TWMS J. App. and Eng. Math. V.12, N.1, 2022, pp. 302-313

DENUMERABLY MANY POSITIVE SOLUTIONS FOR RL-FRACTIONAL ORDER BVP HAVING DENUMERABLY MANY SINGULARITIES

K. RAJENDRA PRASAD[1], MAHAMMAD KHUDDUSH[1], M. RASHMITHA[1], §

Abstract. In this paper, we consider Riemann-Liouville two-point fractional order boundary value problem having denumerably many singularities and determined sufficient conditions for the existence of denumerably many positive solutions by an application of Krasnoselskii's cone fixed point theorem in a Banach space.

Keywords: fractional derivative, homeomorphism, homomorphism, cone, Krasnoselskii's fixed point theorem, positive solutions.

AMS Subject Classification: 47B34, 34A08, 34B18.

1. Introduction

Recently, boundary value problems associated with fractional order differential equations received a significant popularity due to its attractive applications in various fields of chemistry, physics, polymer rheology, aerodynamics, etc. Further, fractional order differential equations have been used for mathematical modeling in potential fields, signal processing, viscoelastic materials, diffusion problems, heat propagation, control theory and many others. In recent years, there are certain research articles on the existence and uniqueness solutions of fractional order nonlinear boundary value problems. Among them we refer few articles, Benchohra et al.[2] studied the class of fractional order boundary value problem using the technique associated with measures of noncompactness. Aghajani et al. [1] studied the solvability of fractional order integro-differential equations by alternative Leray-Schauder fixed point theorem. Li et al. [8] studied on the existence of mild solutions for fractional differential equations with nonlocal conditions. Wang et al. [12] investigated some fractional differential equations by new variant fixed point theorem in Banach spaces. Liang et al. [9] studied the coupled system of nonlinear fractional differential equations by applying Monch type fixed point theorem in a Banach space. Borisut

[1] Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India, 530003.
e-mail: rajendra92@rediffmail.com; ORCID: https://orcid.org/0000-0001-8162-1391.
e-mail: khuddush89@gmail.com; ORCID: https://orcid.org/0000-0002-1236-8334.
e-mail: rashmita.mahanty@gmail.com;ORCID: https://orcid.org/0000-0001-9247-8295.
§ Manuscript received: January 27, 2020; accepted: June 20, 2020.

et al. [3] studied fractional order boundary value problem based on Kransnoselskii's fixed point theorem and Darbo's fixed point theorem.

In 1983, Leibenson [7] is the first to introduced the p-Laplacian equation,

$$\big(\phi_p(x'(t))\big)' = f\big(t, x(t), x'(t)\big),$$

where $\phi_p(\xi) = |\xi|^{p-2}\xi,\ p > 1$. The operator ϕ_p is invertible and $\phi_q(q > 1)$ is its inverse operator such that $q = p/(p-1)$.

The p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional p-Laplacian differential equations. Moreover, the Increasing Homeomorphism and Positive Homomorphism Operators(IHPHO) generalizes and improves the p-Laplacian operator. So research on fractional order boundary value problem with IHPHO has gained momentum.

In [13], Zhao and Liu studied the following fractional differential equation,

$$\begin{gathered}(\phi({}^{C}\mathfrak{D}^{v}_{0+}u(t)))' + h(t)\mathrm{g}(t, u(\theta(t))) = 0,\ t \in (0,1),\\ u(0) = au(1),\ u'(1) = bu'(0) + \lambda[u],\ u^{i}(0) = 0,\ i = 2, \cdots, n-1,\end{gathered}$$

where $2 \le n-1 < v \le n$ and ${}^{C}\mathfrak{D}^{v}_{0+}$is the Caputo fractional derivative. In the sense of a monotone homomorphism, they established some sufficient criteria for the existence of at least two monotone positive solutions by employing the fixed point theorem on cone expansion and compression.

In [4], Ege and Topal considered the fractional boundary value problem with IHPHO,

$$\begin{aligned}&{}^{C}\mathfrak{D}^{q}\big(\phi({}^{C}\mathfrak{D}^{r}z(t))\big) + f\big(t, z(t)\big) = 0,\ \ 0 < q \le 1 < r \le 2,\ 0 < t < 1,\\ &\alpha_1 z(0) - \beta_1 z'(0) = -\gamma_1 z(\xi_1),\ \alpha_2 z(1) + \beta_2 z'(1) = -\gamma_2 z(\xi_2),\ {}^{C}\mathfrak{D}^{r}z(0) = 0.\end{aligned}$$

and established existence of positive solutions by utilizing Krasnoselskii's and Legget–Williams cone fixed point theorems on a Banach space.

Recently, Wang and Zhai [11] studied the fractional order infinite-point boundary value problem,

$$\mathfrak{D}^{\beta}_{0+}\Big[\phi_p\big[\mathfrak{D}^{\alpha}_{0+}z(t) - g(t)\big]\Big] + f(t, z(t)) = 0,\ 0 < t < 1,$$

$$z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0,\ \mathfrak{D}^{\alpha}_{0+}z(0) = 0,\ z^{(j)}(1) = \sum_{k=1}^{\infty}\alpha_k z(\xi_k),\ j = 1, 2, \cdots, n-2,$$

where $n-1 < \alpha \le n$, $n \ge 3$, $0 < \beta \le 1$ and studied existence and uniqueness of solutions by fixed point theorem for $\phi - (h, e)-$ concave operators.

In this paper, we focus on the existence of denumerably many positive solutions for Riemann-Liouville fractional order boundary value problem with IHPHO,

$$\left.\begin{aligned}&\mathfrak{D}^{\varrho}_{0+}\Big[\phi\big[\mathfrak{D}^{\vartheta}_{0+}z(t)\big]\Big] + \omega(t)f(z(t)) = 0,\ 0 < t < 1,\\ &z(0) = \mathfrak{D}^{\vartheta}_{0+}z(0) = 0,\ \mathfrak{D}^{\varrho}_{0+}z(1) +\ z(1) = I^{a}_{0+}z(1),\end{aligned}\right\} \tag{1}$$

where $\mathfrak{D}^{\varrho}_{0+}$, $\mathfrak{D}^{\vartheta}_{0+}$ denote fractional derivatives of Riemann-Liouville type with $0 < \varrho \le 1$, $1 < \vartheta \le 2$, I^{a}_{0+} $(a > 0)$ denotes Riemann-Liouville fractional integral, $\phi : \mathbb{R} \to \mathbb{R}$ is a IHPHO with $\phi(0) = 0$ and $\omega(t) \in L^p[0,1](1 \le p \le \infty)$ has denumerably many singularities in the interval $(0, 1/2)$.

We assume that the following conditions are hold throughout the paper:

(H_1) $f : [0, +\infty) \to [0, +\infty)$ is continuous,

(H_2) $\exists\{t_k\}_{k=1}^{\infty} \ni t_1 < 1/2,\ t_{k+1} < t_k,\ \lim_{k\to\infty} t_k = l^* \geq 0,\ \lim_{t\to t_k} \omega(t) = +\infty$ and $\omega(t) \neq 0$ for all $t \in [0,1]$. Further, for $0 \leq \tau \leq 1$, $0 < \phi(\psi(\tau)) < \infty$, where

$$\psi(\tau) = \phi^{-1}\left(\int_0^{\tau} \frac{(\tau - x)^{\varrho-1}}{\Gamma(\varrho)}\omega(x)dx\right),\ 0 < \varrho \leq 1.$$

2. Kernel and it's Bounds

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions, and construct kernel for the boundary value problem (1). Also, we establish certain lemmas to estimate bounds for the kernel.

Definition 2.1. [6] *The Riemann-Liouville(RL) fractional integral of order δ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$\mathcal{I}_{0+}^{\delta} f(s) = \int_0^s \frac{(s-x)^{\delta-1}}{\Gamma(\delta)} f(x)dx,\ \delta > 0.$$

Lemma 2.1. [6, 10] *The general solution to $\mathfrak{D}_{0+}^{\gamma} f(t) = 0$ with $\gamma \in (m-1, m]$ and $m > 1$ is the function*

$$f(t) = \sum_{k=1}^{m} c_k t^{\gamma-k},$$

where c_k is a real number.

Lemma 2.2. [6, 10] *Let $\gamma > 0$. Then for any function $f : (0,\infty) \to \mathbb{R}$, we have*

$$\mathcal{I}_{0+}^{\gamma}\mathfrak{D}_{0+}^{\gamma} f(t) = f(t) + \sum_{k=1}^{m} c_k t^{\gamma-k},$$

where c_k is a real number and $m \in \mathbb{Z}$ is the smallest integer greater that or equal to γ.

Lemma 2.3. *Suppose $(H_1), (H_2)$ and $\vartheta - \varrho - 1 \geq 0$ hold and let $\kappa \in C^2[0,1]$. The boundary value problem*

$$\mathfrak{D}_{0+}^{\vartheta} z(t) + \kappa(t) = 0,\ t \in (0,1), \tag{2}$$

$$z(0) = 0,\ \mathfrak{D}_{0+}^{\varrho} z(1) +\ z(1) = I_{0+}^{a} z(1), \tag{3}$$

has a unique solution,

$$z(t) = \int_0^1 \mathcal{N}(t,\tau)\kappa(\tau)d\tau + \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\kappa(x)dxd\tau,$$

where

$$\mathcal{N}(t,\tau) = \begin{cases} \frac{kt^{\vartheta-1}(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{kt^{\vartheta-1}(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)} - \frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}, & \tau \leq t, \\ \frac{kt^{\vartheta-1}(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{kt^{\vartheta-1}(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}, & t \leq \tau, \end{cases}$$

$$\chi(t,\tau) = At^{\vartheta-1}(1-\tau)^{a-1},\ A = \frac{k\Gamma(a+\vartheta)}{[\Gamma(a+\vartheta) - k\Gamma(\vartheta)]\,\Gamma(a)},\ \text{and}\ \ k = \frac{\Gamma(\vartheta-\varrho)}{\Gamma(\vartheta-\varrho)+\Gamma(\vartheta)}.$$

Proof. The equivalent fractional integral equation to (2) is given by

$$z(t) = -\int_0^t \frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\kappa(\tau)d\tau + C_1 t^{\vartheta-1} + C_2 t^{\vartheta-2},$$

where C_1 and C_2 are constants. Using boundary conditions (3), we get $C_2 = 0$ and

$$C_1 = \int_0^1 \left[\frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right]\kappa(\tau)d\tau + kI_{0+}^{a} z(1).$$

Therefore,

$$z(t) = \int_0^1 \mathcal{N}(t,\tau)\kappa(\tau)d\tau + kt^{\vartheta-1} I_{0+}^{a} z(1). \tag{4}$$

Integrating above identity (4), we get

$$I_{0+}^{a} z(t) = \int_0^1 \frac{(t-\tau)^{a-1}}{\Gamma(a)} \left(\int_0^1 \mathcal{N}(\tau,x)\kappa(x)dx\right) d\tau + kI_{0+}^{a} z(1) \int_0^t \frac{(t-x)^{a-1}}{\Gamma(a)} x^{\vartheta-1}dx.$$

Substituting $t = 1$, one can obtained

$$\begin{aligned} I_{0+}^{a} z(1) = &\int_0^1 \frac{(1-\tau)^{a-1}}{\Gamma(a)} \left(\int_0^1 \mathcal{N}(\tau,x)\kappa(x)dx\right) d\tau + \frac{k}{\Gamma(a)} \left(\frac{\Gamma(a)\Gamma(\vartheta)}{\Gamma(a+\vartheta)}\right) I_{0+}^{a} z(1) \\ I_{0+}^{a} z(1) = &\frac{\Gamma(a+\vartheta)}{(a+\vartheta) - k\Gamma(\vartheta)} \int_0^1 \frac{(1-\tau)^{a-1}}{\Gamma(a)} \int_0^1 \mathcal{N}(\tau,x)\kappa(x)dxd\tau. \end{aligned}$$

Hence from (4), we get

$$z(t) = \int_0^1 \mathcal{N}(t,\tau)\kappa(\tau)d\tau + \int_0^1 \chi(t,\tau) \int_0^1 \mathcal{N}(\tau,x)\kappa(x)dxd\tau.$$

□

Lemma 2.4. *The kernel $\mathcal{N}(t,\tau)$ have the following properties:*

(i) $\mathcal{N}(t,\tau) \geq 0$ *and continuous on* $[0,1]\times[0,1]$,
(ii) $\mathcal{N}(t,\tau) \leq \mathcal{N}(\tau,\tau)$ *for* $t,\tau \in [0,1]$,
(iii) *there exists* $\xi \in (0,\frac{1}{2})$ *such that* $\xi^{\vartheta-1}\mathcal{N}(1,\tau) \leq \mathcal{N}(t,\tau)$ *for* $t \in [\xi, 1-\xi], \tau \in [0,1]$.

Proof. (i) For $0 < \tau \leq t < 1$, we have $-(t-\tau)^{\vartheta-1} \geq t^{\vartheta-1}(1-t)^{\vartheta-1}$. So,

$$\begin{aligned} \mathcal{N}(t,\tau) \geq &t^{\vartheta-1}\left[-\frac{(1-t)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right] \\ \geq &t^{\vartheta-1}(1-\tau)^{\vartheta-1}\left[-\frac{1}{\Gamma(\vartheta)} + \frac{k}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{-\varrho}}{\Gamma(\vartheta-\varrho)}\right] \\ \geq &t^{\vartheta-1}(1-\tau)^{\vartheta-1}\left[-\frac{1}{\Gamma(\vartheta)} + k\left(\frac{1}{\Gamma(\vartheta)} + \frac{1}{\Gamma(\vartheta-\varrho)}\right)\right] = 0. \end{aligned}$$

Other case is obvious. Moreover, from the definition of $\mathcal{N}(t,\tau)$, it is clear that $\mathcal{N}(t,\tau)$ is continuous on $[0,1]\times[0,1]$.
(ii) For $0 < \tau \leq t < 1$, we have

$$\begin{aligned} \frac{\partial \mathcal{N}(t,\tau)}{\partial t} \leq &\frac{(\vartheta-1)}{\Gamma(\vartheta)}\left[kt^{\vartheta-2}\left(1+\frac{\Gamma(\vartheta)}{\vartheta-\varrho}\right) - (t-\tau)^{\vartheta-2}\right] \\ \leq &\frac{(\vartheta-1)}{\Gamma(\vartheta)}\left[t^{\vartheta-2} - (t-\tau)^{\vartheta-2}\right] < 0. \end{aligned}$$

Other case is clear.
(iii) For $0 < \tau \leq t < 1$ and $t \in [\xi, 1-\xi]$, we have

$$\begin{aligned} \mathcal{N}(t,\tau) = &t^{\vartheta-1}\left[-\frac{(1-(\tau/t))^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right] \\ = &t^{\vartheta-1}\left[-\frac{(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right] \geq t^{\vartheta-1}\mathcal{N}(1,\tau) \geq \xi^{\vartheta-1}\mathcal{N}(1,\tau). \end{aligned}$$

For $0 < t \leq \tau < 1$ and $t \in [\xi, 1-\xi]$, we have

$$\mathcal{N}(t,\tau) = t^{\vartheta-1}\left[\frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right]$$
$$\geq t^{\vartheta-1}\left[-\frac{(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{k(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right] \geq t^{\vartheta-1}\mathcal{N}(1,\tau) \geq \xi^{\vartheta-1}\mathcal{N}(1,\tau).$$

□

Lemma 2.5. *Suppose* $\Gamma(a+\vartheta) > k\Gamma(\vartheta)$ *and* $\xi \in (0, \frac{1}{2}]$. *Then*

$$\min_{t\in[\xi,1-\xi]} \chi(t,\tau) \geq \xi^{\vartheta-1} \max_{t\in[0,1]} \chi(t,\tau).$$

Proof. From Lemma 2.3,

$$\chi(t,\tau) = \frac{k\Gamma(a+\vartheta)}{[\Gamma(a+\vartheta) - k\Gamma(\vartheta)]\,\Gamma(a)} t^{\vartheta-1}(1-\tau)^{a-1}.$$

Then,

$$\frac{\min_{t\in[\xi,1-\xi]}\chi(t,\tau)}{\max_{t\in[0,1]}\chi(t,\tau)} = \frac{\xi^{\vartheta-1}(1-\tau)^{a-1}}{(1-\tau)^{a-1}} = \xi^{\vartheta-1}.$$

□

Lemma 2.6. *Suppose* $(H_1), (H_2)$ *hold, then the unique solution of the problem* (1) *is given by*

$$z(t) = \int_0^1 \mathcal{N}(t,\tau)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0+}\big(\omega(\tau)f(z(\tau))\big)\Big)ds$$
$$+ \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau.$$

Proof. Let $\mathfrak{D}^{\vartheta}_{0+}z(t) = u(t)$ and $v = \phi(u)$. Then by the condition $\mathfrak{D}^{\vartheta}_{0+}z(0) = 0$, we have the following problem

$$\mathfrak{D}^{\varrho}_{0+}v(t) + \omega(t)f(z(t)) = 0,\ v(0) = 0.$$

From Lemma 2.2, we see that $v(t) = ct^{\varrho-1} - \mathcal{I}^{\varrho}_{0+}\big(\omega(t)f(z(t))\big)$. Since $v(0) = 0$, we obtain $v(t) = -\mathcal{I}^{\varrho}_{0+}\big(\omega(t)f(z(t))\big)$. Hence, from the Lemma 2.4, the problem

$$\left.\begin{array}{l}\mathfrak{D}^{\vartheta}_{0+}z(t) = -\phi^{-1}\big(I^{\varrho}_{0+}(\omega(t)f(z(t)))\big),\\ \mathfrak{D}^{\varrho}_{0+}z(1) + z(1) = I^{a}_{0+}z(1),\ z(0) = 0\end{array}\right\} \tag{5}$$

has a unique solution

$$z(t) = \int_0^1 \mathcal{N}(t,\tau)\,\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0+}\big(\omega(s)f(z(\tau))\big)\Big)d\tau$$
$$+ \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau.$$

□

Let $\mathcal{X}$ be the Banach space $C([0,1],\mathbb{R})$ equipped with the norm $\|z\| = \max_{t\in[0,1]} |z(t)|$. Define the cone $\mathcal{P} \subset \mathcal{X}$ by

$$\mathcal{P} = \Big\{z \in \mathcal{X} : z(t) \text{ is nonnegative on } [0,1]\Big\},$$

For any $z \in \mathcal{P}$, define an operator $\mathcal{F} : \mathcal{P} \to \mathcal{X}$ by

$$(\mathcal{F}z)(t) = \int_0^1 \mathcal{N}(t,\tau)\phi^{-1}\Big(\mathcal{I}_{0^+}^{\varrho}\big(\omega(s)f(z(\tau))\big)\Big)d\tau + \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}_{0^+}^{\varrho}\big(\omega(x)f(z(x))\big)\Big)dxd\tau.$$

Lemma 2.7. *Assume that $(H_1),(H_2)$ hold. Then for each $\xi \in (0,1/2)$, $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$ and $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Since $\omega(\tau)f(z(\tau))$ is nonnegative for $\tau \in [0,1]$, $z \in \mathcal{P}$ and $\mathcal{N}(t,\tau) \geq 0$ for all $t,\tau \in [0,1]$, it follows that $\mathcal{F}(z(t)) \geq 0$ for all $t \in [0,1]$, $z \in \mathcal{P}$. Thus $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$. By standard methods and application of Arzela-Ascoli theorem, one can prove the operator $\mathcal{F}$ is completely continuous. □

3. Denumerably Infinitely Many Positive Solutions

In this section, we establish the existence of denumerably many positive solutions for (1) by utilizing the following theorems.

Theorem 3.1. [5] *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and Ω_1, Ω_2 are open sets with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\| \leq \|z\|$, $z \in \mathcal{E} \cap \partial\Omega_1$, *and* $\|\mathcal{A}z\| \geq \|z\|$, $z \in \mathcal{E} \cap \partial\Omega_2$, *or*
(b) $\|\mathcal{A}z\| \geq \|z\|$, $z \in \mathcal{E} \cap \partial\Omega_1$, *and* $\|\mathcal{A}z\| \leq \|z\|$, $z \in \mathcal{E} \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *(Hölder's) Let $h \in L^p[0,1]$ and $g \in L^q[0,1]$, where $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then $hg \in L^1[0,1]$ and $\|hg\|_1 \leq \|h\|_p\|g\|_q$. Further, if $h \in L^1[0,1]$ and $g \in L^\infty[0,1]$, then $hg \in L^1[0,1]$ and $\|hg\|_1 \leq \|h\|_1\|g\|_\infty$.*

Consider the following three possible cases for $\omega \in L^p[0,1] : 0 < p < 1, p = 1$, $p = \infty$.
Firstly, we seek denumerably many positive solutions for the case $p > 1$.

Theorem 3.3. *Suppose $(H_1) - (H_2)$ hold, let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence with $t_{k+1} < \xi_k < t_k$. Let $\{E_k\}_{k=1}^{\infty}$ and $\{O_k\}_{k=1}^{\infty}$ be such that*

$$E_{k+1} < O_k < \alpha O_k < E_k, \ k \in \mathbb{N},$$

where

$$\alpha = \max\left\{\frac{1}{\xi_1^{\vartheta-1}\sigma(\xi_1)\phi^{-1}\Big(\mathcal{I}_{0^+}^{\varrho}\big(\omega(\xi_1)\big)\Big)\int_{\xi_1}^{1-\xi_1}\mathcal{N}(1,x)dx}, \ 1\right\}.$$

Assume that f satisfies

(A1) $f(z) \leq \phi(M_1E_k)$ *for all* $t \in [0,1]$, $0 \leq z \leq E_k$, $M_1 < \dfrac{1}{(1+A)\|G(\tau,\tau)\|_q\|\psi\|_p}$,
(A2) $f(z) \geq \phi(\alpha O_k)$ *for all* $t \in [\xi_k, 1-\xi_k]$, $0 \leq z \leq O_k$.

Then the bvp (1) has denumerably many positive solutions $\{z_k\}_{k=1}^{\infty}$ such that $O_k \leq \|z_k\| \leq E_k$ for $k = 1,2,3\cdots$.

Proof. Let $\Omega_{1,k} = \{z \in \mathcal{X} : \|z\| < E_k\}$, $\Omega_{2,k} = \{z \in \mathcal{X} : \|z\| < O_k\}$ be open subsets of $\mathcal{X}$. One can observe from (H_2) that $l^* < t_{k+1} < \xi_k < t_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. Let $z \in \mathcal{P} \cap \partial\Omega_{1,k}$.

Then, $z(s) \leq E_k = \|z\|$ for all $s \in [0,1]$. By $(A1)$, we have

$$\begin{aligned}
\|\mathcal{F}z\| &= \max_{t\in[0,1]} \left\{ \int_0^1 \mathcal{N}(t,\tau)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\tau)f(z(\tau))\big)\Big)d\tau \right. \\
&\qquad \left. + \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau \right\} \\
&\leq \max_{t\in[0,1]} \left\{ M_1E_k\int_0^1 \mathcal{N}(\tau,\tau)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\tau)\big)\Big)d\tau \right. \\
&\qquad \left. + M_1E_k\int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(x,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(x)\big)\Big)dxd\tau \right\} \\
&\leq M_1E_k \max_{t\in[0,1]} \left\{ 1+\int_0^1 \chi(t,\tau)d\tau \right\} \int_0^1 \mathcal{N}(\tau,\tau)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\tau)\big)\Big)d\tau \\
&\leq (1+A)M_1E_k\int_0^1 \mathcal{N}(\tau,\tau)\psi(\tau)d\tau \leq (1+A)M_1E_k\|\mathcal{N}(\tau,\tau)\|_q\|\psi\|_p \leq E_k.
\end{aligned}$$

Since $E_k = \|z\|$ for $z \in \mathcal{P}\cap\partial\Omega_{1,k}$, we get

$$\|\mathcal{F}z\| \leq \|z\|. \tag{6}$$

Let $t \in [\xi_k, 1-\xi_k]$. Then by $(A2)$, we have

$$\begin{aligned}
\|\mathcal{F}z\| &= \max_{t\in[0,1]} \left\{ \int_0^1 \mathcal{N}(t,\tau)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\tau)f(z(\tau))\big)\Big)d\tau \right. \\
&\qquad \left. + \int_0^1 \chi(t,\tau)\int_0^1 \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau \right\} \\
&\geq \max_{t\in[0,1]} \left\{ \int_{\xi_k}^{1-\xi_k} \mathcal{N}(t,\tau)\phi^{-1}\Big(I^{\varrho}_{0^+}(\omega(s)f(z(\tau)))\Big)d\tau \right. \\
&\qquad \left. + \int_{\xi_k}^{1-\xi_k} \chi(t,\tau)\int_{\xi_k}^{1-\xi_k} \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau \right\} \\
&\geq \max_{t\in[0,1]} \left\{ \int_{\xi_1}^{1-\xi_1} \mathcal{N}(t,\tau)\phi^{-1}\Big(I^{\varrho}_{0^+}(\omega(s)f(z(\tau)))\Big)d\tau \right. \\
&\qquad \left. + \int_{\xi_1}^{1-\xi_1} \chi(t,\tau)\int_{\xi_1}^{1-\xi_1} \mathcal{N}(\tau,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(x)f(z(x))\big)\Big)dxd\tau \right\} \\
&\geq \alpha O_k\xi_1^{\vartheta-1} \max_{t\in[0,1]} \left\{ 1+\int_{\xi_1}^{1-\xi_1} \chi(t,\tau)d\tau \right\} \int_{\xi_1}^{1-\xi_1} \mathcal{N}(1,x)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\xi_1)\big)\Big)dx \\
&\geq \alpha O_k\xi_1^{\vartheta-1}\sigma(\xi_1)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\xi_1)\big)\Big)\int_{\xi_1}^{1-\xi_1} \mathcal{N}(1,x)dx \\
&\geq O_k = \|z\|.
\end{aligned}$$

Thus, if $z \in \mathcal{P}\cap\partial\Omega_{2,k}$, then

$$\|\mathcal{F}z\| \geq \|z\|. \tag{7}$$

It is evident that $0 \in \Omega_{2,k} \subset \bar{\Omega}_{2,k} \subset \Omega_{1,k}$. From (6) and (7), it follows from Theorem 3.1 that the operator $\mathcal{F}$ has a fixed point $z_k \in \mathcal{P} \cap \left(\bar{\Omega}_{1,k} \backslash \Omega_{2,k}\right) \ni O_k \leq \|z_k\| \leq E_k$. The proof is completed. □

For $p = 1$, we have the following theorem.

Theorem 3.4. *Suppose $(H_1) - (H_5)$ hold, let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence with $t_{k+1} < \xi_k < t_k$. Let $\{E_k\}_{k=1}^{\infty}$ and $\{O_k\}_{k=1}^{\infty}$ be such that*

$$E_{k+1} < O_k < \alpha O_k < E_k, \ k \in \mathbb{N},$$

Assume that f satisfies
$(B1)$ $f(z) \leq \phi(M_2 E_k)$ for all $t \in [0,1]$, $0 \leq z \leq E_k$, where

$$M_2 < \min\left\{\frac{1}{(1+A)\|\mathcal{N}(\tau,\tau)\|_{\infty}\|\psi\|_1}, \ \alpha\right\}$$

and $(A2)$. Then the bvp (1) has denumerably many positive solutions $\{z_k\}_{k=1}^{\infty}$. Furthermore, $O_k \leq \|z_k\| \leq E_k$ for each $k \in \mathbb{N}$.

Proof. Let $\|\mathcal{N}(\tau,\tau)\|_q\|\psi\|_p$ be replaced by $\|\mathcal{N}(\tau,\tau)\|_{\infty}\|\psi\|_1$ and repeat the argument above. □

Lastly, the case $p = \infty$.

Theorem 3.5. *Assume that $(H_1) - (H_2)$ hold. Let $\{E_k\}_{k=1}^{\infty}$ and $\{O_k\}_{k=1}^{\infty}$ be such that*

$$E_{k+1} < O_k < \alpha O_k < E_k, \ k \in \mathbb{N},$$

Assume that f satisfies
$(E1)$ $f(z) \leq \phi(M_3 E_k)$ for all $t \in [0,1]$, $0 \leq z \leq E_k$, where

$$M_3 < \min\left\{\frac{1}{(1+A)\|\mathcal{N}(\tau,\tau)\|_1\|\psi\|_{\infty}}, \ \alpha\right\}$$

and $(A2)$. Then the bvp (1) has denumerably many positive solutions $\{z_k\}_{k=1}^{\infty}$ such that $O_k \leq \|z_k\| \leq E_k$ for $k = 1, 2, 3, \cdots$.

Proof. Let $\|\mathcal{N}(\tau,\tau)\|_q\|\psi\|_p$ be replaced by $\|\mathcal{N}(\tau,\tau)\|_1\|\psi\|_{\infty}$ and repeat the argument above. □

4. Examples

In this section, we present an example to check validity of our main results.
Example 4.1. Consider the following fractional order boundary value problem,

$$\left.\begin{aligned} &\mathfrak{D}_{0^+}^{3/4}\big(\phi(\mathfrak{D}_{0^+}^{7/4} z(t))\big) + \omega(t) f(z(t)) = 0, \ t \in (0,1), \\ &z(0) = \mathfrak{D}_{0^+}^{7/4} z(0) = 0, \ z(1) + \mathfrak{D}_{0^+}^{3/4} z(1) = I_{0^+}^{2} z(1), \end{aligned}\right\} \tag{8}$$

where

$$\phi(z) = \begin{cases} \dfrac{z^5}{1+z^2}, & z \leq 0, \\ z^2, & z > 0, \end{cases}$$

$$f(z)=\begin{cases} \frac{11}{5}\times 10^{-8}, & z\in(10^{-4},+\infty),\\ \frac{\frac{9}{2}\times 10^{-(8k+4)}-\frac{11}{5}\times 10^{-8k}}{10^{-2(2k+1)}-10^{-4k}}(z-10^{-4k})+\frac{11}{5}\times 10^{-8k}, & \\ & z\in\left[10^{-2(2k+1)},10^{-4k}\right],\\ \frac{9}{2}\times 10^{-(8k+4)}, & z\in\left(\frac{1}{5^{3/4}}\times 10^{-(4k+2)},10^{-(4k+2)}\right),\\ \frac{\frac{9}{2}\times 10^{-(8k+4)}-\frac{47}{5}\times 10^{-(8k+8)}}{\frac{1}{5^{3/4}}\times 10^{-(4k+2)}-10^{-(4k+4)}}(z-10^{-(4k+4)})+\frac{11}{5}\times 10^{-(8k+8)}, & \\ & z\in\left(10^{-(4k+4)},\frac{1}{5^{3/4}}\times 10^{-(4k+2)}\right],\\ 0, & z=0,\end{cases}$$

and let

$$\omega(t)=\sum_{k=1}^{\infty}\omega_k(t),$$

in which

$$\omega_k(\tau)=\begin{cases} \frac{2^{\varrho}\varrho}{(2k-1)(2k+1)(2^{\varrho}-(2-t_{k+1}-t_k)^{\varrho})}, & 0\le\tau<\frac{t_{k+1}+t_k}{2},\\ \frac{1}{\eta(t_k-\tau)^{1/2}(1-\tau)^{\varrho-1}}, & \frac{t_{k+1}+t_k}{2}\le\tau<t_k,\\ \frac{1}{\eta(\tau-t_k)^{1/2}(1-\tau)^{\varrho-1}}, & t_k<\tau\le\frac{t_k+t_{k-1}}{2},\\ 0, & \frac{t_k+t_{k-1}}{2}<\tau<t_1,\\ \frac{\varrho}{2(2k-1)(2k+1)(1-t_1)^{\varrho}}, & t_1\le\tau\le 1,\end{cases}$$

and

$$\eta=\frac{\sqrt{2}}{6}\left(4\pi^2-27\right),\ t_k=\frac{31}{64}-\sum_{j=1}^{k}\frac{1}{4(j+1)^4},\ k=1,2,3,\cdots.$$

Let

$$p=q=2,\ t_k=\frac{31}{64}-\sum_{j=1}^{k}\frac{1}{4(j+1)^4},\quad\text{for}\quad k=1,2,3,\cdots,\ \xi_k=\frac{1}{2}(t_k+t_{k+1}),$$

then

$$\xi_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32},\ \xi_k>\frac{1}{5}\ t_{k+1}<\xi_k<t_k.$$

Therefore,

$$\xi_k^{\alpha-1}>5^{-3/4},\ k=1,2,3,\cdots.$$

It is easy to see

$$t_1=\frac{15}{32}<\frac{1}{2},\quad\text{for}\quad k=1,2,3,\cdots,\ t_k-t_{k+1}=\frac{1}{4(k+2)^4}.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and $\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90}$, it follows that

$$l^*=\lim_{k\to\infty}t_k=\frac{31}{64}-\sum_{j=1}^{\infty}\frac{1}{4(j+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5},$$

and

$$\begin{aligned}I^{\varrho}_{0^+}\omega(1)&=\frac{1}{\Gamma(\varrho)}\int_0^1(1-\tau)^{\varrho-1}\omega(\tau)d\tau=\frac{1}{\Gamma(\varrho)}\sum_{k=1}^{\infty}\int_0^1(1-\tau)^{\varrho-1}\omega_k(\tau)d\tau\\
&=\frac{1}{\Gamma(\varrho)}\sum_{k=1}^{\infty}\left[\frac{3}{2(2k-1)(2k+1)}+\int_{\frac{t_{k+1}+t_k}{2}}^{t_k}\frac{\sqrt{2}}{\eta(t_k-\tau)^{1/2}}d\tau+\int_{t_k}^{\frac{t_k+t_{k-1}}{2}}\frac{\sqrt{2}}{\eta(\tau-t_k)^{1/2}}d\tau\right]\\
&=\frac{1}{\Gamma(\varrho)}\left\{\frac{3}{4}+\frac{1}{\sqrt{2}\eta}\sum_{k=1}^{\infty}\left[\frac{1}{(k+1)^2}+\frac{1}{(k+2)^2}\right]\right\}=\frac{1}{\Gamma(\varrho)}\left\{\frac{3}{4}+\frac{1}{\sqrt{2}\eta}\left[\frac{\pi^2}{3}-\frac{9}{4}\right]\right\}\\
&=\frac{1}{\Gamma(\varrho)}.\end{aligned}$$

It follows that

$$\phi^{-1}(I^{\varrho}_{0^+}\omega(1))=\frac{1}{\sqrt{\Gamma(\varrho)}},$$

and $\|\psi(1)\|_2=\left[\int_0^1[\phi^{-1}(I^{\varrho}_{0^+}\omega(1))]^2ds\right]^{1/2}=\frac{1}{\sqrt{\Gamma(\varrho)}}\approx 0.9033542711.$

$$\begin{aligned}I^{\varrho}_{0^+}\omega(\xi_1)&=\frac{1}{\Gamma(\varrho)}\int_0^{\xi_1}(\xi_1-s)^{\varrho-1}\omega(s)ds=\frac{1}{\Gamma(\varrho)}\int_0^{\xi_1}(\xi_1-s)^{\varrho-1}\omega_1(s)ds\\
&=\frac{1}{\Gamma(\varrho)}\int_0^{\xi_1}\frac{\varrho(\xi_1-s)^{\varrho-1}}{3[1-(1-\xi_1)^{\varrho}]}ds=\frac{{\xi_1}^{\varrho}}{3\Gamma(\varrho)[1-(1-\xi_1)^{\varrho}]},\end{aligned}$$

$\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\xi_1)\big)\Big)=\frac{{\xi_1}^{\varrho/2}}{\sqrt{3\Gamma(\varrho)[1-(1-\xi_1)^{\varrho}]}}\approx 0.6390712607.$ It follows from a simple calculations, we obtained

$$\begin{aligned}\int_{\xi_1}^{1-\xi_1}\mathcal{N}(1,\tau)d\tau&>\int_{15/32}^{1-15/32}\left[\frac{(k-1)(1-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}+\frac{(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right]d\tau\\
&\approx 1.171656594,\end{aligned}$$

$$\xi_1^{\vartheta-1}\sigma(\xi_1)\phi^{-1}\Big(\mathcal{I}^{\varrho}_{0^+}\big(\omega(\xi_1)\big)\Big)\int_{\xi_1}^{1-\xi_1}\mathcal{N}(1,s)ds>2.11819593108,$$

$$\int_0^1|\mathcal{N}(\tau,\tau)|^2d\tau=\int_0^1\left|\frac{k\tau^{\vartheta-1}(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta)}-\frac{k\tau^{\vartheta-1}(1-\tau)^{\vartheta-\varrho-1}}{\Gamma(\vartheta-\varrho)}\right|^2d\tau=0.44187,$$

and

$$\|\mathcal{N}(\tau,\tau)\|_2=\left[\int_0^1|\mathcal{N}(\tau,\tau)|^2d\tau\right]^{1/2}\approx 0.46842,$$

so that

$$M_1=1.491599442=\frac{1}{(1+A)\|\mathcal{N}(\tau,\tau)\|_2\|\psi(1)\|_2}<\frac{1}{(1+A)\|\mathcal{N}(\tau,\tau)\|_2\|\psi(\tau)\|_2}$$

and

$$\alpha = \max\left\{\frac{1}{\xi_1^{\vartheta-1}\sigma(\xi_1)\phi^{-1}\Big(\mathcal{I}_{0^+}^{\varrho}\big(\omega(\xi_1)\big)\Big)\int_{\xi_1}^{1-\xi_1}\mathcal{N}(1,x)dx},\ 1\right\}$$
$$= \max\{2.11819593108, 1\} = 2.11819593108.$$

Now, taking

$$O_k = 10^{-2(2k+1)} \text{ and } E_k = 10^{-4k},$$

then

$$E_{k+1} = 10^{-(4k+4)} < \frac{1}{5^{3/4}} \times 10^{-(4k+2)} < \xi_k^{\vartheta-1} O_k$$
$$< O_k = 10^{-(4k+2)} < E_k = 10^{-4k},$$

$\alpha O_k = 2.11819593108 \times 10^{-2(2k+1)} < 1.491599442 \times 10^{-4k} = M_1 E_k$.
Also, f satisfies conditions:

$$f(z) \le \phi(M_1 E_k) = M_1^2 E_k^2 = 2.224868895 \times 10^{-8k},\ z \in \left[0, 10^{-4k}\right]$$
$$f(z) \ge \phi(\alpha O_k) = \alpha^2 O_k^2$$
$$= 4.48675400244 \times 10^{-(8k+4)},\ z \in \left[\frac{1}{5^{3/4}} \times 10^{-2(2k+1)}, 10^{-2(2k+1)}\right].$$

Hence, by Theorem 3.2, the bvp (8) has denumerably infinitely many positive solutions $\{z^{[k]}\}_{k=1}^{\infty}$ with $10^{-2(2k+1)} \le \|z^{[k]}\| \le 10^{-4k}$ for $k \in \mathbb{N}$.

5. Conclusion

We derived sufficient conditions for the existence of denumerably many positive solutions for Riemann-Liouville fractional order boundary value problem with denumerably many singularities by using Krasnoselskii's cone fixed point theorem on a Banach space.

Acknowledgement. The authors thank to the reviewers for their appropriate comments and suggestions which help us to improve the manuscript. The author M. Rashmitha is thankful to DST-INSPIRE, Government of India, New Delhi for awarding JRF No.2016/IF 160480.

References

[1] Aghajani, A., Jalilian, Y. and Trujillo, J. J., (2012), On the existence of solutions of fractional integro-differential equations, Fract. Calc. Appl. Anal., 5(1), pp. 44–69.

[2] Benchohra, M., Graef, J. R. and Mostafai, F. Z., (2010), Weak solutions for nonlinear fractional differential equations on reflexive Banach spaces, Electron. J. Qual. Theory. Diff. Eq., 54, pp. 1–10.

[3] Borisut, P., Kumam, P., Ahmed, I. and Sitthithakerngkiet, K., (2019), Nonlinear Caputo Fractional Derivative with Nonlocal Riemann-Liouville Fractional Integral Condition Via Fixed Point Theorems, Symmetry, 11(6), 829.

[4] Ege, S. M. and Topal, F. S., (2017), Existence of positive solutions for fractional order boundary value problems, J. Applied Anal. Comp., 7(2), pp. 702–712.

[5] Guo, D. and Lakshmikantham, V., (1988), Nonlinear Problems in Abstract Cones, Academic Press, San Diego.

[6] Kilbas, A.A., Srivastava, H. M. and Trujillo, J. J., (2006), Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204, Elsevier Science B. V., Amsterdam.

[7] Leibenson, L. S., (1983), General problem of the movement of a compressible fluid in a porous medium, Izvestiia Akademii Nauk Kirgizskoi SSR., 9, pp. 7–10.

[8] Li, K., Peng, J. and Gao, J., (2013), Nonlocal fractional semilinear Differential equations in separable Banach spaces, Electron. J. Differential Equations, 2013(7), pp. 1–7.

[9] Liang, J., Liu, Z. and Wang, X., (2013), Solvability for a couple system of nonlinear fractional differential equations in a Banach space, Fract. Calc. Appl. Anal., 16(1), pp. 51–63.
[10] Podlubny, I., (1999), Fractional differential equations, Academic Press, San Diego.
[11] Wang, L. and Zhai, C., (2019), Unique solutions for new fractional differential equations with p-Laplacian and infinite-point boundary conditions, Int. J. Dynamical Systems and Differential Equations, 9(1), 1–13.
[12] Wang, J. R., Zhou, Y. and Feckan, M., (2019), Abstract Cauchy problem for fractional differential equations, Nonlinear Dyn., 71, DOI 10.1007/s11071-012-0452-9.
[13] Zhao, K., and Liu, J., (2016), Multiple monotone positive solutions of integral BVPs for a higher-order fractional differential equation with monotone homomorphism, Adv. Difference Equ. 2016(1), pp 20.

Kapula Rajendra Prasad received his M.Sc. and Ph.D. degrees from Andhra University, Visakhapatnam, India. Dr. Prasad did his post doctoral work at Auburn University, Auburn, USA. Presently, he is working as a professor and Chairman-Board of Studies, Department of Applied Mathematics, Andhra University. His major research interest includes ordinary differential equations, difference equations, dynamic equations on time scales, fractional order differential equations, mathematical modeling and fixed point theory.

Mahammad Khuddush received his M.Sc., M.Phil. and Ph.D. degrees from Andhra University, Visakhapatnam, India. His major research interest includes ordinary differential equations, dynamic equations on time scales, fractional order differential equations, mathematical modeling and fixed point theory.

Mahanty Rashmita received her M.Sc. from Andhra University, Visakhapatnam, India. Currently, Mrs. Rashmita is a Ph.D. student under the guidance of Prof. K. Rajendra Prasad, Department of Applied Mathematics, Andhra University, Visakhapatnam, India. She is working on fractional order boundary value problems.