int_0^l \frac{\partial^2 y(t)}{\partial t^2}\Big(\frac{L_{l,i}(t) - L_{l,i-1}(t)}{2i-1}\Big)dt \end{aligned}$$

.

Now, let $Q_{l,i}(t) = (2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)$ then we have

$$c_i = \frac{l}{4(2i+3)(2i-1)} \int_0^l \frac{\partial^2 y(t)}{\partial t^2} Q_{l,i}(t) dt,$$

thus $|c_i| \leq \frac{l\alpha}{4(2i+3)(2i-1)} \int_0^l |Q_{l,i}(t)| dt$.
Also we have

$$\begin{aligned} \Big(\int_0^l |Q_{l,i}(t)| dt\Big)^2 &= \Big(\int_0^l |(2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)| dt\Big)^2 \\ &\leq l\Big(\frac{(2i-1)^2 l}{2i+5} + \frac{(4i+2)^2 l}{2i+1} + \frac{(2i+3)^2 l}{2i-3}\Big) \\ &\leq \frac{6l^2(2i+3)^2}{2i-3}. \end{aligned}$$

Thus we obtain $|c_i| \leq \frac{l^2\alpha\sqrt{6}}{4\sqrt{(2i-3)^3}}$. Consequently, $\sum_{i=0}^{\infty} c_i$ is absolutely convergent and thus the expansion of the function converges uniformly. □

Theorem 5.2. *Let $y(t)$ be a continuous function defined on $[0,l]$ with bounded second derivative, say* $\left|\frac{\partial^2 y(t)}{\partial t^2}\right| \leq \alpha$, *then we have the following accuracy estimation*

$$\varepsilon_n \leq \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}},$$

where

$$\varepsilon_n = \Big(\int_0^l (y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t))^2 dt\Big)^{1/2}. \tag{5.1}$$

Proof.

$$\begin{aligned}
\varepsilon_n^2 &= \int_0^l \Big(y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt = \int_0^l \Big(\sum_{i=0}^{\infty} c_i L_{l,i}(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \\
&= \sum_{i=m+1}^{\infty} c_i^2 \int_0^l L_{l,i}^2(t) dt = \sum_{i=m+1}^{\infty} c_i^2 \frac{l}{(2i+1)} \\
&\leq \sum_{i=m+1}^{\infty} \frac{6\alpha^2 l^5}{16(2i-3)^3(2i+1)} \\
&\leq \frac{6\alpha^2 l^5}{16} \sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}
\end{aligned}$$

Then we have, $\varepsilon_n \leq \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}}$. □

6. Error estimation

In this section, we state the estimated error for the weakly singular Volterra integro-differential equation (1.1). Firstly, we define

$$e_m(t) = y(t) - y_m(t) \tag{6.1}$$

If $y_m(t)$ is a good approximation for $y(t)$ then for a given ϵ, Max $|e_m(t)| < \epsilon$. For this purpose, we are looking for an approximation for $e_m(t)$ by using the same method we used for approximation of $y(t)$. Firstly, we obtain from equation (6.1) that

$$y(t) = e_m(t) + y_m(t), \tag{6.2}$$

Therefore by using equations, (6.2) and (1.1)we have,

$$\begin{aligned}
&D(e_m(t) + y_m(t)) = f(t) + \lambda_1 \int_0^t \frac{e_m^{(k)}(s) + y_m^{(k)}(s)}{(t-s)^{\alpha}} ds + \lambda_2 \int_0^t k(t,s)(e_m^{(l)}(s) + y_m^{(l)}(s)) ds, \\
&\sum_{k=0}^{m-1} a_{jk}(e_m(t) + y_m(t))^{(k)}(0) = d_j, \quad m = max(J,l,k), \quad j = 0,1,\ldots,m-1,
\end{aligned}$$

also we have

$$D(e_m(t)) = H_m(t) + \lambda_1 \int_0^t \frac{e_m^{(k)}(s)}{(t-s)^{\alpha}} ds + \lambda_2 \int_0^t k(t,s) e_m^{(l)}(s) ds,$$

$$\sum_{k=0}^{m-1} a_{jk}(e_m(t))^{(k)}(0) = 0, \quad m = max(J, l, k), \quad j = 0, 1, \ldots, m-1,$$

$H_m(t)$ is a perturbation term associated with $y_m(t)$ and can be obtained with following formulae

$$H_m(t) = f(t) - D(y_m(t)) + \lambda_1 \int_0^t \frac{y_m^{(k)}(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t k(t,s) y_m^{(l)}(s) ds,$$

We proceed to find an approximation $(e_{m,N})(t)$ to the $e_m(t)$ in the same as we did for the solutions of equations (1.1) and (1.2) . (N denotes the Tau degree of $e_m(t)$). In fact, we solve the same problem of this time with the unknown function $e_m(t)$. By obtaining the approximation of $e_m(t)$, we actually approximate $y(t) - y_m(t)$.

7. Numerical results and comparisons

In this section, we present five numerical examples to demonstrate the accuracy of the proposed method. The results show that this method, by selecting a few numbers of shifted Legendre polynomials is accurate. Let $t_i = ih$, $i = 0, 1, 2, \ldots, N$, $h = \dfrac{L}{N}$ where N denotes the final space level t_N , $N+1$ is the number of nodes. In order to check the accuracy of the proposed method, the maximum absolute errors and L_2 norm errors between the exact solution $y(t)$ and the approximate solution $y_m(t)$ are given by the following definitions.

Maximum norm error: $\| e_M \|_\infty = max \mid y(t_i) - y_m(t_i) \mid$.

L_2 norm error: $\dfrac{1}{N}\left(\sum_{i=0}^{N} \mid y(t_i) - y_m(t_i) \mid^2\right)^{1/2}$.

Example 1. *As a first application, we offer the following Volterra integro-differential equation with weakly singular kernel*

$$\begin{cases} y''(t) - t^2 y'(t) = 6t - \frac{1}{2}t^4 - \frac{16}{5}x^{\frac{5}{2}} + \int_0^t \dfrac{y'(s)}{\sqrt{t-s}} \mathrm{d}s - \frac{5}{4}\int_0^t tsy''(s)\mathrm{d}s, \quad s \le 1, \quad t > 0, \\ y(0) = 1, \quad y'(0) = 0, \end{cases}$$

The exact solution $y(t) = t^3 + 1$. we have obtained the exact solution in four terms $m = 3$ which shows the high accuracy of the method.

Example 2. *Consider the following Volterra integro-differential equation with weakly singular kernel*

$$\begin{cases} y^{(4)}(t) - y(t) = f(t) + \dfrac{1}{2}\int_0^t \dfrac{y^{(3)}(s)}{\sqrt{t-s}} \mathrm{d}s, \qquad s \le 1, \quad t > 0, \\ y(0) = y'(0) = 1, \quad y''(0) = 2, \quad y'''(0) = 3, \end{cases}$$

with $f(t) = -1 + 4e^t + \frac{1}{2}\left(-\sqrt{t} - \frac{1}{2}e^t\sqrt{\pi}(5+2t)erf(\sqrt{t})\right)$ *where* $erf(t)$ *is the error function.*

The exact solution of this example is $y(t) = 1 + te^t$. The maximum absolute errors and L_2 norm errors between the exact solution $y(t)$ and the approximate solution $y_m(t)$ with various choices of m and two different values $N = 100$, and $N = 50$, are presented in Table 1. We see in this table that the results are accurate for even small choices of m. The graphs of the maximum absolute error function for $m = 9$, $m = 7$ are shown in Figures 1 and 2.

TABLE 1. $\| e_M \|_\infty$ is the Maximum norm error and $\| e_M \|_2$ is L_2 norm error .

	$N=100$		$N=50$	
m	$\| e_M \|_\infty$	$\| e_M \|_2$	$\| e_M \|_\infty$	$\| e_M \|_2$
7	1.49×10^{-6}	8.73×10^{-8}	1.49×10^{-6}	1.23×10^{-7}
9	4.17×10^{-10}	2.00×10^{-11}	4.17×10^{-10}	2.83×10^{-11}
12	3.10×10^{-15}	1.44×10^{-16}	3.10×10^{-15}	2.10×10^{-16}
16	1.33×10^{-15}	4.17×10^{-17}	1.33×10^{-15}	6.21×10^{-17}
15	2.12×10^{-7}	1.27×10^{-8}	2.07×10^{-7}	1.79×10^{-8}

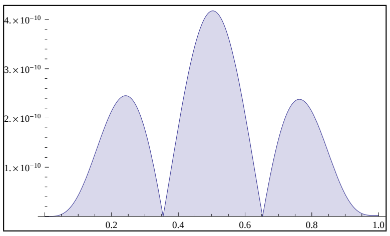

FIGURE 1. The maximum absolute error function for $m = 9$

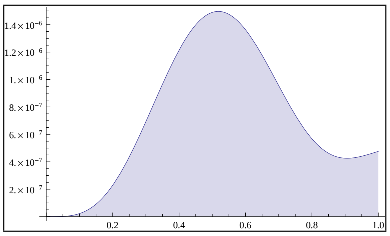

FIGURE 2. The maximum absolute error function for $m = 7$

Example 3. *Consider the following nonlinear Volterra integral equation [18]*

$$\begin{cases} \int_0^t cos(t-s)\, y''(s)\mathrm{d}s = 6(1 - cost), & 0 \le t \le 1, \\ y(0) = y'(0) = 1. \end{cases}$$

The exact solution of this example is $y(t) = t^3$. In this example, we implement the shifted Legendre Tau method to solve this kind of Volterra integral equation solved in [18] by a method based on Haar functions. We show the comparison of the numerical and exact solution for $m = 8$ in Figure 3. In Table 2, we make a comparison of the presented algorithm with the Haar wavelet method proposed in [18]. The maximum absolute errors for different values of m are shown in Table 2. Obviously, the absolute errors of proposed method are low as compared to the absolute errors in [18]. From the results of this table, the best results we have achieved is at $m = 16$. Also, we plot the logarithmic graph of maximum absolute error ($log_{10}Error$) with various values of m in Figure 4.

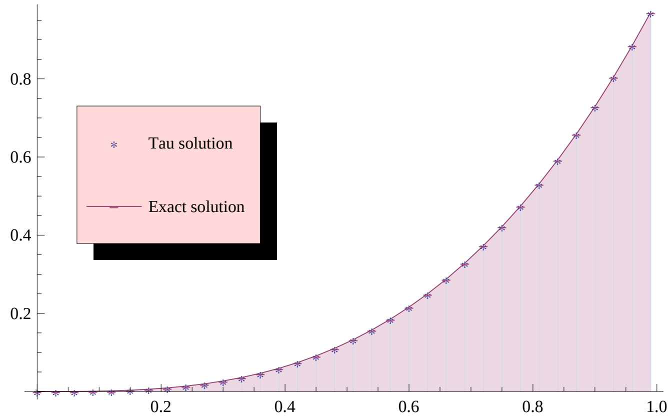

FIGURE 3. Comparison of numerical and exact solutions of Example 3 for $m = 8$

TABLE 2. $\| e_M \|_\infty$ is the Maximum norm error.

	SLT method, N=100	HW method [18]
m	$\| e_M \|_\infty$	$\| e_M \|_\infty$
8	4.31×10^{-6}	8.4×10^{-3}
10	6.34×10^{-4}	–
12	5.50×10^{-10}	–
14	3.38×10^{-12}	–
16	1.50×10^{-14}	2.1×10^{-3}

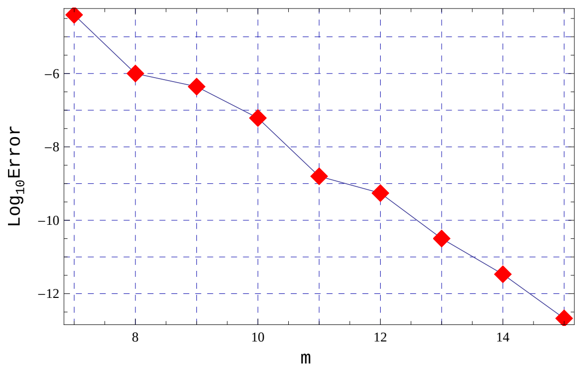

FIGURE 4. The logarithmic graph of maximum absolute error ($log_{10}Error$) with various values of m for Example 3

Example 4. *Consider the following Volterra integro-differential equation*

$$\begin{cases} y^{(4)}(t) + y'(t) - y(t) = f(t) + \int_0^t \dfrac{y'(s)}{\sqrt{t-s}} \mathrm{d}s - \int_0^t y''(s) \mathrm{d}s, \quad 0 \le t \le 1, \\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = 0, \end{cases}$$

where $f(t) = 1 - cos(t) - sin(t) + \sqrt{2\pi}\Big(-cos(t)Fresnels(\sqrt{\frac{2t}{\pi}}) + Fresnelc(\sqrt{\frac{2t}{\pi}})sin(t)\Big).$

The exact solution of this example is $y(t) = cos(t)$. As we expected, Tau method has produced an accurate approximation of the exact solution. The maximum absolute error for different choices of m is shown in Table 3. Also in Figure 5, an illustration of the rate of convergence for the shifted Legendre Tau method with various m is shown.

TABLE 3. Maximal absolute error($|y(t) - y_m(t)|$) for different choices of m.

t	m=n=7	m=n=10	m=n=15
0.1	1.43×10^{-9}	3.06×10^{-14}	2.06×10^{-23}
0.2	1.42×10^{-8}	7.51×10^{-14}	5.95×10^{-23}
0.3	4.18×10^{-8}	7.81×10^{-14}	4.53×10^{-23}
0.4	7.05×10^{-8}	1.90×10^{-13}	2.89×10^{-22}
0.5	8.13×10^{-8}	6.82×10^{-15}	4.29×10^{-22}
0.6	6.61×10^{-8}	1.89×10^{-13}	2.92×10^{-22}
0.7	3.38×10^{-8}	8.86×10^{-14}	4.93×10^{-23}
0.8	3.45×10^{-9}	7.48×10^{-14}	5.67×10^{-23}
0.9	1.19×10^{-8}	3.28×10^{-14}	1.98×10^{-23}
1	1.60×10^{-8}	3.25×10^{-16}	6.79×10^{-29}

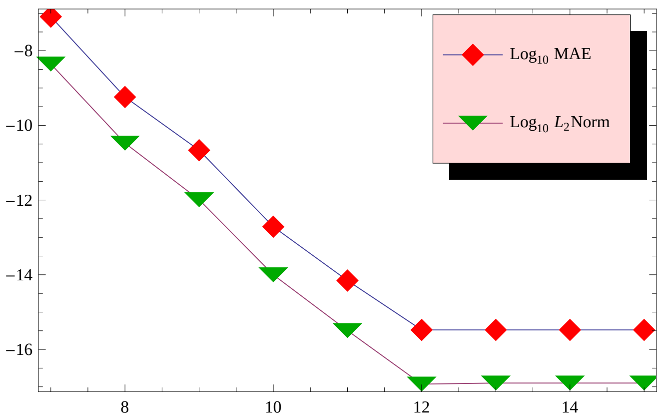

FIGURE 5. An illustration of the rate of convergence for the shifted Legendre Tau method with various m of Example 4

Example 5. *Consider the following Volterra integro-differential equation*

$$\begin{cases} y^{(3)}(t) + y'(t) - y(t) = f(t) + \int_0^t \dfrac{y'(s)}{\sqrt{t-s}}\mathrm{d}s - \int_0^t (t+s)y^{(3)}(s)\mathrm{d}s, & 0 \le t \le 1, \\ y(0) = 1,\ y'(0) = \frac{1}{2},\ y''(0) = \frac{-1}{4}, \end{cases}$$

where $f(t) = \dfrac{3}{8(1+t)^{\frac{5}{2}}} + \dfrac{1}{2\sqrt{1+t}} - \sqrt{1+t} + \frac{1}{4}\Big(2 + t - \dfrac{2(1+2t)}{(1+t)^{\frac{3}{2}}}\Big) - arctan(\sqrt{t})$.

The exact solution of this example is $y(t) = \sqrt{1+t}$. Table 4 shows the Tau error and estimate error discussed in section 6.

TABLE 4. "Error 1" is the Legendre Tau error and "Error 2" is the estimate error that stated in section 6.

t	$m=7$		$m=11$	
	Error 1	Error 2	Error 1	Error 2
0.1	9.34×10^{-8}	9.34×10^{-8}	8.61×10^{-12}	8.61×10^{-12}
0.2	2.21×10^{-7}	2.21×10^{-7}	5.36×10^{-11}	5.36×10^{-11}
0.3	7.42×10^{-8}	7.42×10^{-8}	8.24×10^{-11}	8.24×10^{-11}
0.4	2.07×10^{-7}	2.07×10^{-7}	7.88×10^{-12}	7.88×10^{-12}
0.5	2.94×10^{-7}	2.94×10^{-7}	1.03×10^{-10}	1.03×10^{-10}
0.6	9.02×10^{-8}	9.02×10^{-8}	4.11×10^{-11}	4.11×10^{-11}
0.7	1.81×10^{-7}	1.81×10^{-7}	6.06×10^{-11}	6.06×10^{-11}
0.8	2.40×10^{-7}	2.40×10^{-7}	5.53×10^{-11}	5.53×10^{-11}
0.9	9.13×10^{-8}	9.13×10^{-8}	1.31×10^{-11}	1.31×10^{-11}
1	2.63×10^{-8}	2.63×10^{-8}	5.92×10^{-15}	5.92×10^{-15}

8. Conclusion

In this research, the Volterra integro-differential equation with a weakly singular kernel was solved based on shifted Legendre Tau approximation in conjunction with the operational matrices of partial derivatives and integral parts. The most important section of our method is converting the problem to a linear system of algebraic equations. The performance of the proposed method for the considered problems was measured by calculating the maximum norm error and L_2 norm error. Also, the proposed method was validated numerically by some numerical examples and it was seen that the method exhibits the exponential convergence and produces highly accurate results for smooth solutions.

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