TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 824-831

STATE ESTIMATION OF A CHAOTIC GALTON BOARD MODEL

A. A. MAT DAUD[1], §

Abstract. In this study, the application of the gradient descent of indeterminism (GDI) shadowing filter to a recently proposed chaotic Galton board model is presented and discussed. It is found that the original GDI shadowing filter does not provide stability to the convergence of the indeterminism and the error. The adaptive step-size (AS) is introduced to address the problem. The advantages of AS over fixed step-size (FS) and how it can improve the original shadowing filter are investigated in this study. The AS will guarantee successful implementation of GDI shadowing algorithm and will improve the performance of the original shadowing filter.

Keywords: shadowing filter, Galton board, gradient descent of indeterminism.

AMS Subject Classification: 37N15, 34C28.

1. Introduction

Several methods have been developed for finding shadowing trajectories and state estimation of nonlinear dynamical systems. One of the most powerful methods is the gradient descent filter. One method that has proven very powerful recently is the gradient descent filter [1]. Gradient descent is an optimisation method which minimises a quantity by moving continuously in the steepest descent direction [2]. Shadowing trajectories technique is an important method to determine the quality and the reliability of forecasting models and trajectories of chaotic systems [3].

In this study, the gradient descent algorithm will be used. Gradient descent noise reduction is a technique that attempts to recover the true trajectory, from noisy observations of a non-linear dynamical system for which the dynamics are known [4].

The present study has two main aims. First is to investigate whether the original GDI shadowing filter can successfully be implemented to a discrete, nonlinear dynamical system. The results of this study indicate that the original GDI shadowing filter does not provide stability to the convergence of the error. This requires further investigation on the why the problem occurs and how to overcome it. Therefore, the second aim of this

[1] Faculty of Ocean Engineering Technology and Informatics, University Malaysia Terengganu, Kuala Nerus, 21030, Malaysia.
Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Perth WA, 6009 Australia.
e-mail: auni_aslah@yahoo.com; ORCID: https://orcid.org/0000-0001-5502-0963.

§ Manuscript received: May 27, 2020; accepted: August 03, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.3

.

study is to propose a possible solution, namely by proposing the adaptive step-size (AS) in the algorithm. The implementation of GDI shadowing filter to continuous model, which requires different modifications, has been presented recently [5].

One may argue the significance of the results presented in the manuscript because the improved GDI shadowing filter is only applied to one model, namely the Galton board model.

The issue can be addressed as follows. It is important to note that the original GDI shadowing filter has been successfully applied to other models [6]. Therefore, although the improved GDI version will be only applied to Galton board model in this study, it can be guaranteed to be successfully applied to those models which have been investigated using the original version. In fact, the performance of the improved GDI filter will be at least as good as the original GDI filter, if not better. However, the application of the original GDI filter fails to provide stability in the case of Galton board model, as will be highlighted in this paper. That is the reason why the Galton board model is selected for this study.

The paper is organised as follows. Section 2 introduces a number of required improvements of the shadowing filter. The results of the application of the original and improved algorithm on the Galton board model for state estimation are presented, compared and discussed in Section 3. The conclusion of this study follow in Section 4.

1.1. **Galton board.** A Galton board (also called a quincunx) is a mechanical device which consists of two upright, parallel plates, a smooth wooden board and a glass sheet, with many interleaved horizontal rows of equally spaced pins. The pins are arranged on a board in a hexagonal array, as shown in Figure 1. At the top of the quincunx device, there is a funnel into which small lead balls are released. At the bottom of the device there is a row of narrow rectangular compartments or bins where the balls are collected. The whole installation is covered with a glass sheet from the front to allow viewing. Galton claimed that the Galton board is a random system in which each ball has an equal probability of going to either side of every pin that it strikes, giving rise to a bell-shaped distribution of lead. That is, the final exit distribution of the balls at the bottom of the device approximates a Binomial or Gaussian distribution. Nowadays, it is often a classroom and textbook demonstration of probability theory, Brownian motion and statistical mechanics. A more detailed discussion on the Galton board can be found in the literature [7, 8, 9, 10]. The details on the assumptions employed and the formulation and derivation of the governing equations of the Galton board model are not presented in this study (refer [11, 12, 13, 14]).

In the Galton board, the pins are equally spaced a distance H apart horizontally, and a distance V apart vertically. Every other row is offset by $H/2$, that is, half the pin spacing. The pins are arranged in a repeated series of 'quincunx' patterns, that is, a cross pattern of a dice's five face. Galton chose $V/H = 1$, or $H = V$. Unlike previous quincunx models, the value $V = 1.27\,\text{cm}$ will be used throughout this study as it appears that this is the correct value, based on a photograph of the original Galton board.

The lead ball is assumed to be spherical and the pins are assumed to be cylindrical. It is assumed that a ball is in contact with a pin when the distance between the centre of the ball and the centre of the pin is equal to R. That is, R is the combined radius of lead ball and pin. The impacts between the lead ball and the pin are assumed to be instantaneous but inelastic. It is assumed that there is no air resistance.

Several chaotic Galton board models have been proposed recently [11, 15]. In this paper, the most complex model in [11] will be used to assess the performance of the GDI shadowing filter. In the model, the normal and tangential coefficients of restitution (a

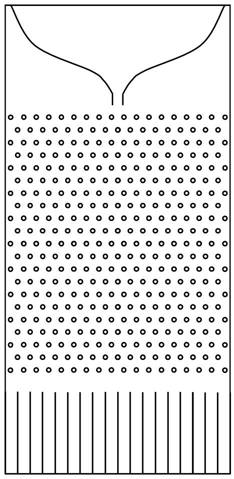

FIGURE 1. Schematic of a Galton board.

measure of the elasticity of the collision between the ball and the pins) are defined as constants

$$e_N = -\frac{v'_N}{v_N}, \quad (1)$$

$$e_T = \frac{v'_T}{v_T}, \quad (2)$$

where v'_N and v_N are the normal component of the rebound and incident velocity, respectively. v'_T and v_T are the tangential components of the rebound and incident velocity, respectively.

2. Improved GDI shadowing filter

2.1. **Shadowing filter.** A guideline and analysis of the gradient descent of indeterminism (GDI) shadowing filter has been published [6]. The original GDI shadowing filter using gradient descent will not be provided and discussed in this paper (refer [4, 16, 17] for more details).

For Galton board model, we compute the adjoint product by using a combination of *direct numerical approximation* and *λI-approximation method.* The computation of the adjoint product requires some truncation because there could be problem in computing adjoint with grazing collisions. Such collisions will cause problems in computing the adjoint product using numerical differentiation because it will give a very large value of Jacobian elements, as the changes of the velocities of the ball (before and after) are very small.

The computation of the adjoint product are as described in the following. Firstly, we take the adjoint A as $0.5I$, where I is the identity matrix. We compute the Jacobian, J by using numerical differentiation, that is, by adding small perturbations in each coordinate direction. Then J is transposed. Finally, any element of J' which is greater than a truncation threshold T will be truncated and substituted by the corresponding element in A. The computation can be expressed in the following equation

$$A_{i,j} = \begin{cases} J_{i,j}, & \text{if } |J_{i,j}| < T \\ 0.5, & i = j \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

T can be any small number less than 1. We have considered some values of T and we found by experiment that a value of 0.5 generally provides good convergence. A smaller T will truncate most elements of A, while a greater A may allow some elements that will cause instabilities. The initial findings are also supported by and in agreement with previous studies [2, 3].

However, it is important to note that the determination of optimal values for T may also depend on the nature (discrete or continuous) of system under investigation, which worth a separate, new study.

2.2. **Definitions of measured quantities.** In the paper we investigate the quality of the estimated states Z_m based on the four defined quantities [6]; the indeterminism I_m, the final state mismatch magnitude $I_{n,m}$, the root mean square error of states E_m and the last point error $E_{n,m}$.

A fixed step-size Δ is used in the iteration of the original GDI shadowing filter. The choice of Δ will determine the convergence rate and the stability of the algorithm. In general, a small step-size value will provide stability, with a slow convergence rate. On the other hand, a large value of step-size will give a fast convergence rate but the stability is not ensured. The principle criteria of selecting the optimal value of Δ is that the convergence of the indeterminism, should be continuously decreasing.

In this study, we introduce AS to address the difficulties. AS is defined as a number of different step-size values in the iterative GDI shadowing filter, which are varied or adapted (but must always be decreasing) to avoid the failure of convergence problem. It can reduce the rate of convergence by decreasing the step-size value.

GDI shadowing filter can fail to converge if the value of the step-size is too large, and the convergence will be slow if the value of the step-size is too small. Failure here refers to the non-decreasing indeterminism as the number of iteration increases. Therefore, selecting the largest step-size value which provide a decrease of indeterminism may not necessarily be a good idea, but if a certain step-size value does not provide a decrease, when a smaller step-size does, then the step-size value must be decreased.

2.3. **The implementation.** The essential idea in this study is to adapt the step-size value according to the convergence of the indeterminism. The aim is to provide a fast convergence rate and not compromise the stability. The reduction of the step-size value is based on the values of current and previous indeterminism. If the indeterminism of current iteration is larger than or equal to the indeterminism of previous iteration, then the step-size value is divided by a factor k in the following iteration and the state estimates of the previous iteration will be filtered again using a new decreased step-size value. However, if the indeterminism decreases, the step-size value remains unchanged for the next iteration.

However, there are two questions that need to be answered before the AS can be employed; the appropriate value of k and initial step-size Δ_0. Inappropriate choices of the values will result in very slow convergence and hence increase the amount of iteration. The questions will be addressed in the following sections.

2.4. **The appropriate value for the adaptive factor.** The value of k should not be too large because if $k \gg 1$ then the step-size will become very small very quickly and provide very slow progress to convergence. Furthermore, several values of k were initially considered for Galton board model, but it is found by experiment that there was no significant difference. It is found that the filter may work for arbitrary values of k bigger than 1. Generally, $k = 2$ provides the fastest convergence rate. The initial findings are also supported by and in agreement with previous studies [5, 14].

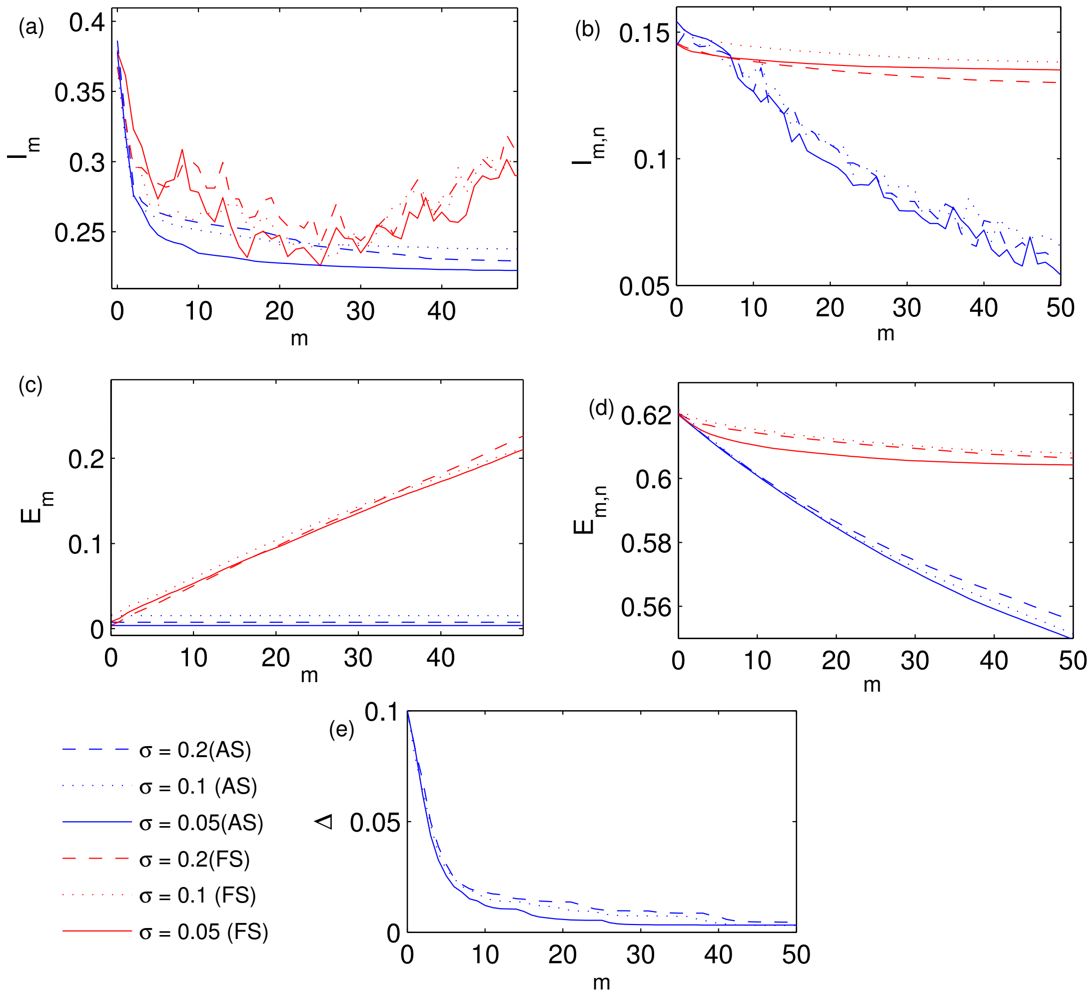

FIGURE 2. The average values of (a) I_m, (b) E_m, (c) $I_{n,m}$ and (d) $E_{n,m}$, as a function of the number of iteration, m with noise level,$\sigma = 0.1$, 0.2 and 0.3, $k = 0.5$, and the initial step-size, $2\Delta/(n-1) = 0.1$, for Galton board model, e) The average value of the term $2\Delta/(n-1)$ as a function of m.

However, it is important to note that the determination of optimal values for T may also depend on the nature (discrete or continuous) of system under investigation, which worth a separate, new study.

2.5. **The value of initial step-size.** Employing AS will ensure the convergence, that is, the indeterminism will always be decreasing or at least, non-increasing. The only issue to be considered is the convergence rate. The algorithm with AS works for any values $0 < \Delta_0 < 1$, but it is found that, in most cases, a value around $\Delta_0 = 0.1$ provides the fastest convergence rate.

As has been noted for T, the determination of optimal values for k may depend on the nature of system under investigation. Further studies are recommended for that purpose.

3. RESULTS AND DISCUSSIONS

3.1. **State estimation using FS.** To assess the performance of the GDI shadowing filter for the Galton board model, we applied the filter to the Galton board model when $R = 0.26$, $e_N = 0.49$, and $e_T = 0.2$. We computed average values of I_m, $I_{n,m}$, E_m and $E_{n,m}$ for the observations (our artificial data) where $2\Delta/(n-1) = 0.1$, $m = 50$, and $\sigma = 0.05, 0.1$ and 0.2. Small Gaussian errors of observations are used (although the noise size relative to the pin spacing are quite large) because large errors could be a problem with the ball bouncing the wrong way. For example, a ball that hit near the top of a pin and bounces to the right of the pin might bounce to the left if large noise are added to the point of impact. The results showed and discussed in Fig. 2 are qualitatively similar for different model parameter values. This robustness to changes in the values of the model parameters imply that Galton board model is in the dynamical regime that is difficult to predict.

Fig. 2 shows the average value of I_m, $I_{n,m}$, E_m and $E_{n,m}$ for 50 trajectories plotted as a function of the number of iteration m. It is found that basic GDI shadowing filter with selected step-size, does not work for Galton board model. The figure shows that the filter provides around 10% reduction of $I_{n,m}$ and 2.5% reduction of $E_{n,m}$. However, the figure clearly shows that the GDI shadowing filter does not provide stability, that is, average value of I_m reaches a minimum at around $m = 5$ and starts to increases beyond that, for all of the three noise levels. Furthermore, E_m increases as the number of iterations increase. This should not happen and shows that the selection of the step-size value is not correct. We attempt to address this failure by repeating the experiment using smaller step-size, that is $2\Delta/(n-1) = 0.05$. However, we found that it is still unstable and still fails with this step-size value, that is, I_m and E_m still increase, although not until $m > 5$.

3.2. **State estimation using AS.** To assess the performance of the GDI shadowing filter with AS, we repeated the experiment using the same initial observation data used in the previous experiment and computed the average values for I_m, E_m, $I_{n,m}$ and $E_{n,m}$. It is important to note that the comparison is made between the AS filter and the FS filter of one stepsize value only as the selected value has been found to be optimal [6]. The results for Galton board model are plotted in Fig. 2. The Figures clearly show that, using the AS, the average value of all four quantities decreases or at least does not increase, as the number of iterations, m increases. Observe in Fig. 2a and 2c that AS works whereas the FS fails. Note that in Fig. 2b, the FS initially has a faster rate of convergence, it is eventually surpassed by the AS after less that 10 iterations. Using AS, I_m is strictly decreasing for all noise levels, and E_m is not increasing. In Fig. 2b and 2d, (where the FS has no convergence problem) the AS performs slightly better than the FS. Fig. 2e shows the average value of the term $2\Delta/(n-1)$ as the number of iteration m increases. Note that, on average, big step-sizes are used in the first ten iterations, and smaller step-sizes are used beyond that.

4. Conclusions

The current study provides two main results: the implementation of the GDI shadowing filter to the most complex and recently developed Galton board model and the improvement of the original GDI shadowing filter with the AS. The results reveal that the original GDI shadowing filter does not provide stability to the convergence of the indeterminism and the error. We have also introduced and proposed a new simple way in choosing the step-size value to address the convergence problem. Our approach is to adapt the step-size sequence in the a gradient descent algorithm so as to ensure that the indeterminism is reduced at each iteration. It is designed to eliminate the uncertainty in selection of the optimal step-size value, and to improve the convergence rate while maintaining the stability.

The results of this study show that the performance of the AS is significantly better than the FS, even for the cases where the FS works. The AS, with arbitrary values of k greater than 1, will ensure the average value of I_m to be strictly decreasing, or at least non-increasing. Although the FS in some cases may initially provide higher convergence rate, it is eventually outperformed by the AS. A larger step-size will generally give faster convergence, but the convergence of the indeterminism may not be strictly decreasing. Futhermore, it is known that for the cases where a large, fixed step-size does not work, a smaller step-size will guarantee the convergence of the indeterminism. However, the GDI shadowing filter with AS will provide better convergence of the indeterminism, after the same number of iterations. The adaptive step-size will guarantee successful implementation of the GDI shadowing algorithm and will improve the performance of the original shadowing filter.

In future, further study is required to quantitatively determine the optimal values of T and k, and develop a standard statistical test or procedure for that purpose, taking into consideration the nature (discrete or continuous) of the system under study.

Acknowledgment. I would like to thank my supervisors, Prof. Kevin Judd and Dr. Thomas Stemler. The research was carried out at The University of Western Australia (UWA)[14]. The study was sponsored by Malaysian Ministry of Higher Education.

References

[1] Judd, K., (2003), Nonlinear state estimation, indistinguishable states and the extended Kalman Filter, Physica D, 183, pp. 273-281.

[2] Judd, K., Reynolds, C. and Rosmond, T., (2004), Towards shadowing in operational weather prediction, Technical Report Technical Report NRL/MR/7530-04-18, Naval Research Laboratory, Monterey, CA, USA.

[3] Judd, K, Smith, L. A. and Weisheimer, A., (2004), Gradient free descent: shadowing and state estimation using limited derivative information, Physica D, 190, pp. 153-166.

[4] Ridout, D. and Judd, K., (2002), Convergence properties of a gradient descent noise reduction, Physica D, 165, pp. 26-47.

[5] Mat Daud, A. A., (2016), State estimation and forecasting of the ski-slope model using an improved shadowing filter, International Journal of Bifurcation and Chaos, 26, pp. 1650056.

[6] Stemler, T. and Judd, K., (2009), A guide to using shadowing filters for forecasting and state estimation, Physica D, 238, pp. 1260-1273.

[7] Chris, P., (2006), Bagatelle as the inspiration for Galton's Quincunx, Journal of the British Society for the History of Mathematics, 102–110, 21(2).

[8] Galton, F., (1889), Natural Inheritance, 1889, MacMillan.

[9] Kunert, J. and Montag, A. and Phimann, S., (2001), The quincunx: history and mathematics, Statistical Papers, pp. 143-169, 42.

[10] Stigler, S. M., (1989), Francis Galton's Account of the Invention of Correlation, Statistical Science, pp. 73–79, 4(2).

[11] Mat Daud, A. A., (2014), Mathematical modelling and symbolic dynamics analysis of three new galton board models, Communications in Nonlinear Science and Numerical Simulation, 19, pp. 3476-3491.

[12] Judd, K., (2007), Galton's Quincunx: Random walk or chaos?, International Journal of Bifurcation and Chaos, 17, pp. 4463-4467.

[13] Kozlov, V. V. and Mitrofanova, M. Y., (2003), Galton Board, Regular and Chaotic Dynamics, 8, pp. 431-439.

[14] Mat Daud, A. A., State estimation and forecasting using a shadowing filter applied to quincunx and ski-slope models, PhD thesis, (2012).

[15] Mat Daud, A. A., (2017), An Improved Mathematical Model of Galton Board With Velocity-Dependent Restitution, Journal of Computational and Nonlinear Dynamics, 12, pp. 060901–1-060901–3.

[16] Judd, K. and Smith, L. A., (2004), Indistinguishable states II: The imperfect model scenario, Physica D, 196, pp. 224-242.
[17] Judd, K., Smith, L. A., (2001), Indistinguishable states I: Perfect model scenario, Physica D, 151, pp. 125-141.

Auni Aslah Mat Daud started his career as a mathematics tutor at Kolej Universiti Sains dan Teknologi Malaysia (currently known as Universiti Malaysia Terengganu - UMT) in 2006 upon graduating with a First Class Honours (B.Sc. in Mathematics) from Universiti Teknologi Malaysia in 2005. He served as a part-time tutor at the University of Western Australia during his doctoral studies at the university. He became a lecturer at UMT in 2012 following the completion of his PhD. He was promoted to senior lecturer in 2014 His research interests are mathematical modelling and analysis of linear and nonlinear dynamical systems.