TWMS J. App. and Eng. Math. V.12, N.2, 2022, pp. 445-455

TOTAL ROMAN DOMINATION NUMBER OF GRAPHS

D. K. THAKKAR[1], S. M. BADIYANI[1], §

Abstract. In this paper a characterization of minimal total roman dominating functions has been proved. It has also been proved that if v is a pendant vertex and its neighbor u is not a pendant vertex then $f(v) = 0$ or 1 for any minimal total roman dominating function f. The operation of vertex removal has been considered. A necessary and sufficient condition under which the total roman domination number increases (under this operation) has been proved. Similarly a necessary and sufficient condition has been proved under which the total roman domination number decreases. Some examples has been given with miscellaneous results. We characterize those graphs for which the total roman domination number is equal to the number of vertices of the graph.

Keywords: Total Dominating Set, Total Roman Dominating Function, Total Roman Domination Number, Minimal Total Roman Dominating Function, Minimum Total Roman Dominating Function.

AMS Subject Classification: (2010), 05C69.

1. Introduction

Total domination in graphs is one of most prominent topic of domination theory which was studied by Cockayne E. J., Hedetniemi S.T. and Dawes R. M. in [4]. A previous survey on total domination in graphs can also be found in [9]. Recently, another concept of domination theory called roman domination have made a rapid progress and was studied by several authors [6,7,10,11,12]. In the year 2004, Cockayne E. J., Dreyer P. A., Jr. Hedetniemi, S. M. and Hedetniemi, S. T. in [5] studied the graph theoretic properties of this variant of domination number of a graph. By combining these two concepts, a stronger concept called total roman domination was introduced and studied by [1] in 2016 and later on Amjai J., Sheikholeslami S. M., et.al. proved several results on bounds for total roman domination number and total roman domination number of trees in [2,3]. A literature on the domination theory in graphs has been surveyed and detailed in [8].

We consider the concept of total roman domination and prove several results related to this concepts. First, we characterize a minimal total roman dominating function. Further, we consider the effect of removing a vertex on the total roman domination number of the graph. In particular, we prove a necessary and sufficient condition under which total roman domination number increases when this operation is performed. Similarly, we

[1] Department of Mathematics, Saurashtra University Campus, University Road, Rajkot-360005, India.
e-mail: dkthakkar1@yahoo.co.in; ORCID: https://orcid.org/0000-0002-3887-4972.
e-mail: sankycolors@gmail.com; ORCID: https://orcid.org/0000-0001-9241-5091.

§ Manuscript received: May 28, 2019; accepted: January 16, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.2

prove a necessary and sufficient condition under which the total roman domination number decreases when this operation is performed. We prove that the total roman domination number of a graph G is n ($n = |V(G|)$ if and only if the maximum degree of the graph does not exceed 2.

2. Preliminaries and Notations

In this paper we consider only those graphs which are simple, undirected, finite and without isolated vertices. If G is a graph, $V(G)$ will denote the vertex set of graph G and $E(G)$ will denote the edge set of graph G. For a graph G the minimum and maximum degree of the vertices are denoted by $\delta(G)$ and $\Delta(G)$ respectively. If v is a vertex of G then $N(v)$ denotes the set of vertices adjacent to v and $N[v] = N(v) \bigcup \{v\}$. If S is a set then $|S|$ will denote the cardinality of the set S.

Suppose $f : V(G) \to R$ and $g : V(G) \to R$ are two functions then $f \leq g$ if $f(v) \leq g(v); \forall v \in V(G)$. Also $f < g$ if $f \leq g$ and for some vertex $w, f(w) < g(w)$. A function $f : V(G) \to R$ is said to be a minimal function with respect to some property P if (i) f has property P (ii) there is no function g such that $g < f$ and g has property P. If G is a graph and $v \in V(G)$ then $G - v$ will denote the subgraph obtained by removing the vertex v from G. The total roman domination number of the graph G is denoted as$\gamma_{tR}(G)$, whereas the domination number of the graph G is denoted as $\gamma(G)$.
If $f : V(G) \to \{0, 1, 2\}$ is a function then we write,

$$V_2(f) = \{v \in V(G)/f(v) = 2\}$$
$$V_1(f) = \{v \in V(G)/f(v) = 1\}$$
$$V_0(f) = \{v \in V(G)/f(v) = 0\}$$

Obviously the above sets are mutually disjoint and their union is the vertex set $V(G)$. The weight of this function $f = \sum_{(v \in V(G))} f(v)$. This number is denoted as $w(f)$.

Definition 2.1[6]: A function $f : V(G) \to \{0, 1, 2\}$ is called a "*roman dominating function*" if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a minimum roman dominating function is called the "*roman domination number*" of G which is denoted by $\gamma_R(G)$.

Definition 2.2[1]: A roman dominating function $f : V(G) \to \{0, 1, 2\}$ is called a "*total roman dominating function*" if $V_1(f) \bigcup V_2(f)$ has no isolated vertices.

Definition 2.3[1]: A total roman dominating function with minimum weight is called a "*minimum total roman dominating function*" and its weight is called "*total roman domination number*" of a graph and it is denoted by$\gamma_{tR}(G)$.

Definition 2.4[1]: A function $f : V(G) \to \{0, 1, 2\}$ is called a "*minimal total roman dominating function*" if:
(i)f is a total roman dominating function.
(ii)Whenever $g : V(G) \to \{0, 1, 2\}$ and $g < f$ then g is not a total roman dominating function.

Definition 2.5[7]: Let G be a graph, $S \subset V(G)$ and $v \in S$. The *External Private Neighborhood* of v with respect to S is the set,

$$expn[v, S] = \{u \in V(G) - S/N[u] \bigcap S = \{v\}\}$$

3. Minimal Total Roman Dominating Functions

Proposition 3.1. *Let G be a graph and $v \in V(G)$ be a pendant vertex and suppose its neighbor u is not a pendant vertex then $f(v) = 1$ or $f(v) = 0$ if f is a minimal total roman dominating function.*

Proof. If $f(v) = 2$ then $f(u) = 1$ or $f(u) = 2$.
Now define $g : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v) = 1 \text{ and } g(w) = f(w); \forall w \neq v$$

Then g is a total roman dominating function with $g < f$ which contradicts the minimality of f. Therefore $f(v) = 1$ or $f(v) = 0$. □

Remark 3.2. *Let G be a graph and $f : V(G) \rightarrow \{0, 1, 2\}$ is a minimal total roman dominating function. Suppose $v \in V(G)$ is a pendant vertex and its neighbor u is also a pendant vertex then $f(v) = 1$ and $f(u) = 1$.* □

Now we characterize the minimal total roman dominating function.

Theorem 3.3. *Let G be a graph and $f : V(G) \rightarrow \{0, 1, 2\}$ is a total roman dominating function then f is minimal if and only if the following conditions are satisfied:*

i) *If $f(v) = 1$ then one of the following holds:*
  a) *$N(v) \bigcap V_2(f) = \phi$.*
  b) *there is a vertex u in $V_1(f) \bigcup V_2(f)$ such that u is adjacent to only one vertex in $V_1(f) \bigcup V_2(f)$ namely v.*

ii) *If $f(v) = 2$ then one of the following holds:*
  a) *there is a vertex z such that $f(z) = 0$ and $z \in expn[v; V_2(f)]$.*
  b) *There is a vertex x such that $f(x) = 1$ or 2 and x is adjacent to only one vertex in $V_1(f) \bigcup V_2(f)$ namely v.*
  c) *$N(v) \bigcap V_2(f) = \phi$.*

Proof. Suppose f is a minimal total roman dominating function on G.
i) Suppose $f(v) = 1$. Define $g : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v) = 0 \text{ and } g(w) = f(w); \; \forall w \neq v$$

Then g cannot be a total roman dominating function as $g < f$.Then one of the following two possibilities holds.
a) There is a vertex z such that $g(z) = 0$ and z is not adjacent to any vertex x for which $g(x) = 2$ then $z = v$ and v is not adjacent to any vertex for which $f(x) = 2$. Therefore $N(v) \bigcup V_2(f) = \phi$.
b) $V_1(g) \bigcup V_2(g)$ has an isolated vertex. If $x \in V_1(g)$ then $g(x) = 1$ implies $f(x) = 1$ and therefore x is adjacent to some vertex y for which $f(y) = 1$ or 2. Also note that $v \notin V_1(g) \bigcup V_2(g)$. Let u be an isolated vertex in $V_1(g) \bigcup V_2(g)$ then $g(u) = 1$ or 2. Therefore $f(u) = 1$ or 2 then $u \in V_1(f) \bigcup V_2(f)$. Therefore there is a vertex w in $V_1(f) \bigcup V_2(f)$ which is adjacent to u. Now $f(w) = 1$ or 2. Since u is isolated in $V_1(g) \bigcup V_2(g)$, $v \notin V_1(g)$ and $v \in V_1(f)$, we have $w = v$. Therefore u is adjacent to only one vertex in $V_1(f) \bigcup V_2(f)$ namely v.
ii) Suppose $f(v) = 2$. Define $g : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v) = 1 \text{ and } g(w) = f(w); \; \forall w \neq v$$

Then g cannot be a total roman dominating function. Also note that $V_1(g)\bigcup V_2(g)$ does not have any isolated vertex. Therefore there is a vertex z such that $g(z) = 0$ and z is not adjacent to any vertex x for which $g(x) = 2$. However $f(z) = 0$ and z is adjacent to some vertex u for which $f(u) = 2$. Therefore $u = v$ and z is adjacent to only one vertex of $V_2(f)$ namely v. Therefore $z \in expn[v; V_2(f)]$.
Suppose $f(v) = 2$. Define $g : V(G) \to \{0, 1, 2\}$ as follows:

$$g(v) = 0 \text{ and } g(w) = f(w); \forall w \neq v$$

Then g cannot be a total roman dominating function. Then one of the following two possibilities holds.
a) $V_1(g)\bigcup V_2(g)$ has an isolated vertex. Suppose x is an isolated vertex of $V_1(g)\bigcup V_2(g)$. Then $g(x) = f(x) = 1$ or $g(x) = f(x) = 2$. Now x is adjacent to some vertex y of $V_1(f)\bigcup V_2(f)$. Also x is not adjacent to any vertex of $V_1(g)\bigcup V_2(g)$. Therefore x is adjacent to only one vertex $V_1(f)\bigcup V_2(f)$ namely v.
b) There is a vertex x such that $g(x) = 0$ and x is not adjacent to any vertex of $V_2(g)$. If $x \neq v$ then $g(x) = f(x)$ and x is adjacent to some vertex y of $V_2(f)$. Since x is not adjacent to any vertex of $V_2(g)$ we have $y = v$. Therefore x is adjacent to only one vertex of $V_2(f)$ namely v. Therefore $x \in expn[v; V_2(f)]$.
c) If $x = v$ then $g(v) = 0$ and v is not adjacent to any vertex of $V_2(g)$. Therefore v is not adjacent to any vertex of $V_2(f)$.
Thus $N(v)\bigcap V_2(f) = \phi$.
Conversely suppose f is a total roman dominating function which satisfies the given conditions. Let $g : V(G) \to \{0, 1, 2\}$ be a function such that $g < f$ then for some $v \in V(g)$ we have $g(v) < f(v)$.
Case-i):- Suppose $f(v) = 1$ then $g(v) = 0$.
Suppose $N(v)\bigcap V_2(f) = \phi$. Since $g < f, N(v)\bigcap V_2(g) = \phi$. Therefore g cannot be a roman dominating function and hence g cannot be a total roman dominating function.
Now suppose there is a vertex u in $V_1(f)\bigcup V_2(f)$ such that u is adjacent to only one vertex in $V_1(f)\bigcup V_2(f)$ namely v. Note that $V_1(g)\bigcup V_2(g)$ is a subset of $V_1(f)\bigcup V_2(f)$ and $v \notin V_1(g)\bigcup V_2(g)$ therefore u is not adjacent to any vertex of $V_1(g)\bigcup V_2(g)$. Therefore g is not a total roman dominating function.
Case-ii):- a) Suppose $f(v) = 2$ and $g(v) = 1$.
Now $V_1(g)\bigcup V_2(g)$ does not have any isolated vertex and also there is a vertex z such that $f(z) = 0$ and $z \in expn[v; V_2(f)]$ then $g(z) = 0$ and z is not adjacent to any vertex of $V_2(g)$ because $V_2(g)$ is a subset of $V_2(f)$ and $v \notin V_2(g)$. Therefore g cannot be a roman dominating function and hence g cannot be a total roman dominating function.
b) Suppose $f(v) = 2$ and $g(v) = 0$.
Suppose there is a vertex x in $V_1(f)\bigcup V_2(f)$ such that x is adjacent to only one vertex in $V_1(f)\bigcup V_2(f)$ namely v. Consider the set $V_1(g)\bigcup V_2(g)$ which is a subset of $V_1(f)\bigcup V_2(f)$ and $v \notin V_1(g)\bigcup V_2(g)$. Since $f(x) = 1$ or 2, $g(x) = 1$ or 2 and therefore $x \in V_1(g)\bigcup V_2(g)$ and x is not adjacent to any vertex of $V_1(g)\bigcup V_2(g)$. Thus g is not a total roman dominating function.
Now suppose $N(V)\bigcap V_2(f) = \phi$. Therefore $N(v)\bigcap V_2(g) = \phi$. Thus v is not adjacent to any vertex of $V_2(g)$ and since $g(v) = 0$, g cannot be a roman dominating function and therefore g cannot be a total roman dominating function.
Now suppose there is a vertex z such that $f(z) = 0$ and $z \in expn[v; V_2(f)]$ then $g(z) = 0$. Note that $v \notin V_2(g)$. Since $z \in expn[v; V_2(f)]$ and $V_2(g)$ is a subset of $V_2(f)$, z is not adjacent to any vertex of $V_2(g)$. Therefore g cannot be a roman dominating function and hence g cannot be a total roman dominating function.

From all the above cases it follows that if $g < f$ then g cannot be a total roman dominating function and therefore f is a minimal total roman dominating function. □

Example 3.4. *Let G be a graph with $|V(G)| = n$ and v be a vertex such that $degree(v) = \Delta(G) \geq 2$. Define $f : V(G) \to \{0, 1, 2\}$ as follows:*
$f(v) = 2$.
$f(u) = 1$; for some neighbor u of v.
$f(w) = 0$; for every other neighbor w of v.
$f(x) = 1$; for all other vertices.
Then f is a minimal total roman dominating function on G.
Also $w(f) = n - \Delta(G) + 2$. Thus we have a minimal total roman dominating function with $w(f) = n - \Delta(G) + 2$.
Similarly if $\delta(G) \geq 2$ then there is a minimal total roman dominating function f with $w(f) = n - \delta(G) + 2$.
In particular if G is a k-regular graph (for $k \geq 2$) then there is a total roman dominating function f with $w(f) = n - k + 2$. □

4. Minimum Total Roman Dominating Functions

We begin with the following proposition.

Proposition 4.1. *Let G be a graph. If $\gamma_{tR}(G) = n$ then $\Delta(G) \leq 2$.*

Proof. From the above example 3.4 it follows that,

$$n = \gamma_{tR}(G) \leq n - \Delta(G) + 2$$

Which implies $\Delta(G) \leq 2$. □

Proposition 4.2. *Let G be a path graph P_n with n vertices (for $n \geq 2$) then $\gamma_{tR}(G) = n$.*

Proof. Suppose $\gamma_{tR}(G) < n$.
a) First suppose that $\gamma_{tR}(G) = n-1$. Let $f : V(G) \to \{0, 1, 2\}$ be a minimum total roman dominating function on G with $w(f) = n - 1$. Since $\gamma_{tR}(G) = n - 1 < n$, there is a vertex v_1 and a vertex v_2 adjacent to v_1 such that $f(v_1) = 0$ and $f(v_2) = 2$. Now consider the vertices different from v_1 and v_2 which are total $n - 2$ vertices and the total weight of these vertices is $n - 3$. Therefore again there is a vertex v_3 and a vertex v_4 adjacnet to v_3 such that $f(v_3) = 0$ and $f(v_4) = 2$. By continuing in this way we will either consider all the vertices if n is even or $n - 1$ vertices if n is odd.
Case-1: Suppose n is even then $n = 2j$ and there are j vertices where the value of the function is 0 and j vertices where the value of the function is 2. Therefore $w(f) = 2j = n$. This contradicts our assumption that $w(f) = \gamma_{tR}(G) = n - 1$.
Case-2: Suppose n is odd. In this case we will obtain $n - 1$ vertices such that the value of the function at each of $(\frac{n-1}{2})$ is 0 and the value of the function at each of the other $(\frac{n-1}{2})$ is 2. Therefore there is a vertex x different from all the above $n - 1$ vertices such that $f(x) = 0$ (as $w(f) = \gamma_{tR}(G) = n - 1$). Therefore there are total $(\frac{n+1}{2})$ vertices where the value of the function is 0.
Let $S = \{v \in V(G)/f(v) = 2\}$ then $|S| = (\frac{n-1}{2})$. Since f is a total roman dominating function $V_1(f) \bigcup V_2(f) = V_2(f) = S$ is a total dominating set of G and $|S| = \frac{n-1}{2} < \frac{n}{2}$; this is a contradiction because $\gamma_t(G) \geq \frac{n}{2}$ for a path graph G with n vertices.

Therefore in both the cases we get the contradiction so there is no minimum total roman dominating function f whose weight is $n-1$ and hence there is no total roman dominating function whose weight is $n-1$.
b) Suppose that $\gamma_{tR}(G) = n-2$. Let $f : V(G) \to \{0,1,2\}$ be a minimum total roman dominating function with $w(f) = n-2$. Let x be any vertex such that $f(x) = 0$. Define $g : V(G) \to \{0,1,2\}$ as follows:

$$g(x) = 1 \text{ and } g(y) = f(y); \forall y \neq x$$

Then g is a total roman dominating function and $w(g) = n-1$; which contradicts the above conclusion that there cannot be such a function with weight $n-1$. Therefore there is no minimum function f whose weight is $n-2$.
c) Similarly we can prove that there is no minimum function $h : V(G) \to \{0,1,2\}$ whose weight is less than n.
Thus $\gamma_{tR}(G) < n$ is not possible.
Now consider the function $h : V(G) \to \{0,1,2\}$ as follows:

$$h(v) = 1; \forall v \in V(G)$$

Then h is a total roman dominating function with $w(h) = n$. Therefore h is a minimum total roman dominating function and hence $\gamma_{tR}(G) = n$. □

Proposition 4.3. *Let $n \geq 3$ then $\gamma_{tR}(C_n) \leq \gamma_{tR}(P_n)$.*

Proof. Let f be a minimum total roman dominating function on P_n then f is also total roman dominating function on C_n.
Therefore $\gamma_{tR}(C_n) \leq w(f) = \gamma_{tR}(P_n)$. □

Proposition 4.4. *Let $n \geq 3$ then $\gamma_{tR}(C_n) = \gamma_{tR}(P_n)$.*

Proof. Let f be a minimum total roman dominating function for C_n.
Consider $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$ where v_i is adjacent to v_{i+1} for every $i = 1, 2, 3, ..., n-1$ and v_n is adjacent to v_1.
Case-1: Suppose $f(v_i) = 1$;for $i = 1, 2, 3, ..., n$. Now consider the same function $f : V(P_n) \to \{0,1,2\}$ where $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$ where v_i is adjacent to v_{i+1} for $i = 1, 2, 3, ..., n-1$. Obviously f is a total roman dominating function on P_n. Therefore $\gamma_{tR}(P_n) \leq w(f) = \gamma_{tR}(C_n)$.
Case-2: For every function $f : V(C_n) \to \{0,1,2\}$, $f(x) \neq 1$ for some $x \in V(C_n)$. Let g be a minimum total roman dominating function on C_n then, $g(v) = 0$ for some vertex $v \in V(C_n)$ and $g(u) = 2$ for some neighbor u of v. Without loss of generality we may assume that $g(v_1) = 0$ and $g(v_n) = 2$. Note that $g(v_{(n-1)}) \neq 0$ as v_n will be isolated vertex in $V_1(g) \bigcup V_2(g)$. Now define $g' : V(C_n) \to \{0,1,2\}$ as follows:

$$g'(v_1) = 1, g'(v_n) = 1 \text{ and } g'(w) = g(w); \forall w \neq \{v_1, v_n\}$$

Then g' is a total roman dominating function with $w(g') = w(g)$. Therefore g' is a minimum function on C_n.
Now consider the same function $g' : V(P_n) \to \{0,1,2\}$. Then g' is a total roman dominating function on P_n. Therefore $\gamma_{tR}(P_n) \leq w(f) = \gamma_{tR}(C_n)$.
Thus from both the cases and from the above proposition 4.3 it follows that $\gamma_{tR}(P_n) = \gamma_{tR}(C_n)$. □

Proposition 4.5. *For $n \geq 3$ then $\gamma_{tR}(C_n) = n$.*

Proof. From the above proposition 4.2 and proposition 4.4 it follows that $\gamma_{tR}(C_n) = \gamma_{tR}(P_n) = n$. □

Remark 4.6. *Let G be a graph with $\Delta(G) \leq 2$ then G is a path or a cycle or disjoint union of paths or disjoint union of cycles or disjoint union of both paths and cycles.* □

Proposition 4.7. *Let G be a graph with $\Delta(G) \leq 2$ then $\gamma_{tR}(G) = n$ where $n = |V(G)|$.*

Proof. We may note that if G is a disjoint union of graphs $H_1, H_2, H_3, ..., H_k$ then a function $f : V(G) \to \{0, 1, 2\}$ is a minimum total roman dominating function if and only if its restrictions f_i are minimum total roman dominating functions on H_i respectively (for $i = 1, 2, ..., k$). Since $\Delta(G) \leq 2$, G is either a path or a cycle or disjoint union of paths or disjoint union of cycles or disjoint union of both paths and cycles. If G is a cycle or a path then by proposition 4.5 and propostion 4.2 $\gamma_{tR}(G) = n$. Suppose G is a disjoint union of graphs $H_1, H_2, H_3, ..., H_k$ where each H_i is a cycle or path (for $k \geq 2$). Suppose f_i is a minimum function for H_i (for $i = 1, 2, 3, ..., k$). Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(x) = f_i(x); \forall x \in V(H_i)$$

Then f is a minimum total roman dominating function on G.
Also $\gamma_{tR}(G) = w(f) = \sum_{(i=1)}^{k} w(f_i) = \sum_{(i=1)}^{k} |V(H_i)| = |V(G)| = n$.
Thus the result is proved. □

Example 4.8. *Consider the Wheel graph W_6 with six vertices $\{v_0, v_1, v_2, v_3, v_4, v_5\}$. Define $f : V(G) \to \{0, 1, 2\}$ as follows:*

$$f(v_0) = 2, f(v_1) = 1 \text{ and } f(v_i) = 0 \ ; \forall i = 2, 3, 4, 5.$$

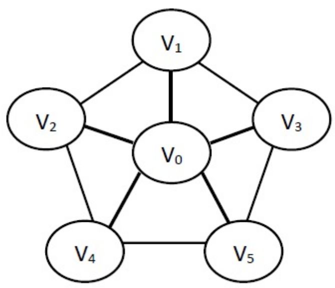

FIGURE 1. (W_6)

Then f is a minimum total roman dominating function and $w(f) = 3$. Therefore $\gamma_{tR}(W_6) = 3 < 6 = |V(W_6)|$. Thus the assumption that $\Delta \leq 2$ cannot be dropped in proposition 4.7 . □

By combining the proposition 4.1 to proposition 4.7 we have proved the following theorem.

Theorem 4.9. *Let G be a graph without isolated vertices then $\gamma_{tR}(G) = n$ if and only if $\Delta(G) \leq 2$.* □

5. Vertex Removal and Total Roman Domination Number

Now we state and prove a necessary and sufficient condition under which the total roman domination number increases when a vertex is removed from the graph.

Theorem 5.1. *Let G be a graph and $v \in V(G)$ be such that $G - v$ does not have isolated vertices then $\gamma_{tR}(G - v) > \gamma_{tR}(G)$ if and only if the following conditions are satisfied:*

i *$f(v) = 2$ for every minimum total roman dominating function f on G.*
ii *There is no function g defined on $V(G-v)$ such that $V_2(g)$ is a subset of $V(G) - N[v]$, $w(g) \leq \gamma_{tR}(G)$ and g is a total roman dominating function on $G-v$.*

Proof. Suppose $\gamma_{tR}(G-v) > \gamma_{tR}(G)$.
i) Suppose $f(v) = 0$. Consider the restriction h of f on $G-v$.
It is obvious that h is a total roman dominating function on $G-v$. Then $\gamma_{tR}(G-v) \leq w(h) = w(f) = \gamma_{tR}(G)$. i.e. $\gamma_{tR}(G-v) \leq \gamma_{tR}(G)$. Which is a contradiction. Therefore $f(v) = 0$ is not possible.
Now suppose $f(v) = 1$. There is a neighbor u of v such that $f(u) = 1$ or 2. Now consider the restriction h of f on $G-v$. It is obvious that h is a total roman dominating function on $G-v$ and $\gamma_{tR}(G-v) \leq w(h) < w(f) = \gamma_{tR}(G)$. i.e.$\gamma_{tR}(G-v) < \gamma_{tR}(G)$. Which is a contradiction. Therefore $f(v) = 1$ is also not possible.
Thus $f(v) = 2$ for every minimum total roman dominating function f on G.
ii) Suppose there is a total roman dominating function g on $G-v$ such that $w(g) \leq \gamma_{tR}(G)$, $V_2(g)$ is a subset of $V(G)-N[v]$ then $\gamma_{tR}(G-v) \leq w(g) \leq \gamma_{tR}(G)$. i.e.$\gamma_{tR}(G-v) \leq \gamma_{tR}(G)$. Which is again a contradiction. Therefore such a function g does not exists.
Conversely suppose conditions *i)* and *ii)* are satisfied.
First suppose that $\gamma_{tR}(G-v) = \gamma_{tR}(G)$. Let g be a minimum total roman dominating function on $G-v$ then $w(g) = \gamma_{tR}(G-v) = \gamma_{tR}(G)$. Suppose v is adjacent to some vertex w such that $g(w) = 2$.
Now define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f(v) = 0 \text{ and } f(x) = g(x); \forall x \neq v$$

Then obviously f is a total roman dominating function on G.
Also $w(f) = w(g) = \gamma_{tR}(G-v) = \gamma_{tR}(G)$. Therefore f is a minimum total roman dominating function on G with $f(v) = 0$ which contradicts condition *i)*.Then v cannot be adjacent to any vertex w for which $g(w) = 2$. Therefore $V_2(g)$ is a subset of $V(G) - N[v]$ and also $w(g) \leq \gamma_{tR}(G)$ and g is a total roman dominating function on $G-v$ which again contradicts condition *ii)*. Therefore $\gamma_{tR}(G-v) = \gamma_{tR}(G)$ is not possible.
Suppose $\gamma_{tR}(G-v) < \gamma_{tR}(G)$.
Let g be a minimum total roman dominating function on $G-v$. Suppose v is adjacent to some vertex w for which $g(w) = 2$.
Now define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f(v) = 0 \text{ and } f(x) = g(x); \forall x \neq v$$

Then f is a total roman dominating function on G and $w(f) = w(g) = \gamma_{tR}(G-v) < \gamma_{tR}(G)$. i.e. $w(f) < \gamma_{tR}(G)$. Which is a contradiction.
Therefore v cannot be adjacent to any vertex w for which $g(w) = 2$. Thus $V_2(g)$ is a subset of $V(G) - N[v]$, $w(g) \leq \gamma_{tR}(G)$ and g is a total roman dominating function on $G-v$, which contradicts condition *ii)*.Therefore $\gamma_{tR}(G-v) < \gamma_{tR}(G)$ is also not possible.
Therefore it must be true that $\gamma_{tR}(G-v) > \gamma_{tR}(G)$. □

Example 5.2. *Consider the Graph G with five vertices $\{v_0, v_1, v_2, v_3, v_4\}$. Define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows:*

$$f(v_0) = 2 \text{ and } f(v_i) = 0; \text{for } i = 1, 2, 3, 4$$

Then f is a minimum total roman dominating function of G and $\gamma_{tR}(G) = 2$. Now consider the subgraph $G - v_0$. Define $g : V(G - v_0) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v_1) = 2 = g(v_2) \text{ and } g(v_3) = 0 = g(v_4)$$

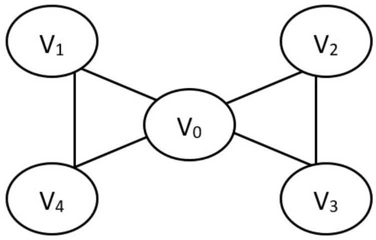

FIGURE 2. G

Then g is a minimum total roman dominating function of $G - v_0$ and the total roman domination number of $G - v_0$ is 4 $\gamma_{tR}(G - v_0) = 4$.
Thus $\gamma_{tR}(G - v_0) > \gamma_{tR}(G)$. □

Corollary 5.3. *Let G be a graph and $v \in V(G)$ be a pendant vertex whose neighbor u is not a pendant vertex then $\gamma_{tR}(G - v) \leq \gamma_{tR}(G)$.*

Proof. From proposition 3.1 $f(v) = 0$ or 1 for every minimal total roman dominating function f on G. Therefore by condition $i)$ of the theorem 5.1 $\gamma_{tR}(G-v) > \gamma_{tR}(G)$ is not possible. Thus $\gamma_{tR}(G - v) \leq \gamma_{tR}(G)$. □

Corollary 5.4. *Let G be a graph and $v \in V(G)$ is a pendant vertex whose neighbor u is not a pendant vertex. If there is a minimum total roman dominating function f on G such that $f(v) = 1$ then $\gamma_{tR}(G - v) < \gamma_{tR}(G)$.*

Proof. Since $f(v) = 1$ and $V_1(f) \bigcup V_2(f)$ does not have isolated vertices we have $f(u) = 1$ or 2. Now consider the restriction g of f on $G - v$. Then g is a total roman dominating function on $G - v$.
Also $w(g) < w(f) = \gamma_{tR}(G)$. Therefore $\gamma_{tR}(G - v) \leq w(g) < w(f) = \gamma_{tR}(G)$. i.e. $\gamma_{tR}(G - v) < \gamma_{tR}(G)$. □

Corollary 5.5. *Let G be a graph and $v \in V(G)$ is a pendant vertex and suppose its neighbor u has degree 2. If there is a minimum total roman dominating function f on G such that $f(v) = 0$ then $\gamma_{tR}(G - v) < \gamma_{tR}(G)$.*

Proof. Since $f(v) = 0, f(u) = 2$. Let w be the neighbor of u such that $w \neq v$ and $f(w) = 1$ or 2. Now define $g : V(G - v) \to \{0, 1, 2\}$ as follows:

$$g(u) = 1 \text{ and } g(x) = f(x); \forall x \neq u$$

Then g is a total roman dominating function on $G - v$.
Also $\gamma_{tR}(G - v) \leq w(g) < w(f) = \gamma_{tR}(G)$. i.e. $\gamma_{tR}(G - v) < \gamma_{tR}(G)$. □

Now we state and prove a necessary and sufficient condition under which the total roman domination number decreases when a vertex is removed from the graph.

Theorem 5.6. *Let G be a graph and $v \in V(G)$ be such that $G - v$ does not have isolated vertices then $\gamma_{tR}(G-v) < \gamma_{tR}(G)$ if and only if there is a minimum total roman dominating function $f : V(G) \to \{0, 1, 2\}$ such that f satisfies the following condition:*
$f(v) = 1$ and for every neighbor u for which $f(u) = 1$ there is a vertex w adjacent to u such that $w \neq v$ and $f(w) = 1$ or 2.

Proof. Suppose $\gamma_{tR}(G - v) < \gamma_{tR}(G)$. Consider $g : V(G) \to \{0, 1, 2\}$ be a minimum total roman dominating function on $G - v$.

Case-1: There is a neighbor u of v such that $g(u) = 1$ then for such vertex u there is a neighbor w,$g(w) = 1$ or 2. Now define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(v) = 1 \text{ and } f(x) = g(x); \forall x \neq v$$

Then f is a minimum total roman dominating function on G such that $f(v) = 1$ and if u is a neighbor of v such that $f(u) = 1$ then $g(u) = 1$ and therefore there is a neighbor w of u such that $g(w) = f(w) = 1$ or 2.

Case-2: Suppose for every neighbor u of v,$g(u) = 0$. Select any neighbor u of v. Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(v) = 1, f(u) = 1, f(x) = g(x); \forall x \neq \{u, v\}$$

Then f is a minimum total roman dominating function on G. Since $g(u) = 0$ there is a vertex w of $G - v$ such that $g(w) = f(w) = 2$. Thus the condition is satisfied by f.

Conversely suppose the condition is satisfied. Define $g : V(G - v) \to \{0, 1, 2\}$ as follows:

$$g(x) = f(x); \forall x \in V(G - v)$$

Then g is a function such that $w(g) < w(f)$. Let x be any vertex for which $g(x) = 0$ then $f(x) = 0$. Since f is a total roman dominating function there is a vertex z adjacent to x for which $g(z) = f(z) = 2$. Therefore g is a total roman dominating function on $G - v$. Thus $\gamma_{tR}(G - v) \leq w(g) < w(f) = \gamma_{tR}(G)$. i.e. $\gamma_{tR}(G - v) < \gamma_{tR}(G)$. □

Remark 5.7. *i) Consider cycle C_n with n – vertices. The total roman domination number of C_n is equal to n. Let v be a vertex of C_n. Now C_{n-v} is the path graph $P_{(n-1)}$ and its total roman domination number is $n - 1$. Thus for any vertex v of C_n, $\gamma_{tR}(C_{n-v}) < \gamma_{tR}(C_n)$.*
ii) Consider a path graph P_n with end vertices v and u.
a) If there is a minimum total roman dominating function f such that $f(v) = 1$ then $\gamma_{tR}(P_{n-v}) < \gamma_{tR}(P_n)$ by corollary 5.4.
b) If for every minimum total roman dominating function f,$f(v) = 0$ then also $\gamma_{tR}(P_{n-v}) < \gamma_{tR}(P_n)$ by corollary 5.5. □

Acknowledgement. The authors would like to extend their gratitude to the referees for their valuable suggestions and comments which improves the quality of this paper.

References

[1] Ahangar H. A., Henning M. A., Samodivkin V. and Yero I. G., (2016), Total Roman Domination in Graphs, Appl. Anal. Discrete Math., 10, pp. 501-517.
[2] Amjadi J., Sheikholeslami S. M., Soroudi M., Nordhaus-Gaddum, Bounds for Total Roman Domination, J. Comb. Optim. DOI: 10.1007/s10878-017-0158-5.
[3] Amjadi J., Nazari-Moghaddam S., Sheikholeslami S. M., Iran I. R., Volkmann L., (2017), Total Roman Domination Number of trees, Australian Journal of Combinatorics, 69(2), pp. 271-285.
[4] Cockayne E. J., Dawes R. M. and Hedetniemi S. T., (1980), Total Domination in Graphs, Networks, 10(3), pp. 211-219.
[5] Cockayne E. J., Dreyer P. A., Jr. Hedetniemi S. M. and Hedetniemi S. T., (2004), Roman Domination in graphs, Discrete Math., 278, pp. 11-22.
[6] Dreyer P. A., (2000), Jr. Dissertation Director: Fred S Roberts, Application and Variations of domination in graphs, New Brunswick, New Jersey.
[7] Haynes T. W., Hedetniemi S. T., Slater P. J., (1998), Fundamental of Domination In graphs, Marcel Dekker, New York.
[8] Haynes T. W., Hedetniemi S. T., Slater P. J., (1998), Domination In graphs Advanced Topics, Marcel Dekker, New York.
[9] Henning M. A. and Yeo A., (2013), Total Domination in Graphs, (Springer Monographs in Mathematics), ISBN: 978-1-4614-6525-6, (Online).

[10] Revelle C. S., Rosing K. E., (2000), Defenders Imperium Romanum: a classical problem in military strategy, Amer.Math.Monthly, 107(7), pp. 585-594.
[11] Stewart I., (1999), Defend the Roman Empire!, Sci. Amer., 281(6), pp. 136-139.
[12] Thakkar D. K. and Badiyani S. M., (2017), On the Roman Domination Number of Graphs, International Journal of Mathematical Archive, 8(2), pp. 184-188.

D. K. Thakkar is a retired professor from the Department of Mathematics, Saurashtra University, Rajkot (Gujarat). His area of specialization is colouring and domination in graph theory.

Dr. S. M. Badiyani is currently working as an assistant professor in Department of Mathematics, Saurashtra University, Rajkot (Gujarat). He is pursuing his Ph.D. under the guidance of Prof. D. K. Thakkar and doing his research in the field of domination theory in graphs.