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A MINIMAX INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. A minimax inequality of Ky Fan for mapping with non-compact domain is given. Moreover, some minimax inequalities are obtained.

Keywords: KKM property, transfer upper semicontinuous, transfer lower semicontinuous, minimax theorem.

AMS Subject Classification: 83-02, 99A00.

1. Introduction and Preliminaries

In 1998, Tarafdar and Watson [7] gave a minimax inequality for mapping with non-compact domain by using the coincidence theorem. Then Zhang [9] obtained a minimax inequality for mapping with noncompact domain under different assumptions. Our goal in this paper is to obtain a minimax inequality for mapping with noncompact domain by using the KKM-type mapping. This result improves the results in [5, 6, 7, 9].

Let D be a nonempty set, we shall denote by 2^D the family of all subsets of D, by $\mathcal{F}(D)$ the family of all nonempty finite subsets of X. Let X and Y be topological spaces. A multi-valued map $F : X \to 2^Y$ is called:

(i) closed if its graph, that is $\{(x, y) : y \in F(x)\}$ is a closed subset of $X \times Y$,

(ii) compact if $\overline{F(X)}$ is a compact subset of Y,

(iii) transfer closed valued if for any $x \in X$ and $y \notin F(x)$ there exists $x' \in X$ such that $y \notin \overline{F(x')}$.

Suppose that f is a real-valued bifunction on $X \times Y$ and γ is a real number, f, is called γ-transfer upper semicontinuous (u.s.c.) in the second variable if for each $(x, y) \in X \times Y$ with $f(x, y) > \gamma$ there exists $x' \in X$ and a neighborhood $U(y)$ of y in Y such that $f(x', z) > \gamma$ for all $z \in U(y)$. The bifunction f is said to be γ-transfer lower semicontinuous (l.s.c.) in the second variable, if $-f$ is γ-transfer u.s.c. in the second variable.

A convex space X is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Let X be a convex space and Y a topological space. If $G : X \to 2^Y$ and $T : X \to 2^Y$ are two multi-valued mappings such that for any $A \in \mathcal{F}(X)$, $T(coA) \subseteq G(A)$, then G is said to be generalized

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KKM mapping with respect to T. Let $T : X \to 2^Y$ be a multi-valued mapping such that if $G : X \to 2^Y$ is a generalized KKM mapping with respect to T, then the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property, in this case we say that T has the KKM property. Denote

$$\mathrm{KKM}(X, Y) := \{T : X \to 2^Y : T \textit{ has the } \mathrm{KKM} \textit{ property}\}.$$

The class $\mathrm{KKM}(X, Y)$ was introduced and studied by Chang and Yen [1]. Also, they have shown that $\mathrm{KKM}(X, Y)$ contains admissible mappings, Kakutani mappings, and etc; see [1]. The following theorem plays a key role in the next section.

Theorem 1.1 ([4], Theorem 2.1)**.** *Let X be a convex space, Y a topological space, and $T \in KKM(X, Y)$ be compact. Suppose that $F : X \to 2^Y$ is transfer closed valued and generalized KKM mapping with respect to T. Then $(\bigcap_{x\in X} F(x)) \cap \overline{T(X)} \neq \emptyset$.*

2. Main results

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Theorem 2.1. *Let X be a convex space and Y a topological space and $T \in KKM(X, Y)$ be compact. Suppose that $f, g : X \times Y \to \mathbb{R}$ are two real valued bifunctions, such that:*
(i) $f(x, y) \leq g(x, y)$, for all $(x, y) \in X \times Y$,
(ii) $f(x, y)$ is quasiconvex on X, for all $y \in Y$,
(iii) g is γ-transfer u.s.c. in the second variable for any $\gamma \in \mathbb{R}$. Then

$$\inf_{y\in T(x)} f(x, y) \leq \sup_{y\in Y} \inf_{x\in X} g(x, y). \tag{1}$$

Proof. First, we can assume that the left-hand side of (1) is not $-\infty$, choose a real number γ, such that

$$\inf_{y\in T(x)} f(x, y) > \gamma. \tag{2}$$

Suppose that $F(x) = \{y \in Y : f(x, y) \geq \gamma\}$ and $G(x) = \{y \in Y : g(x, y) \geq \gamma\}$ for each $x \in X$. We show that F is a generalized KKM mapping with respect to T and G is transfer closed valued. If $A = \{x_1, ..., x_n\} \in F(X)$ and $(T(coA))$ is not a subset of $F(A)$, then there is a point $\bar{x} \in coA$ and $\bar{y} \in T(\bar{x})$ such that $\bar{y} \notin F(A)$. Hence, $f(x_i, \bar{y}) < \gamma$ for all $i = 1, \ldots, n$. From condition (ii) we have $f(\bar{x}, \bar{y}) < \gamma$ which is contracted by (2). Now, assume that $(x, y) \in X \times Y$ with $y \notin G(x)$, then $g(x, y) < \gamma$. By assumption (iii) there exist $x' \in X$ and a neighborhood $U(y)$ of y such that $g(x', z) < \gamma$ for all $z \in U(y)$. Thus $U(y) \cap G(x') = \emptyset$ and so $y \notin \overline{G(x')}$. Finally, using condition (i), for each $x \in X$, $F(x) \subseteq G(x)$. Therefore, G and T satisfy in all of the conditions Theorem 1.1, thus

$$(\bigcap_{x\in X} G(x)) \cap \overline{T(X)} \neq \emptyset.$$

That is, there exists $y' \in \overline{T(X)}$ such that $g(x, y') \geq \gamma$ for any $x \in X$. This shows that

$$\sup_{y \in Y} \inf_{x \in X} g(x, y) \geq \gamma,$$

and so, (1) holds. □

Remark 2.1. *(a) The above theorem was obtained by Zhang et al. [[10], Theorem 8] in the case where Y is a compact convex subset of a topological vector space and T is an upper semicontinuous set-valued mapping such that $T(x)$ is a compact convex subset of Y, for each $x \in X$.*

As a consequence of Theorem 2.1 we obtain the following result which improves Theorem 2.1 of [9].

Corollary 2.1. *Let E_1 , E_2 be two topological vector spaces and $X \subset E_1$ and $Y \subset E_2$ be nonempty convex subsets. Let $f, g : X \times Y \to \mathbb{R}$ satisfying in conditions (i-iii) of Theotem 2.1. Suppose that $T : X \to 2^Y$ is an u.s.c. multi-valued mapping such that*
(1) $T(x)$ is a compact convex subset of Y, for all $x \in X$,
(2) for each $x \in X$, there exists a subset $V_x \subset Y$, such that $V_x \cap T(x) \neq \emptyset$ and $K = \bigcup_{x \in X} V_x \subset Y$ is compact convex subset. Then

$$\inf_{y \in T(x)} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Proof. Let $\bar{T} : X \to 2^Y$ be defined as

$$\bar{T}(x) = T(x) \cap K, \ for\ all\ x \in X.$$

Then $\bar{T}$ is compact and compact convex valued and so $\bar{T} \in \mathrm{KKM}(X, Y)$. Therefore, by Theorem 2.1 we have

$$\inf_{y \in \bar{T}(x)} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Since $\bar{T}(x) \subset T(x)$ for each $x \in X$, then

$$\inf_{y \in \bar{T}(x)} f(x, y) \geq \inf_{x \in T(X)} f(x, y).$$

This completes the proof of Corollary 2.1. □

If in Theorem 2.1, $f = g$, then we obtain the following minimax inequality which improves Theorem 1 of [3].

Theorem 2.2. *Let X, Y and T be as in Theorem 2.1. Suppose that $f, g : X \times Y \to \mathbb{R}$ satisfies in condition (ii) and γ-transfer u.s.c. in the second variable for any $\gamma \in \mathbb{R}$. Then*

$$\inf_{y \in T(x)} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

According to Theorem 2.1 and Theorem 2.2 we prove the following results, which are refinement of Fan' minimax inequality [[2], Theorem 5].

Theorem 2.3. *Let E be a topological vector space and X and Y be nonempty convex subsets of E. Suppose that K is a nonempty compact subset of E such that $X \subset K$, $K \subset Y$. Let $f, g : X \times Y \to \mathbb{R}$ be two real valued bifunctions satisfying in conditions (i-iii) of Theorem 2.1. Then*

$$\inf_{x \in X)} f(x, x) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Proof. The proof follows from Theorem 2.1 by taking $T(x) = \{x\}$ for all $x \in X$. □

Theorem 2.4. *Let X, Y and K be as in Theorem 2.3. Let $f, g : X \times Y \to \mathbb{R}$ satisfying in hypothesis of Theorem 2.2. Then*

$$\inf_{x \in X)} f(x, x) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

Remark 2.2. *Theorems 2.3 and 2.4 improve Theorems 2.4 and 2.5 of* [9]*, Theorems 10 and 11 of* [10] *and Theorem 5 of* [8]*.*

As an application of Theorem 2.1, we present the following result.

Theorem 2.5. *Let X, Y and T as given in Theorem 2.1. Suppose that $f, g : X \times Y \to \mathbb{R}$ are two real valued bifunctions such that:*
(i) $f(x, y) \leq g(x, y)$, for all $(x, y) \in X \times Y$,
(ii) $g(x, y)$ is quasiconcave on X, for all $y \in Y$,
(iii) f is γ-transfer l.s.c. in the second variable for any $\gamma \in \mathbb{R}$. Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{y \in T(x)} g(x, y).$$

Proof. If $f_1 = -g$ and $g_1 = f$, then f_1, g_1 are satisfy in all of the condition Theorem 2.1. Therefore,

$$\inf_{y \in T(x)} f_1(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g_1(x, y),$$

that is,

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{y \in T(x)} g(x, y).$$

□

Remark 2.3. *From the above theorem we can establish the analogous results to Corollary 2.1, Theorem 2.2, 2.3 and 2.4.*

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