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A NOVEL THIRD KIND CHEBYSHEV WAVELET COLLOCATION METHOD FOR THE NUMERICAL SOLUTION OF STOCHASTIC FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In the formulation of natural processes like emissions, population development, financial markets, and the mechanical systems, in which the past affect both the present and the future, Volterra integro-differential equations appear. Moreover, as many phenomena in the real world suffer from disturbances or random noise, it is normal and healthy for them to go from probabilistic models to stochastic models. This article introduces a new approach to solve stochastic fractional Volterra integro-differential equations based on the operational matrix method of Chebyshev wavelets of third kind and stochastic operational matrix of Chebyshev wavelets of third kind. Also, we have given the convergence and error analysis of the proposed method. A variety of numerical experiments are carried out to demonstrate our theoretical findings.

Keywords: Stochastic Volterra integro-differential equations, Chebyshev wavelets of third kind, Brownian motion, stochastic operational matrix of Chebyshev wavelets of third kind.

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1. Introduction

In mathematical modeling of many physical phenomena, including mechanic, economic, dynamic reactor, etc., stochastic Volterra integral equations arise. Such systems also appear in the study of the growth model for biological populations, and the study of behavior in physics and technological dynamism in more realistic systems [1, 2, 3, 4]. Most specifically those systems additive noise under certain probability rules, such as Gaussian white noise. Thus, it is normal, in the most complex situations, to use stochastic Volterra integral equations [5, 6]. The analysis of stochastic differential equations, therefore, constitutes an relevant area of study. These differential equations are very unusual in straightforward solutions and computational methods need to be used to overcome these issues. Recently, computational techniques have outcome as a very efficient and powerful computational methodology for simulating complex or smooth physical phenomena [7, 8, 9]. Such methods have been most recently employed for the solution of time partial diffusion systems

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in [10, 11, 12, 13, 14, 15, 16], while M. Asgari employed block pulse function to obtain the numerical solution of stochastic fractional Volterra integro-differential equations (SF-VIDE) [17]. This pilot study aims to develop the collocation technique for the numerical solution of SFVIDE. Let us consider the following SFVIDE [17] for this reason,

$$D^{\alpha}y(x) = f(x) + \int_0^x k_1(x,t)y(t)dt + \int_0^x k_2(x,t)y(t)dW(t), \tag{1}$$

with initial conditions,

$$y^{(i)}(0) = y_i\,,\ i = 0,1,2,...,n-1,\ \ n-1 < \alpha \le n,\ \ n \in N,$$

where, $W(x)$ is a Brownian motion and $y(x)$ is the unknown stochastic process, and solution of (1), it is adapted to $\{F_t,\, t \ge 0\}$.

Wavelets are mathematical functions that divide data into frequency components and analyze individual components in their respective resolution. As a statistical tool, wavelets can be used to obtain data from the variety of data types like seismic waves, earthquakes, signal processing, nuclear engineering, acoustics, and astronomy. Many researchers have paid great attention to it and it has been applied in a various technical fields. These wavelets which are obtained from orthogonal polynomials, in particular, are regularly used in the quest for the approximate solution of various types of integral, differential, and integro-differential equations. some of them are found in [18, 19, 20, 21]. The fractional order operational matrices of integration of Haar wavelet, Bernoulli wavelet, Chebyshev wavelet, and the Legendre wavelet have been used in the last decade to solve differential equations of fractional order [22, 23, 24, 25, 26]. Similarly, stochastic operational matrices of fractional order integration of Chebyshev wavelets have been used to solve stochastic differential equations of fractional order [27, 28].

Encouraged by most of these work, we approximate equation (1) using Chebyshev wavelets of third kind [29]. There are four types of Chebyshev polynomials and they are well known [30]. There is indeed a great focus on Chebyshev polynomials of the first and second kinds and their various implementations in the literature, for instance, see [31, 32]. There are, however, few studies focusing on third and fourth type Chebyshev wavelets. Here, we stretch the importance of Chebyshev wavelets of third kind to form a stochastic operational matrix of integration (SOMI) of Chebyshev wavelets of third kind. This SOMI of Chebyshev wavelets of third kind is used to acquire the approximate solution of equation (1).

The remaining paper is structured as follows. Section 2 provides some basic definitions and characteristics of stochastic calculus, wavelets, Chebyshev wavelets of third kind, and fractional calculus. Also, in this section, SOMI of Chebyshev wavelets of third kind are obtained. The proposed method of solution is given to estimate the solution of fractional integro-differential equations in section 3. Computational experiments are presented to show the efficiency and reliability of the proposed method in section 5. Convergence and Error analysis of the proposed method is studied in 4. Finally, in Section 6 the conclusion of the article is given.

2. Properties of stochastic calculus, Fractional calculus, wavelets, and Third kind Chebyshev wavelets

2.1. **Brownian Motion.** For definitions of Brownian motion see [33].

2.2. **Itô Integrals.** If we consider the following ordinary differential equation (ODE):

$$\frac{dy(x)}{dx} = g(x,y), \quad dy(x) = g(x,y)dx, \tag{2}$$

satisfying the initial conditions $y(0) = y_0$ can be written in integral form as follows:

$$y(x) = y_0 + \int_0^x g(s, y(s))ds, \tag{3}$$

where $y(x) = y(x, y_0, x_0)$ is the solution satisfying the initial conditions $y(x_0) = y_0$. For example:

$$\frac{dy(x)}{dx} = a(x)y(x), \quad y(0) = y_0. \tag{4}$$

If we take the ODE (4) and consider that $a(x)$ is not deterministic but instead a stochastic parameter, we get a stochastic differential equation (SDE). The parameter $a(x)$ is given as:

$$a(x) = g(x) + h(x)\xi(x), \tag{5}$$

where $\xi(x)$ denotes a white noise process. and therefore, we get:

$$\frac{dY(x)}{dx} = g(x)Y(x) + h(x)Y(x)\xi(x). \tag{6}$$

If we let $dW(x) = \xi(x)dx$ and use equation (6) in the differential form, $dW(x)$ represents the Brownian motion's differential form and we get:

$$dY(x) = g(x)Y(x)dx + h(t)Y(x)dW(x). \tag{7}$$

In order to explain stochastic integral equations, let us consider the following example:

$$\begin{aligned}
g(x,w) &= W(x,w)\int_0^T W(x,w)dW(x,w) \\
&= \lim_{N\to\infty}\sum_{i=1}^N W(x_{i-1},w)\left(W(x_i,w) - W(x_{i-1},w)\right) \\
&= \lim_{N\to\infty}\left[\frac{1}{2}\sum_{i=1}^N (W^2(x_i,w) - W^2(x_i-1,w)) - \frac{1}{2}\sum_{i=1}^N (W(x_i,w) - W(x_{i-1},w))^2\right] \\
&= -\frac{1}{2}\lim_{N\to\infty}\sum_{i=1}^N (W(x_i,w) - W(x_{i-1},w))^2 + \frac{1}{2}W^2(T,w).
\end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[\lim_{N\to\infty}\sum_{i=1}^N (W(x_i,w) - W(x_{i-1},w))^2] &= \lim_{N\to\infty}\sum_{i=1}^N \mathrm{E}[(W(x_i,w) - W(x_{i-1},w))^2] \\
&= \lim_{N\to\infty}\sum_{i=1}^N (x_i - x_{i-1}) \\
&= T.
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}[\lim_{N\to\infty}\sum_{i=1}^N (W(x_i,w) - W(x_{i-1},w))^2] &= \lim_{N\to\infty}\sum_{i=1}^N \mathrm{var}[(W(x_i,w) - W(x_{i-1},w))^2] \\
&= 2\lim_{N\to\infty}(x_i - x_{i-1})^2.
\end{aligned}$$

By reducing the partition, the variance becomes zero,

$$\begin{aligned}\lim_{N\to\infty}\sum_{i=1}^{N}(x_i-x_{i-1})^2 &\leq \max_i(x_i-x_{i-1})\lim_{N\to\infty}\sum_{i=1}^{N}(x_i-x_{i-1})\\ &=(x_i-x_{i-1})T\\ &=0, \end{aligned} \tag{9}$$

since $x_{i-1}-x_i \to 0$. Since the expected value of $\sum_{i=1}^{N}(x_i-x_{i-1})^2$ is T and the variance becomes zero, we get

$$\sum_{i=1}^{N}(W(x_i,w)-W(x_{i-1},w))^2 = T. \tag{10}$$

The stochastic integral has the solution:

$$\int_0^T W(x,w)dW(x,w) = \frac{1}{2}W^2(T,w)-\frac{1}{2}T. \tag{11}$$

This is contradictory to our normal calculus intuition. For deterministic integral $\int_0^T x(t)dt = \frac{1}{2}x^2(t)$, but the the Itô integral varies by the term $-\frac{1}{2}T$. This illustration illustrates that differentiation rules and integration rules in the stochastic calculus (especially the chain rule), must be reformulated.

Properties of Itô Integrals:

- $\text{var}\left[\int_0^T g(x,w)dW(x,w)\right] = \int_0^T E[g^2(x,w)]dt.$

There are two important properties in calculating the variance of the Itô integrals:

- $\left[\left(\int_0^T g(x,w)dW(x,w)\right)^2\right] = \int_0^T E\left[g^2(x,w)\right]dt.$
- $\int_0^T E\left[g^2(x,w)\right]dt < \infty.$

The second property is the condition of existence for Itô integrals.

2.3. **Fractional calculus.** For detailed study of fractional calculus see [34].

2.4. **Third kind Chebyshev wavelets.** Third kind Chebyshev wavelets with four arguments [36] k, n, m, and x are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{1}{\pi}}C_m(2^kx-2n+1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}},\\ 0, & Otherwise, \end{cases} \tag{12}$$

where, $k>0$, $n=1,2,...,2^{k-1}$, x denotes the time and m denotes the degree of third kind Chebyshev polynomials. In equation (12), $C_m(x)$ are Chebyshev polynomials of third kind whose degree is m with weight function $w(x)=\sqrt{\frac{1+x}{1-x}}$ on $[-1,\ 1]$ and satisfy the recursive formula:

$$C_0(x)=1,\ C_1(x)=2x-1,$$

$$C_{m+1}(x) = 2xC_m(x)-C_{m-1}(x),\ \ m=1,2,3,...$$

For instance, for $k=2$ and $M=2$, we get

$$\left.\begin{aligned}\psi_{1,0}(x) &= \tfrac{2}{\sqrt{\pi}}\\ \psi_{1,1}(x) &= \tfrac{2}{\sqrt{\pi}}(8x-3)\end{aligned}\right\} 0 \leq x < \frac{1}{2},$$

$$\left.\begin{aligned}\psi_{1,0}(x) &= \tfrac{2}{\sqrt{\pi}} \\ \psi_{1,1}(x) &= \tfrac{2}{\sqrt{\pi}}\,(8x-7)\end{aligned}\right\}\frac{1}{2} \leq x < 1.$$

When concerned with Chebyshev wavelets of third kind, the weight function $w(x) = \sqrt{\frac{1+x}{1-x}}$ must be dilated and truncated as $w(x) = w(2^k x - 2n + 1)$.

2.5. **Function approximation.** Let us expand $f(x) \in L^2[0,1)$ with respect to the Chebyshev wavelets of third kind as,

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} v_{n,m}\psi_{n,m}(x). \tag{13}$$

If we truncate the infinite series given above, we get

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} v_{n,m}\psi_{n,m}(x) = V^T\psi(x) = f_{\hat{m}}(x), \tag{14}$$

where, the $\hat{m} \times 1$ ($\hat{m} = 2^{k-1}M$) matrices V and $\psi(x)$ are given as follows:

$$V = \left[v_{1,0}, v_{1,1}, ..., v_{1,M-1}, v_{2,0}, ..., v_{2,M-1}, ..., v_{2^{k-1},0}, ..., v_{2^{k-1},M-1}\right]^T, \tag{15}$$

and

$$\begin{aligned}\psi(x) = [&\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), ..., \psi_{2,M-1}(x), \\ &..., \psi_{2^{k-1},0}(x), ..., \psi_{2^{k-1},M-1}(x)]^T.\end{aligned} \tag{16}$$

2.6. **Operational matrix of integration (OMI) and SOMI of Chebyshev wavelets of third kind.** OMI P of Chebyshev wavelets of third kind are derived in [36] as,

$$\int_0^x \psi(t)dt = P\psi(x), \tag{17}$$

where

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & F \\ 0 & 0 & \cdots & 0 & L \end{bmatrix},$$

where the $M \times M$ matrices L and F are given by

$$L = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ -2 & -\frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 & 0 \\ \frac{5}{6} & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{6} & \cdots & 0 & 0 \\ -\frac{7}{12} & 0 & -\frac{1}{6} & -\frac{1}{24} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{M-2}\frac{2M-3}{(M-1)(M-2)} & 0 & 0 & 0 & \ddots & -\frac{1}{2(M-2)(M-1)} & \frac{1}{2(M-1)} \\ (-1)^{M-1}\frac{2M-1}{M(M-1)} & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-1)} & -\frac{1}{2M(M-1)} \end{bmatrix}.$$

If M is even,

$$F=\begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{\frac{M}{2}} & 0 & 0 & \cdots & 0 \\ \alpha_{\frac{M}{2}} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\alpha_i=\frac{2}{2i-1},\ i=1,2,...,\frac{M}{2}$, and if M is odd,

$$F=\begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{\frac{M+1}{2}-1} & 0 & 0 & \cdots & 0 \\ \alpha_{\frac{M+1}{2}-1} & 0 & 0 & \cdots & 0 \\ \alpha_{\frac{M+1}{2}} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\alpha_i=\frac{2}{2i-1},\ i=1,2,...,\frac{M+1}{2}$. And the fractional OMI P_α Chebyshev wavelets of third kind are derived in [36] as,

$$[P_\alpha]_{2^{k-1}M\times 2^{k-1}M}=[\psi]_{2^{k-1}M\times 2^{k-1}M}\,[F_\alpha]_{2^{k-1}M\times 2^{k-1}M}\,\left[\psi^{-1}\right]_{2^{k-1}M\times 2^{k-1}M},$$

where,

$$F_\alpha=\frac{1}{\hat{m}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{\hat{m}-1} \\ 0 & 1 & \zeta_1 & \zeta_2 & \cdots & \zeta_{\hat{m}-2} \\ 0 & 0 & 1 & \zeta_1 & \cdots & \zeta_{\hat{m}-3} \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 & \zeta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where, $\zeta_i=(i+1)^{\alpha+1}-2i^{\alpha+1}+(i-1)^{\alpha+1},\ i=1,2,...,\hat{m}-1$. And therefore,

$$I^\alpha f(x)\simeq F^T P_\alpha \psi(x).$$

For instance, if $k=2$ and $M=2$, we get

$$P=\frac{1}{4}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 2 & 0 \\ -2 & -\frac{1}{4} & 2 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -2 & -\frac{1}{4} \end{bmatrix}_{4\times 4},$$

and for $\alpha=0.5$,

$$[P_\alpha]_{4\times 4}=\begin{bmatrix} 0.6877 & 0.1558 & 0.3669 & -0.0738 \\ -0.6232 & 0.0645 & -0.3281 & 0.0388 \\ 0 & 0 & 0.6877 & 0.1558 \\ 0 & 0 & -0.6232 & 0.0645 \end{bmatrix}_{4\times 4}.$$

Now, we derive the SOMI of Chebyshev wavelets of third kind as follows:
The stochastic integral of $\psi(x)$ can be obtained as follows:

$$\int_0^x \psi(t)dW(t) = P_s\psi(x), \tag{18}$$

where the matrix P_s (of order $\hat{m} \times \hat{m}$) is the SOMI of Chebyshev wavelets of third kind. For $M = 2$ and $k = 2$, we have

$$\begin{aligned}\int_0^x \psi_{1,0}(t)dW(t) &= \begin{cases} \frac{2}{\sqrt{\pi}}W(x), & 0 \le x < 1/2 \\ \frac{2}{\sqrt{\pi}}W\left(\frac{1}{2}\right), & 1/2 \le x < 1 \end{cases} \\ &\simeq W\left(\frac{1}{4}\right)\psi_{1,0}(x) + W\left(\frac{1}{2}\right)\psi_{2,0}(x),\end{aligned} \tag{19}$$

$$\begin{aligned}\int_0^x \psi_{1,1}(t)dW(t) &= \begin{cases} \frac{2}{\sqrt{\pi}}\left((8x-3)W(x) - \int_0^x W(t)dt\right), & 0 \le x < 1/2 \\ \frac{2}{\sqrt{\pi}}\left(W\left(\frac{1}{2}\right) - \int_0^{1/2} W(t)dt\right), & 1/2 \le x < 1 \end{cases} \\ &\simeq \left(-\int_0^{1/4} W(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{4}\right)\psi_{1,1}(x) \\ &+ \left(W\left(\frac{1}{2}\right) - \int_0^{1/2} W(t)dt\right)\psi_{2,0}(x),\end{aligned} \tag{20}$$

$$\begin{aligned}\int_0^x \psi_{2,0}(t)dW(t) &= \begin{cases} 0, & 0 \le x < 1/2 \\ \frac{2}{\sqrt{\pi}}\left(W(x) - W\left(\frac{1}{2}\right)\right), & 1/2 \le x < 1 \end{cases} \\ &\simeq \left(W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right)\right)\psi_{2,0}(x),\end{aligned} \tag{21}$$

$$\begin{aligned}\int_0^x \psi_{2,1}(t)dW(t) &= \begin{cases} 0, & 0 \le x < 1/2 \\ \frac{2}{\sqrt{\pi}}\left(W\left(\frac{1}{2}\right) - \int_0^{1/2} W(t)dt\right), & 1/2 \le x < 1 \end{cases} \\ &\simeq \left(-\int_0^{1/4} W(t)dt\right)\psi_{2,0}(x) + W\left(\frac{3}{4}\right)\psi_{2,1}(x).\end{aligned} \tag{22}$$

Using equations (19) to (22), we get

$$\int_0^x \psi(t)dW(t) = \begin{bmatrix} \int_0^x \psi_{1,0}(t)dW(t) \\ \int_0^x \psi_{1,1}(t)dW(t) \\ \int_0^x \psi_{2,0}(t)dW(t) \\ \int_0^x \psi_{2,1}(t)dW(t) \end{bmatrix}.$$

Therefore,

$$\int_0^x \psi(t)\ dW(t) = \underbrace{\begin{bmatrix} W\left(\frac{1}{4}\right) & 0 & W\left(\frac{1}{2}\right) & 0 \\ \left(-\int_0^{1/4} W(t)dt\right) & W\left(\frac{1}{4}\right) & \left(W\left(\frac{1}{2}\right) - \int_0^{1/2} W(t)dt\right) & 0 \\ 0 & 0 & \left(W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right)\right) & 0 \\ 0 & 0 & \left(-\int_0^{1/4} W(t)dt\right) & W\left(\frac{3}{4}\right) \end{bmatrix}}_{P_S}\psi(x).$$

We have derived the SOMI of Chebyshev wavelets of third kind $k = 2$ and $M = 2$ ($\hat{m} = 4$). In the same way for the different values of k and M we can derive the SOMI of Chebyshev wavelets of third kind.

3. Third kind Chebyshev wavelet stochastic operational matrix method

In this section, an efficient direct method to solve SFVIDE is provided using results in the previous section. We can rewrite equation (1) in an integral form using definitions of fractional differentiation and integral:

$$y(x) = f_0(x) + I^{\alpha}(f(x)) + I^{\alpha}\left(\int_0^x k_1(x,t)y(t)dt\right) + I^{\alpha}\left(\int_0^x k_2(x,t)y(t)dW(t)\right), \quad (23)$$

where, $f_0(x) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0^+)$. Approximating $f_0(x)$, $y(x)$, $f(x)$, and $k_i(x,t)$, $i = 1, 2$ with respect to Chebyshev wavelets of third kind as follows:

$$y(x) \simeq V^T \psi(x) = V \psi^T(x), \quad (24)$$

where V is given in equation (15) and is the unknown vector to be determined.

$$f_0(x) \simeq F_0^T \psi(x) = F_0 \psi^T(x), \quad (25)$$

$$f(x) \simeq F^T \psi(x) = F \psi^T(x), \quad (26)$$

$$k_1(x,t) \simeq \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x), \quad (27)$$

$$k_2(x,t) \simeq \psi^T(x) K_2 \psi(t) = \psi^T(t) K_2^T \psi(x), \quad (28)$$

where V, F_0 and F are third kind Chebyshev wavelet coefficient vectors and K_1, K_2 are third kind Chebyshev wavelet matrices. Using equations (27), (28) and remark given in [33], an integral part of (23) is approximated as,

$$\begin{aligned} I^{\alpha}\left(\int_0^x k_1(x,t)y(t)dt\right) &\simeq I^{\alpha}\left(\psi^T(x) K_1 \int_0^x \psi(t)\,\psi^T(t) V dt\right) \\ &= I^{\alpha}\left(\psi^T(x) K_1 \int_0^x \tilde{V}\psi(t)dt\right) \\ &\simeq I^{\alpha}\left(\psi^T(x) K_1 \tilde{V} P \psi(x)\right) \\ &= I^{\alpha}\left(B_1^T \psi(x)\right) \\ &= B_1^T P_{\alpha} \psi(x). \end{aligned} \quad (29)$$

Similarly, for Itô integral, we get

$$\begin{aligned} I^{\alpha}\left(\int_0^x k_2(x,t)y(t)dW(t)\right) &\simeq I^{\alpha}\left(\psi^T(x) K_2 \int_0^x \psi(t)\,\psi^T(t) V dW(t)\right) \\ &= I^{\alpha}\left(\psi^T(x) K_2 \int_0^x \tilde{V}\psi(t)dW(t)\right) \\ &\simeq I^{\alpha}\left(\psi^T(x) K_2 \tilde{V} P_S \psi(x)\right) \\ &= I^{\alpha}\left(B_2^T \psi(x)\right) \\ &= B_2^T P_{\alpha} \psi(x), \end{aligned} \quad (30)$$

where $\tilde{V}$is a $\hat{m}$-vector given in the remark [33] for the vector V defined in equation (15). B_1 and B_2 are $\hat{m}$-vectors containing diagonal elements of matrices $K_1\tilde{V}P$ and $K_2\tilde{V}P_S$ respectively. Substituting equations (24), (25), (26), (29), and (30), we get

$$V^T\psi(x) = F_0^T\psi(x) + F^T P_\alpha\psi(x) + B_1^T P_\alpha\psi(x) + B_2^T P_\alpha\psi(x), \tag{31}$$

that is

$$V - P_\alpha^T(B_1 + B_2) = \bar{F}, \tag{32}$$

where $\bar{F} = P_\alpha^T F + F_0$. Equation (32) is a linear system of equations. V is the unknown vector obtained by solving the linear system of equations (32). The solution of SFVIDE (23) is obtained by substituting the vector V in equation (24).

4. Convergence and Error analysis

Theorem 4.1. *Let $y(x)$ and $y^*(x)$ be the exact and approximate solutions of (1)-(12), respectively. Let us assume that*

(1) $\| y(x) \| < \infty$,

(2) $\| k_i \| \leq \kappa_i$, $\kappa_i \in \mathbb{R}$, $\| k_1 \|^2 + \| k_2 \|^2 \neq \frac{\Gamma^2(\alpha)}{3}$,

then, $\| y(x) - y^*(x) \| \to 0$, *where*

$$\| y(x) \|^2 = E\left[| y |^2\right].$$

Proof. See [37]. □

5. Computational experiments

Test problem 5.1. *We consider the SFVIDE* [37]

$$D^\alpha y(x) = \frac{\Gamma(2)x^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{x^3}{3} + \int_0^x ty(t)dt + \int_0^x y(t)dW(t)\,, \tag{33}$$

satisfying the initial condition $y(0) = 0$. SFVIDE (5.1) does not have an exact solution. To obtain the numerical solution of this SFVIDE, the third kind Chebyshev wavelet method described in section 3 is applied. Table 1 shows the approximate solution obtained by the third kind Chebyshev wavelet method for various values of α for $\hat{m} = 8$ and figure 1 shows the approximate solution obtained by the third kind Chebyshev wavelet method for various values of α for $\hat{m} = 8$ of test problem 5.1.

6. Conclusion

In the formulation of natural processes, for instance, population growth, pollution, financial markets, and mechanical structures, SFVIE arise in which the past affect both the present and the future. Therefore, considering that several phenomena in the natural world suffer from disturbances or random noise, switching from the probabilistic models to stochastic models is common and safe for them. There are usually no exact solutions of these models. And so in this article, we opt for approximate solution of these equations using Chebyshev wavelets of third kind. A new SOMI of Chebyshev wavelets of third kind is obtained. With the help of existing fractional OMI Chebyshev wavelets of third kind and the obtained SOMI of Chebyshev wavelets of third kind, we obtain the solution of SFVIDE. The computational experiments show the method presented is efficient and accurate.

TABLE 1. Approximate solution obtained by the method described for different values of x, α for $\hat{m} = 8$ of test problem 5.1.

x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.0625	0.5999	0.5885	0.5877
0.1875	0.7261	0.7115	0.7106
0.3125	0.8147	0.8049	0.8043
0.4375	0.8944	0.8847	0.8750
0.5625	0.9764	0.9661	0.9569
0.6875	1.0864	1.0767	1.0662
0.8125	1.1997	1.1805	1.1752
0.9375	1.2993	1.2809	1.2713

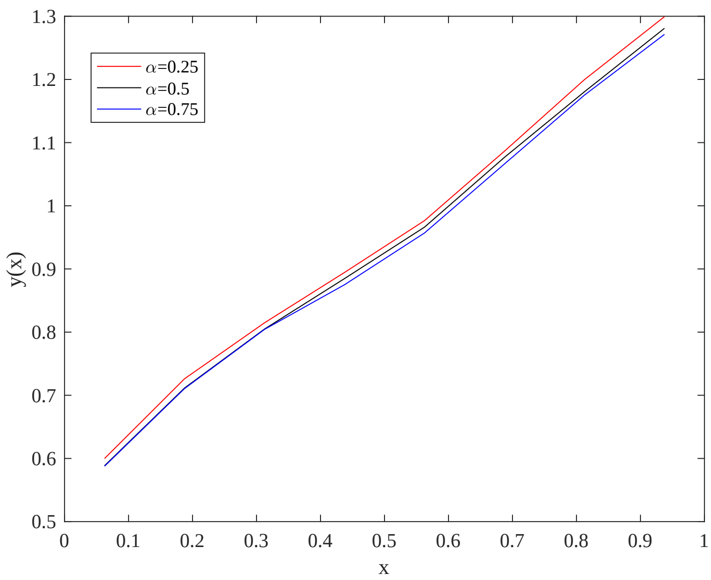

FIGURE 1. An approximate solution obtained by the third kind Chebyshev wavelet method for certain values of α and $\hat{m} = 8$ of test problem 5.1.

Test problem 5.2. *Let us consider the SFVIDE* [37]

$$D^{\alpha}y(x) = \frac{7}{12}x^4 - \frac{5}{6}x^3 + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^x (x+t)y(t)dt + \int_0^x ty(t)dW(t)\,, \quad (34)$$

with the initial condition $y(0) = 0$. *This SFVIDE does not have an exact solution. To obtain the numerical solution of this SFVIDE, the third kind Chebyshev wavelet method described in section 3 is applied. Table 2 shows the approximate solution obtained by the third kind Chebyshev wavelet method for different values of* α *for* $\hat{m} = 8$ *and figure 2 shows the approximate solution obtained by the third kind Chebyshev wavelet method for different values of* α *for* $\hat{m} = 8$ *of test problem 5.2.*

Table 2. Approximate solution obtained by the method described for different values of x, α for $\hat{m} = 8$ of test problem 5.2.

x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.0625	0.0400	0.0324	0.0256
0.1875	0.1600	0.1444	0.1296
0.3125	0.3600	0.3364	0.3136
0.4375	0.6400	0.6084	0.5776
0.5625	1.0000	0.9604	0.9216
0.6875	1.4400	1.3924	1.3456
0.8125	1.9600	1.9044	1.8496
0.9375	2.5600	2.4964	2.4336

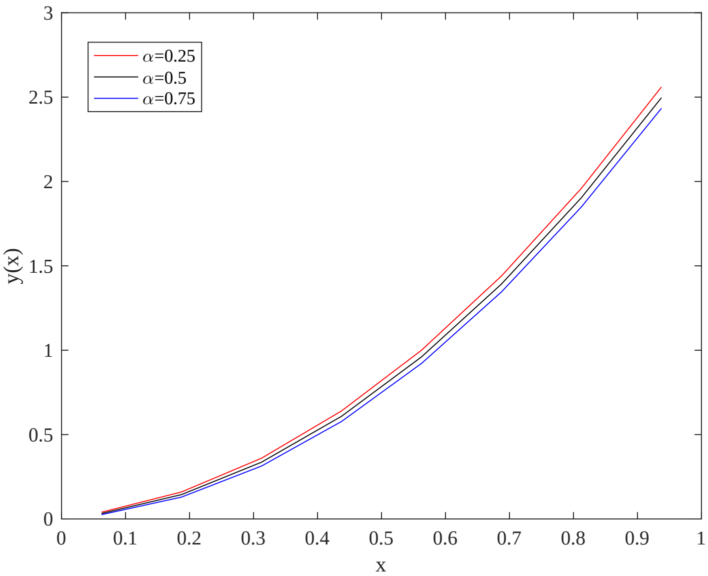

Figure 2. An approximate solution obtained by the third kind Chebyshev wavelet method for certain values of α and $\hat{m} = 8$ of test problem 5.2.

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