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IDENTIFYING CYCLIC AND $(1+2v)$-CONSTACYCLIC CODES OVER $\mathbb{Z}_4[v]/\langle v^3-1\rangle$ WITH $\mathbb{Z}_4$-LINEAR CODES

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ABSTRACT. This paper studies cyclic and $(1+2v)$-constacyclic codes over the ring $\mathbb{Z}_4[v]/\langle v^3-1\rangle$. By introducing three different Gray maps, we show that the Gray images of cyclic codes are quasi-cyclic codes over $\mathbb{Z}_4$ and that of $(1+2v)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$. Moreover, we show that the Gray image of skew $(1+2v)$-constacyclic code is a quasi-cyclic code over $\mathbb{Z}_4$.

Keywords: Cyclic code, Gray map, constacyclic code, quasi-cyclic code, skew constacyclic code.

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1. Introduction

The study of linear codes over finite rings has been initiated since the early 1970s. The discovery of specific good binary non-linear codes from cyclic codes over $\mathbb{Z}_4$ via the Gray map in [7] has a paradigm shift in the studies of codes towards finite rings. Since then, the finite ring $\mathbb{Z}_4$ and its extension rings occupy a special place in coding theory. Cyclic codes are the important class of linear codes over finite rings and have been studied extensively by many researchers on various rings [1, 6, 14].

In recent years various researchers have done extensive research on cyclic codes and their generalizations such as skew-cyclic codes, constacyclic codes, skew-constacyclic codes and other codes over different finite rings [2, 3, 4, 8, 10, 12, 13, 17]. The rings of order 16 that are extensions of $\mathbb{Z}_4$ have been immensely studied by many researchers after the introduction of linear and cyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4, u^2=0$ in [22]. For instance, Özen et al. [15] studied cyclic and constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2-1\rangle$ and determined the form of the generators of cyclic codes and spanning sets. They also proved that the $\mathbb{Z}_4$-image of a $(2+u)$-constacyclic code of odd length is a cyclic code over $\mathbb{Z}_4$ and provided examples

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with better parameters than previously known $\mathbb{Z}_4$-linear codes. In [20], Shi et al. studied $(1+2u)$-constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2-1\rangle$ and showed that the Gray images of $(1+2u)$-constacyclic codes of length n over the ring are cyclic codes of length $2n$ over $\mathbb{Z}_4$.

In [9], Islam and Prakash considered the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$, where $u^2=v^2=uv=vu=0$ of order 64 and determined the generator polynomials and minimal spanning set for cyclic codes over the ring. Further, the authors proved that the Gray images of $(1+2u)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$. Later on, Dertli and Cengellenmis [5] introduced the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$, $u^2=u, v^2=v, uv=vu=0$ and studied the Gray images of cyclic, constacyclic, quasi-cyclic and their skew codes over the ring. Moreover, they determined cyclic DNA and skew cyclic DNA codes over the ring. In [16], Özen et al. studied cyclic and constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^3-u^2\rangle$ and their Gray images. They developed the structure of generator polynomial of cyclic and constacyclic codes with odd length over this ring and constructed several new and optimal codes in terms of the Lee, Euclidean and Hamming weight in reference to the database. More recently, Islam and Prakash [11] discussed $(1+2u+2v+2uv)$-constacyclic and skew $(1+2u+2v+2uv)$-constacyclic codes over the non-chain ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4+uv\mathbb{Z}_4$, where $u^2=u, v^2=v, uv=vu$ and obtained that the Gray images of $(1+2u+2v+2uv)$-constacyclic and skew $(1+2u+2v+2uv)$-constacyclic codes over the ring are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$. Moreover, they proved that $(1+2u+2v+2uv)$-constacyclic codes of odd length n are principally generated and computed several new linear codes over $\mathbb{Z}_4$.

In this paper, we consider the finite commutative ring $R=\mathbb{Z}_4[v]/\langle v^3-1\rangle$ and introduce three distinct Gray maps and study their images of cyclic and $(1+2v)$-constacyclic codes over $\mathbb{Z}_4$. This article intends to establish relations among the known linear codes such as cyclic codes, skew-cyclic codes, quasi-cyclic codes, constacyclic codes, skew-constacyclic codes or permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$ via the newly introduced Gray maps obtained as $\mathbb{Z}_4$-images of cyclic and $(1+2v)$-constacyclic codes over the ring R. The paper is organized as follows. In Section 2, we study the characteristic properties of the ring R and introduce three distinct maps called Gray maps on R^n. In Section 3, we investigate the properties of cyclic codes with the help of newly introduced Gray maps. $(1+2v)$-constacyclic codes over R and their Gray images are discussed in Section 4. Moreover, $(1+2v)$-constacyclic codes of odd length n over the ring R with Nechaev's permutation and other permutation are considered. In Section 5, we study skew-constacyclic codes and their $\mathbb{Z}_4$-images, and Section 6 concludes the paper.

2. Preliminaries

Let R denotes the commutative ring $\mathbb{Z}_4[v]/\langle v^3-1\rangle$ which has characteristic 4 and 64 elements. This ring can be considered as $\mathbb{Z}_4+v\mathbb{Z}_4+v^2\mathbb{Z}_4$ with $v^3=1$. Any element r of R can be written as $r=x+vy+v^2z$, where $x,y,z\in\mathbb{Z}_4$. There are 24 units in R - $1,3,v,3v,v^2,3v^2,1+2v,1+2v^2,2+v,2+3v,2+v^2,2+3v^2,3+2v,3+2v^2,v+2v^2,2v+v^2,2v+3v^2,3v+2v^2,1+2v+2v^2,2+v+2v^2,2+2v+v^2,2+2v+3v^2,2+3v+2v^2,3+2v+2v^2$. The set of units $V=\{1,3,1+2v,1+2v^2,3+2v,3+2v^2,1+2v+2v^2,3+2v+2v^2\}$ in R satisfies $\lambda^2=1$ for all $\lambda\in V$. The unit $(1+2v)$ of the ring R is used in the study of this manuscript. The ring R has nine ideals along with the chain conditions which can be described as follows

$\langle 0\rangle \subset \langle 2+2v+2v^2\rangle \subset \langle 1+v+v^2\rangle \subset \langle 1+v+3v^2\rangle \subset R$,

$\langle 0\rangle \subset \langle 2+2v\rangle \subset \langle 2\rangle \subset \langle 1+v\rangle \subset R$,

$\langle 0\rangle \subset \langle 2+2v+2v^2\rangle \subset \langle 2\rangle \subset \langle 1+v\rangle \subset R$, and

$\langle 0\rangle \subset \langle 2+2v\rangle \subset \langle 1+3v\rangle \subset \langle 1+v\rangle \subset R$.

Clearly, R is a finite non-chain ring. Also, R is a principal ideal ring with two maximal ideals $\langle 1+v\rangle$ and $\langle 1+v+3v^2\rangle$.

A linear code C of length n over R is an R-submodule of R^n and elements of the code are called codewords. A linear code C of length n over R is said to be a cyclic code if it is invariant under the cyclic shift operator σ, *i.e.*, $\sigma(C) = C$, where $\sigma(c_0, c_1, ..., c_{n-1}) = (c_{n-1}, c_0, c_1, ..., c_{n-2})$ for all $(c_0, c_1, ..., c_{n-1}) \in C$. Let λ be a unit in R. A linear code C of length n over R is said to be a λ-constacyclic code if it is invariant under the constacyclic shift operator τ_λ, *i.e.*, $\tau_\lambda(C) = C$, where $\tau_\lambda(c_0, c_1, ..., c_{n-1}) = (\lambda c_{n-1}, c_0, c_1, ..., c_{n-2})$ for all $(c_0, c_1, ..., c_{n-1}) \in C$. Moreover, a λ-constacyclic code of length n over R can be identified as an ideal of the quotient ring $R[\alpha]/\langle \alpha^n - \lambda\rangle$ by the correspondence

$$c = (c_0, c_1, ..., c_{n-1}) \to c(\alpha) = c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1}(mod\langle \alpha^n - \lambda\rangle).$$

Definition 2.1. *(Islam and Prakash* [9]*). Let σ be the cyclic shift operator and $n = mk$. Then, the quasi-cyclic shift operator $\rho_k : \mathbb{Z}_4^n \to \mathbb{Z}_4^n$ is defined by*

$$\rho_k(c^1|c^2|...|c^k) = (\sigma(c^1)|\sigma(c^2)|...|\sigma(c^k)|),$$

where $c^i \in \mathbb{Z}_4^m$ for $i = 1, 2, ..., k$. A linear code C of length n over $\mathbb{Z}_4$ is said to be a quasi-cyclic code of index k if and only if $\rho_k(C) = C$.

We introduce three distinct Gray maps on the ring R as follows. Firstly, we take a Gray map ϕ_1 from R to $\mathbb{Z}_4^2$ as

$$\phi_1 : R \to \mathbb{Z}_4^2$$

defined by

$$\phi_1(x + vy + v^2z) = (x + 2y, x + 2y + 2z) \quad \forall\ x, y, z \in \mathbb{Z}_4.$$

Clearly, ϕ_1 is a $\mathbb{Z}_4$-linear map but not bijective. This map can be extended to R^n component-wise as follows:

$$\phi_1 : R^n \to \mathbb{Z}_4^{2n}$$

$$\begin{aligned}\phi_1(r_0, r_1, ..., r_{n-1}) = (&x_0 + 2y_0, x_1 + 2y_1, ..., x_{n-1} + 2y_{n-1}, x_0 + 2y_0 + 2z_0,\\ &x_1 + 2y_1 + 2z_1, ..., x_{n-1} + 2y_{n-1} + 2z_{n-1}),\end{aligned} \tag{1}$$

where $r_j = x_j + vy_j + v^2z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$.

The second Gray map $\phi_2 : R^n \to \mathbb{Z}_4^{2n}$ is defined by

$$\phi_2(r_0, r_1, ..., r_{n-1}) = (2x_0, 2x_1, ..., 2x_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ..., 2y_{n-1} + 2z_{n-1}), \tag{2}$$

where $r_j = x_j + vy_j + v^2z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$.

Further, we consider another Gray map on R^n as

$$\phi_3 : R^n \to \mathbb{Z}_4^{3n}$$

defined by

$$\phi_3(r_0, r_1, ..., r_{n-1}) = (y_0, y_1, ..., y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-1} + y_{n-1}, \\ 2z_0, 2z_1, ..., 2z_{n-1}), \quad (3)$$

where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$.

We recall that the Lee weight $w_L(x)$ of any $x \in \mathbb{Z}_4$ is $\min\{|x|, |4-x|\}$. Thus, the Lee weights of 0,1,2,3 are, respectively, 0,1,2,1. The Lee weight of a vector $x' \in \mathbb{Z}_4^n$ is defined as the rational sum of the Lee weight of its coordinates. The Lee weight for $r = (r_0, r_1, ..., r_{n-1}) \in R^n$ is given by $w_L(r) = \sum_{j=0}^{n-1} w_L(r_j)$, where $w_L(r) = w_L(\phi_1(r))$ is the Lee weight for any $r \in R$ and the Lee distance for the code C is defined by $d(C) = \min\{d_L(r, r') \mid r \neq r', r, r' \in C\}$, where $d_L(r, r') = w_L(r - r')$. Now, $d_L(r, r') = w_L(r - r') = w_L(\phi_1(r - r')) = w_L(\phi_1(r) - \phi_1(r')) = d_L(\phi_1(r), \phi_1(r')), \forall r, r' \in R^n$. Hence, ϕ_1 is a distance preserving map from R^n(Lee distance) to $\mathbb{Z}_4^{2n}$(Lee distance). Similarly, ϕ_2 and ϕ_3 are also distance preserving maps from R^n to $\mathbb{Z}_4^{2n}$ and $\mathbb{Z}_4^{3n}$, respectively.

3. Cyclic codes over R

In the present section, we discuss the algebraic properties of cyclic codes over the ring R and their $\mathbb{Z}_4$-Gray images. It is obtained that the first and second Gray images of cyclic codes of length n over R are quasi-cyclic codes of length $2n$ over $\mathbb{Z}_4$ and the third Gray image is a quasi-cyclic code of length $3n$ over $\mathbb{Z}_4$.

Proposition 3.1. *For any $r \in R^n$, we have $\phi_1(\sigma(r)) = \rho_2(\phi_1(r))$, where ϕ_1 is the Gray map defined in equation (1), σ is the cyclic shift operator and ρ_2 is the quasi-cyclic shift operator on R^n given in the preliminaries.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Therefore, we have

$$\begin{aligned}\phi_1(\sigma(r)) &= \phi_1(r_{n-1}, r_0, r_1, ..., r_{n-2}) \\ &= (x_{n-1} + 2y_{n-1}, x_0 + 2y_0, x_1 + 2y_1, ..., x_{n-2} + 2y_{n-2}, x_{n-1} + 2y_{n-1} + 2z_{n-1}, \\ &\quad x_0 + 2y_0 + 2z_0, x_1 + 2y_1 + 2z_1, ..., x_{n-2} + 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

And,

$$\begin{aligned}\rho_2(\phi_1(r)) &= \rho_2(x_0 + 2y_0, x_1 + 2y_1, ..., x_{n-1} + 2y_{n-1}, x_0 + 2y_0 + 2z_0, x_1 + 2y_1 + 2z_1, ..., \\ &\quad x_{n-2} + 2y_{n-2} + 2z_{n-2}, x_{n-1} + 2y_{n-1} + 2z_{n-1}) \\ &= (x_{n-1} + 2y_{n-1}, x_0 + 2y_0, x_1 + 2y_1, ..., x_{n-2} + 2y_{n-2}, x_{n-1} + 2y_{n-1} + 2z_{n-1}, \\ &\quad x_0 + 2y_0 + 2z_0, x_1 + 2y_1 + 2z_1, ..., x_{n-2} + 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

Hence, $\phi_1(\sigma(r)) = \rho_2(\phi_1(r))$. □

Theorem 3.1. *The Gray image $\phi_1(C)$ of a cyclic code C of length n over R is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code of length n over R, $\sigma(C) = C$. Applying ϕ_1 on both sides and using Proposition 3.1, we have $\rho_2(\phi_1(C)) = \phi_1(C)$. This implies that $\phi_1(C)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$. □

Proposition 3.2. *For any $r \in R^n$, we have $\phi_2(\sigma(r)) = \rho_2(\phi_2(r))$, where ϕ_2 is the Gray map defined in equation (2), σ is the cyclic shift operator and ρ_2 is the quasi-cyclic shift operator on R^n given in the preliminaries.*

Proof. Similar to the proof of Proposition 3.1. □

Theorem 3.2. *The Gray image $\phi_2(C)$ of a cyclic code C of length n over R is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Similar to the proof of Theorem 3.1. □

Proposition 3.3. *For any $r \in R^n$, we have $\phi_3(\sigma(r)) = \rho_3(\phi_3(r))$, where ϕ_3 is the Gray map defined in equation (3), σ is the cyclic shift operator and ρ_3 is the quasi-cyclic shift operator on R^n given in the preliminaries.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Therefore, we have

$$\begin{aligned}\phi_3(\sigma(r)) &= \phi_3(r_{n-1}, r_0, r_1, ..., r_{n-2})\\ &= (y_{n-1}, y_0, y_1, ..., y_{n-2}, 2x_{n-1} + y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2},\\ &\quad 2z_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}).\end{aligned}$$

And,

$$\begin{aligned}\rho_3(\phi_3(r)) &= \rho_3(y_0, y_1, ..., y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-1} + y_{n-1}, 2z_0, 2z_1, ..., 2z_{n-1})\\ &= (y_{n-1}, y_0, y_1, ..., y_{n-2}, 2x_{n-1} + y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2},\\ &\quad 2z_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}).\end{aligned}$$

Hence, $\phi_3(\sigma(r)) = \rho_3(\phi_3(r))$. □

Theorem 3.3. *The Gray image $\phi_3(C)$ of a cyclic code C of length n over R is a quasi-cyclic code of length $3n$ with index 3 over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code of length n over R, $\sigma(C) = C$. On applying ϕ_3 and using Proposition 3.3, we get $\rho_3(\phi_3(C)) = \phi_3(C)$. This shows that $\phi_3(C)$ is a quasi-cyclic code of length $3n$ with index 3 over $\mathbb{Z}_4$. □

Considering Φ_1 as the permutation version of the above Gray map ϕ_1, we define Φ_1 as follows

$$\begin{aligned}\Phi_1(r_0, r_1, ..., r_{n-1}) &= (\phi_1(r_0), \phi_1(r_1), ..., \phi_1(r_{n-2}), \phi_1(r_{n-1}))\\ &= (x_0 + 2y_0, x_0 + 2y_0 + 2z_0, x_1 + 2y_1, x_1 + 2y_1 + 2z_1, ..., x_{n-2} + 2y_{n-2},\\ &\quad x_{n-2} + 2y_{n-2} + 2z_{n-2}, x_{n-1} + 2y_{n-1}, x_{n-1} + 2y_{n-1} + 2z_{n-1}), \qquad (4)\end{aligned}$$

where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$.

Proposition 3.4. *For any $r \in R^n$, we have $\Phi_1(\sigma(r)) = \sigma^2(\Phi_1(r))$, where Φ_1 is the map defined in equation (4) and σ is the cyclic shift operator as given in the preliminaries.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Then

$$\begin{aligned}\Phi_1(\sigma(r)) &= \Phi_1(r_{n-1}, r_0, ..., r_{n-2})\\ &= (x_{n-1} + 2y_{n-1}, x_{n-1} + 2y_{n-1} + 2z_{n-1}, x_0 + 2y_0, x_0 + 2y_0 + 2z_0, x_1 + 2y_1,\\ &\quad x_1 + 2y_1 + 2z_1, ..., x_{n-2} + 2y_{n-2}, x_{n-2} + 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

And,

$$\begin{aligned}\sigma^2(\Phi_1(r)) &= \sigma^2(x_0+2y_0, x_0+2y_0+2z_0, x_1+2y_1, x_1+2y_1+2z_1, ..., x_{n-2}+2y_{n-2},\\ &\quad x_{n-2}+2y_{n-2}+2z_{n-2}, x_{n-1}+2y_{n-1}, x_{n-1}+2y_{n-1}+2z_{n-1})\\ &= (x_{n-1}+2y_{n-1}, x_{n-1}+2y_{n-1}+2z_{n-1}, x_0+2y_0, x_0+2y_0+2z_0, x_1+2y_1,\\ &\quad x_1+2y_1+2z_1, ..., x_{n-2}+2y_{n-2}, x_{n-2}+2y_{n-2}+2z_{n-2}).\end{aligned}$$

Hence, $\Phi_1(\sigma(r)) = \sigma^2(\Phi_1(r))$. □

Theorem 3.4. *Let C be a cyclic code of length n over R. Then $\Phi_1(C)$ is equivalent to a 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code of length n over R, $\sigma(C) = C$. Applying Φ_1 on both sides and using Proposition 3.4, we have $\sigma^2(\Phi_1(C)) = \Phi_1(C)$. This shows that $\Phi_1(C)$ is equivalent to a 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$. □

Remark 3.1. *Taking Φ_2 and Φ_3 as the permutation versions of the other Gray maps ϕ_2 and ϕ_3, respectively, we can obtain analogous results of Proposition 3.4 and Theorem 3.4.*

4. Constacyclic codes over R

In this section, we investigate the relationships between the Gray images of $(1+2v)$-constacyclic codes over R and some well-known linear codes over $\mathbb{Z}_4$. It is obtained that the Gray images of $(1+2v)$-constacyclic codes over R are cyclic, quasi-cyclic and permutation equivalent to a quasi-cyclic codes over $\mathbb{Z}_4$. Moreover, we discuss $(1+2v)$-constacyclic codes of odd length n over R with Nechaev's permutation and other permutation.

Proposition 4.1. *For any $r \in R^n$, we have $\phi_1(\tau_{(1+2v)}(r)) = \sigma(\phi_1(r))$, where ϕ_1 is the Gray map given in equation (1), $\tau_{(1+2v)}$ is the $(1+2v)$-constacyclic shift operator and σ is the cyclic shift operator on R^n given in the preliminaries.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Clearly, $(1+2v)(x_{n-1}+vy_{n-1}+v^2z_{n-1}) = (x_{n-1}+2z_{n-1}) + v(2x_{n-1}+y_{n-1}) + v^2(2y_{n-1}+z_{n-1})$ and $\phi_1\left((1+2v)(x_{n-1}+vy_{n-1}+v^2z_{n-1})\right) = (x_{n-1}+2y_{n-1}+2z_{n-1}, x_{n-1}+2y_{n-1})$. Therefore, we have

$$\begin{aligned}\phi_1(\tau_{(1+2v)}(r)) &= \phi_1((1+2v)r_{n-1}, r_0, ..., r_{n-2})\\ &= (x_{n-1}+2y_{n-1}+2z_{n-1}, x_0+2y_0, x_1+2y_1, ..., x_{n-2}+2y_{n-2}, x_{n-1}+2y_{n-1},\\ &\quad x_0+2y_0+2z_0, x_1+2y_1+2z_1, ..., x_{n-2}+2y_{n-2}+2z_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma(\phi_1(r)) &= \sigma(x_0+2y_0, x_1+2y_1, ..., x_{n-2}+2y_{n-2}, x_{n-1}+2y_{n-1}, x_0+2y_0+2z_0,\\ &\quad x_1+2y_1+2z_1, ..., x_{n-2}+2y_{n-2}+2z_{n-2}, x_{n-1}+2y_{n-1}+2z_{n-1})\\ &= (x_{n-1}+2y_{n-1}+2z_{n-1}, x_0+2y_0, x_1+2y_1, ..., x_{n-2}+2y_{n-2}, x_{n-1}+2y_{n-1},\\ &\quad x_0+2y_0+2z_0, x_1+2y_1+2z_1, ..., x_{n-2}+2y_{n-2}+2z_{n-2}).\end{aligned}$$

Hence, $\phi_1(\tau_{(1+2v)}(r)) = \sigma(\phi_1(r))$. □

Theorem 4.1. *The Gray image $\phi_1(C)$ of a $(1+2v)$-constacyclic code C of length n over R is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a $(1+2v)$-constacyclic code of length n over R, $\tau_{(1+2v)}(C) = C$. Applying ϕ_1 on both sides and using Proposition 4.1, we have $\sigma(\phi_1(C)) = \phi_1(C)$. This implies that $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Proposition 4.2. *For any $r \in R^n$, we have $\phi_2(\tau_{(1+2v)}(r)) = \rho_2(\phi_2(r))$, where ϕ_2 is the Gray map defined in equation (2), $\tau_{(1+2v)}$ is the $(1+2v)$-constacyclic shift operator and ρ_2 is the quasi-cyclic shift operator on R^n given in the preliminaries.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Clearly, $(1+2v)(x_{n-1} + vy_{n-1} + v^2 z_{n-1}) = (x_{n-1} + 2z_{n-1}) + v(2x_{n-1} + y_{n-1}) + v^2(2y_{n-1} + z_{n-1})$ and $\phi_2\left((1+2v)(x_{n-1} + vy_{n-1} + v^2 z_{n-1})\right) = (2x_{n-1}, 2y_{n-1} + 2z_{n-1})$. Therefore, we have

$$\begin{aligned}\phi_2(\tau_{(1+2v)}(r)) &= \phi_2((1+2v)r_{n-1}, r_0, r_1, ..., r_{n-2})\\ &= (2x_{n-1}, 2x_0, 2x_1, ..., 2x_{n-2}, 2y_{n-1} + 2z_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ...,\\ &\quad 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_2(\phi_2(r)) &= \rho_2(2x_0, 2x_1, ..., 2x_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ..., 2y_{n-1} + 2z_{n-1})\\ &= (2x_{n-1}, 2x_0, 2x_1, ..., 2x_{n-2}, 2y_{n-1} + 2z_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ...,\\ &\quad 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

Hence, $\phi_2(\tau_{(1+2v)}(r)) = \rho_2(\phi_2(r))$. □

Theorem 4.2. *The Gray image $\phi_2(C)$ of a $(1+2v)$-constacyclic code C of length n over R is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a (1+2v)-constacyclic code of length n over R, $\tau_{(1+2v)}(C) = C$. Applying ϕ_2 on both sides and using Proposition 4.2, we have $\rho_2(\phi_2(C)) = \phi_2(C)$. This shows that $\phi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

Proposition 4.3. *For any $r \in R^n$, we have $\phi_3(\tau_{(1+2v)}(r)) = \omega(\rho_3(\phi_3(r)))$, where ϕ_3 is the Gray map defined in equation (3), $\tau_{(1+2v)}$ is the (1+2v)-constacyclic shift operator and ρ_3 is the quasi-cyclic shift operator on R^n given in the preliminaries and ω is the permutation of $\mathbb{Z}_4^{3n}$ defined by $\omega(a_1, a_2, ..., a_{3n}) = (a_{\mu(1)}, a_{\mu(2)}, ..., a_{\mu(3n)})$ with the permutation $\mu = (1, n+1)$ of $\{1, 2, 3, ..., 3n\}$.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Then

$$\begin{aligned}\phi_3(\tau_{(1+2v)}(r) &= \phi_3((1+2v)r_{n-1}, r_0, r_1, ..., r_{n-2})\\ &= (2x_{n-1} + y_{n-1}, y_0, y_1, ..., y_{n-2}, y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2},\\ &\quad 2z_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}).\end{aligned}$$

And, we have

$$\begin{aligned}\rho_3(\phi_3(r)) &= \rho_3(y_0, y_1, ..., y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-1} + y_{n-1}, 2z_0, 2z_1, ..., 2z_{n-1})\\ &= (y_{n-1}, y_0, y_1, ..., y_{n-2}, 2x_{n-1} + y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2},\\ &\quad 2z_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}).\end{aligned}$$

On applying the permutation ω, we get

$$\begin{aligned}\omega(\rho_3(\phi_3(r))) &= (2x_{n-1} + y_{n-1}, y_0, y_1, ..., y_{n-2}, y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2},\\ &\quad 2z_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}).\end{aligned}$$

Hence, $\phi_3(\tau_{(1+2v)}(r)) = \omega(\rho_3(\phi_3(r)))$.

□

Theorem 4.3. *The Gray image $\phi_3(C)$ of a $(1+2v)$-constacyclic code C of length n over R is permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. Since C is a $(1+2v)$-constacyclic code of length n over R, $\tau_{(1+2v)}(C)=C$. Applying ϕ_3 on both sides and by Proposition 4.3, we have $\omega(\rho_3(\phi_3(C)))=\phi_3(C)$. This implies that $\phi_3(C)$ is permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$. □

Let C be a $(1+2v)$-constacyclic codes of odd length n over R. Obviously, $(1+2v)^n=1$ if n is an even integer and $(1+2v)^n=(1+2v)$ if n is an odd integer. Based on the results established in [2, 3, 9, 10, 15, 20], analogous results are given below without proofs.

Theorem 4.4. *A mapping $\beta: R[\alpha]/\langle\alpha^n-1\rangle \longrightarrow R[\alpha]/\langle\alpha^n-\lambda\rangle$ defined by $\beta(a(\alpha))=a(\lambda\alpha)$ is a ring isomorphism, if n is an odd integer.*

Corollary 4.1. *For any odd integer n, I is an ideal of $R[\alpha]/\langle\alpha^n-1\rangle$ if and only if $\beta(I)$ is an ideal of $R[\alpha]/\langle\alpha^n-\lambda\rangle$.*

Corollary 4.2. *Let C be a subset of R^n and $\overline{\beta}$ be a permutation of R^n, defined by $\overline{\beta}(c_0,c_1,...,c_{n-1})=(c_0,\lambda c_1,...,\lambda^{n-1}c_{n-1})$. Then C is a cyclic code of odd length n over R if and only if $\overline{\beta}(C)$ is a λ-constacyclic code over R.*

Definition 4.1 (Qian et al. [18]). *Let n be an odd integer and $\zeta=(1,n+1)(3,n+3)...(2i+1,n+2i+1)...(n-2,2n-2)$ be a permutation of the set $\{0,1,2,...,2n-1\}$. Then the Nechaev's permutation π is permutation of $\mathbb{Z}_4^{2n}$ defined by*

$$\pi(r_0,r_1,...,r_{2n-1})=(r_{\zeta(0)},r_{\zeta(1)},...,r_{\zeta(2n-1)}).$$

Theorem 4.5. *For any $r\in R^n$, we have $\phi_1(\overline{\beta}(r))=\pi(\phi_1(r))$, where ϕ_1 is the Gray map defined in equation (1), $\overline{\beta}$ is the permutation of R^n with $\lambda=(1+2v)$ and π is the Nechaev's permutation as given before.*

Proof. Let $r=(r_0,r_1,...,r_{n-1})\in R^n$, where $r_j=x_j+vy_j+v^2z_j\in R$ and $x_j,y_j,z_j\in\mathbb{Z}_4$ for $j=0,1,...,n-1$. Clearly, $(1+2v)(x_j+uy_j+u^2z_j)=(x_j+2z_j)+v(2x_j+y_j)+v^2(2y_j+z_j)$ and $\phi_1\left((1+2v)(x_j+vy_j+v^2z_j)\right)=(x_j+2y_j+2z_j,x_j+2y_j)$. Therefore, we have

$$\begin{aligned}\phi_1(\overline{\beta}(r)) &= \phi_1(r_0,(1+2v)r_1,...,(1+2v)^{n-2}r_{n-2},(1+2v)^{n-1}r_{n-1})\\ &= (x_0+2y_0,x_1+2y_1+2z_1,...,x_{n-2}+2y_{n-2}+2z_{n-2},x_{n-1}+2y_{n-1},\\ &\quad x_0+2y_0+2z_0,x_1+2y_1,...,x_{n-2}+2y_{n-2},x_{n-1}+2y_{n-1}+2z_{n-1}).\end{aligned}$$

And,

$$\begin{aligned}\pi(\phi_1(r)) &= \pi(x_0+2y_0,x_1+2y_1,...,x_{n-2}+2y_{n-2},x_{n-1}+2y_{n-1},x_0+2y_0+2z_0,\\ &\quad x_1+2y_1+2z_1,...,x_{n-2}+2y_{n-2}+2z_{n-2},x_{n-1}+2y_{n-1}+2z_{n-1})\\ &= (x_0+2y_0,x_1+2y_1+2z_1,...,x_{n-2}+2y_{n-2}+2z_{n-2},x_{n-1}+2y_{n-1},\\ &\quad x_0+2y_0+2z_0,x_1+2y_1,...,x_{n-2}+2y_{n-2},x_{n-1}+2y_{n-1}+2z_{n-1}).\end{aligned}$$

Hence, $\phi_1(\overline{\beta}(r))=\pi(\phi_1(r))$. □

Corollary 4.3. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\phi_1(C)=\widetilde{C}$), then $\pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. By Corollary 4.2, $\overline{\beta}(C)$ is a $(1+2v)$-constacyclic code over R as C is a cyclic code. From Theorem 4.1, we see that $\phi_1(\overline{\beta}(C))$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. Thus, by Theorem 4.5, $\pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Theorem 4.6. *For any $r \in R^n$, we have $\phi_3(\overline{\beta}(r)) = \gamma(\phi_3(r))$, where ϕ_3 is the Gray map given in equation (3), $\overline{\beta}$ is the permutation of R^n given in Corollary 4.3 with $\lambda = (1+2v)$ and γ is the permutation of $\mathbb{Z}_4^{3n}$ defined by $\gamma(c_1, c_2, ..., c_{3n}) = (c_{\nu(1)}, c_{\nu(2)}, ..., c_{\nu(3n)})$ with the permutation $\nu = (2, n+2)(4, n+4)...(n-1, 2n-1)$ of $\{1, 2, 3, ..., 3n\}$.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2 z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n-1$. Therefore, we have

$$\begin{aligned}\phi_3(\overline{\beta}(r)) &= \phi_3(r_0, (1+2v)r_1, ..., (1+2v)r_{n-2}, r_{n-1}) \\ &= (y_0, 2x_1 + y_1, y_2, ..., 2x_{n-2} + y_{n-2}, y_{n-1}, 2x_0 + y_0, y_1, ..., y_{n-2}, 2x_{n-1} + y_{n-1}, \\ &\quad 2z_0, 2z_1, ..., 2z_{n-2}, 2z_{n-1}).\end{aligned}$$

And,

$$\begin{aligned}\gamma(\phi_3(r)) &= \gamma(y_0, y_1, ..., y_{n-2}, y_{n-1}, 2x_0 + y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2}, 2x_{n-1} + y_{n-1}, \\ &\quad 2z_0, 2z_1, 2z_2, ..., 2z_{n-2}, 2z_{n-1}) \\ &= (y_0, 2x_1 + y_1, ..., 2x_{n-2} + y_{n-2}, y_{n-1}, 2x_0 + y_0, y_1, 2x_2 + y_2, ..., y_{n-2}, \\ &\quad 2x_{n-1} + y_{n-1}, 2z_0, 2z_1, ..., 2z_{n-2}, 2z_{n-1}).\end{aligned}$$

Hence, $\phi_3(\overline{\beta}(r)) = \gamma(\phi_3(r))$. □

Corollary 4.4. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\phi_3(C) = \widetilde{C}$), then $\gamma(\widetilde{C})$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$.*

Proof. By Corollary 4.2, $\overline{\beta}(C)$ is a $(1+2v)$-constacyclic code over R. Using Theorem 4.3 and Theorem 4.6, it is obtained that $\gamma(\widetilde{C})$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$. □

5. Skew-constacyclic codes and their $\mathbb{Z}_4$-images

Let θ be an automorphism on the ring R defined by $\theta(x + vy + v^2 z) = x + vz + v^2 y$ $\forall\ x, y, z \in Z_4$, where $\theta(x) = x\ \forall\ x \in \mathbb{Z}_4$, $\theta(v) = v^2$ and $\theta(v^2) = v$. Obviously, the order of the automorphism is 2 as $\theta^2(a) = a\ \forall\ a \in R$. The set $R[\alpha; \theta] = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mid a_j \in R,\ j = 0, 1, ..., n-1\}$ is a non-commutative skew polynomial ring under the usual addition of polynomials and multiplication of polynomials, which is defined as $(a\alpha^s)(b\alpha^t) = a\theta^s(b)\alpha^{s+t}$. By taking, $\lambda = (1+2v)$ we can identify each vector $r = (r_0, r_1, r_2, ..., r_{n-1}) \in R^n$ with a polynomial $r(\alpha) \in R[\alpha; \theta]/\langle \alpha^n - \lambda \rangle$ by the following correspondence

$$r = (r_0, r_1, ..., r_{n-1}) \to r(\alpha) = r_0 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1} (mod\langle \alpha^n - \lambda \rangle).$$

Definition 5.1. *A non-empty subset C of R^n is called a skew-cyclic code of length n over R if C is an R-submodule of R^n, and $\sigma_\theta(c_0, c_1, ..., c_{n-1}) = (\theta(c_{n-1}), \theta(c_0), ..., \theta(c_{n-2})) \in C$ for any $(c_0, c_1, ..., c_{n-1}) \in C$.*

Definition 5.2 (Islam and Prakash [10]). *A non-empty subset C of R^n is called a skew λ-constacyclic code of length n over R if it satisfies the following conditions:*

(i). *C is an R-submodule of R^n, and*
(ii). *if $(c_0, c_1, ..., c_{n-1}) \in C$, then $\tau_{\theta,\lambda}(c_0, c_1, ..., c_{n-1}) = (\theta(\lambda c_{n-1}), \theta(c_0), ..., \theta(c_{n-2})) \in C$.*

If $\lambda = 1$, then $\tau_{\theta,\lambda}$ is called a skew-cyclic shift operator.

Definition 5.3. *A non-empty subset C of R^n is called a skew 2-quasicyclic code of length n over R if C is an R-submodule of R^n, and $\sigma_\theta^2(c_0, c_1, ..., c_{n-1}) = (\theta^2(c_{n-2}), \theta^2(c_{n-1}), \theta^2(c_0), ..., \theta^2(c_{n-3})) \in C$ for any $(c_0, c_1, ..., c_{n-1}) \in C$.*

Theorem 5.1 (Islam and Prakash [10])**.** *Let C be a linear code of length n over R. Then C is a skew λ-constacyclic code over R if and only if C is a left $R[\alpha;\theta]$-submodule of $R[\alpha;\theta]/\langle \alpha^n - \lambda \rangle$.*

Proposition 5.1. *For any $r \in R^n$, we have $\phi_2(\tau_{\theta,\lambda}(r)) = \rho_2(\phi_2(r))$, where ϕ_2 is the Gray map defined in equation (2), ρ_2 is the quasi-cyclic shift operator as given in the preliminaries and $\tau_{\theta,\lambda}$ is the skew λ-constacyclic shift operator with $\lambda = (1 + 2v)$.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n - 1$. Now, $\theta(x_j + vy_j + v^2z_j) = x_j + vz_j + v^2y_j$ and $\theta((1 + 2v)(x_{n-1} + vy_{n-1} + v^2z_{n-1})) = (x_{n-1} + 2z_{n-1}) + v(2y_{n-1} + z_{n-1}) + v^2(2x_{n-1} + y_{n-1})$. Therefore, we have

$$\begin{aligned}\phi_2(\tau_{\theta,\lambda}(r)) &= \phi_2(\theta(\lambda r_{n-1}), \theta(r_0), \theta(r_1), ..., \theta(r_{n-2}))\\ &= \phi_2((x_{n-1} + 2z_{n-1}) + v(2y_{n-1} + z_{n-1}) + v^2(2x_{n-1} + y_{n-1}), x_0 + vz_0 + v^2y_0,\\ &\quad x_1 + vz_1 + v^2y_1, ..., x_{n-2} + vz_{n-2} + v^2y_{n-2})\\ &=(2x_{n-1}, 2x_0, 2x_1, ..., 2x_{n-2}, 2y_{n-1} + 2z_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ...,\\ &\quad 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

From Proposition 4.2, we have

$$\begin{aligned}\rho_2(\phi_2(r)) = &(2x_{n-1}, 2x_0, 2x_1, ..., 2x_{n-2}, 2y_{n-1} + 2z_{n-1}, 2y_0 + 2z_0, 2y_1 + 2z_1, ...,\\ &2y_{n-2} + 2z_{n-2}).\end{aligned}$$

Hence, $\phi_2(\tau_{\theta,\lambda}(r)) = \rho_2(\phi_2(r))$. □

Theorem 5.2. *The Gray image $\phi_2(C)$ of a skew $(1+2v)$-constacyclic code C of length n over R is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a skew $(1 + 2v)$-constacyclic code of length n over R, $\tau_{\theta,\lambda}(C) = C$. Applying ϕ_2 on both sides and using Proposition 5.1, we have $\rho_2(\phi_2(C)) = \phi_2(C)$. This shows that $\phi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

Using the permutation version Φ_2 of the Gray map given in Section 3, we have obtained the following results.

Proposition 5.2. *For any $r \in R^n$, we have $\Phi_2(\sigma_\theta(r)) = \sigma_\theta^2(\Phi_2(r))$, where Φ_2 is the permutation version of Gray map ϕ_2 and σ_θ is the skew-cyclic shift operator as given before.*

Proof. Let $r = (r_0, r_1, ..., r_{n-1}) \in R^n$, where $r_j = x_j + vy_j + v^2z_j \in R$ and $x_j, y_j, z_j \in \mathbb{Z}_4$ for $j = 0, 1, ..., n - 1$. Then,

$$\begin{aligned}\Phi_2(\sigma_\theta(r)) &= \Phi_2(\theta(r_{n-1}), \theta(r_0), \theta(r_1), ..., \theta(r_{n-2}))\\ &= (2x_{n-1}, 2y_{n-1} + 2z_{n-1}, 2x_0, 2y_0 + 2z_0, 2x_1, 2y_1 + 2z_1, ..., 2x_{n-2}, 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_\theta^2(\Phi_2(r)) &= \sigma_\theta^2(2x_0, 2y_0 + 2z_0, 2x_1, 2y_1 + 2z_1, ..., 2x_{n-2}, 2y_{n-2} + 2z_{n-2}, 2x_{n-1},\\ &\quad 2y_{n-1} + 2z_{n-1})\\ &= (2x_{n-1}, 2y_{n-1} + 2z_{n-1}, 2x_0, 2y_0 + 2z_0, 2x_1, 2y_1 + 2z_1, ..., 2x_{n-2}, 2y_{n-2} + 2z_{n-2}).\end{aligned}$$

Hence, $\Phi_2(\sigma_\theta(r)) = \sigma_\theta^2(\Phi_2(r))$. □

Theorem 5.3. *Let C be a skew-cyclic code of length n over R. Then the image $\Phi_2(C)$ is permutation equivalent to a skew 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a skew-cyclic code of length n over R, $\sigma_\theta(C) = C$. Applying Φ_2 and using Proposition 5.2, we have $\Phi_2(\sigma_\theta(C)) = \sigma_\theta^2(\Phi_2(C)) = \Phi_2(C)$. This implies that $\Phi_2(C)$ is permutation equivalent to a skew 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$. □

6. Conclusion

In this paper, we discussed algebraic structures of cyclic and $(1+2v)$-constacyclic codes over the ring $\mathbb{Z}_4[v]/\langle v^3-1\rangle$. We have shown that the Gray images of cyclic codes are quasi-cyclic codes over $\mathbb{Z}_4$ and that of $(1+2v)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$. It is also proved that Gray image of a skew $(1+2v)$-constacyclic code is permutation equivalent to a skew 2-quasicyclic code over $\mathbb{Z}_4$.

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References

[1] Abualrub, T. and Siap, I., (2007), Cyclic codes over the ring $\mathbb{Z}_2+u\mathbb{Z}_2$ and $\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2$, Des. Codes Cryptogr., 42(3), pp. 273-287.

[2] Aydin, N., Cengellenmis, Y. and Dertli, A., (2017), On some constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2-1\rangle$, their $\mathbb{Z}_4$ images and new codes, Des. Codes Cryptogr., https://doi.org/ 10.1007/s10623-017-0392-y.

[3] Bayram, A. and Siap, I., (2013), Structure of codes over the ring $\mathbb{Z}_3[v]/\langle v^3-v\rangle$, Appl. Algebra Engrg. Comm. Comput., 24, pp. 369-386.

[4] Cengellenmis, Y., Dertli, A. and Aydin, N., (2018), Some constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2\rangle$, new Gray maps and new quaternary codes, Algebra Colloq., 25(3), pp. 369-376.

[5] Dertli, A. and Cengellenmis, Y., (2019), On the codes over the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$ cyclic, constacyclic, quasi-cyclic codes, their skew codes, cyclic DNA and skew cyclic DNA codes, Prespacetime Journal, 10(2), pp. 196-213.

[6] Gao, J., (2015), Some results on linear codes over $\mathbb{F}_p+u\mathbb{F}_p+u^2\mathbb{F}_p$, J. Appl. Math. Comput., 47, pp. 473-485.

[7] Hammons, A. R., Kumar, P. V., Calderbank, A. R., Sloane, N. J. A. and Solé, P., (1994), The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes, IEEE Trans. Inform. Theory, 40, pp. 301-319.

[8] Islam, H., Bag, T. and Prakash, O., (2019), A class of constacyclic codes over $\mathbb{Z}_4[u]/\langle u^k\rangle$, J. Appl. Math. Comput., 60(1-2), pp. 237–251.

[9] Islam, H. and Prakash, O., (2018), A study of cyclic and constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$, Int. J. Inf. Coding Theory, 5(2), pp. 155-168.

[10] Islam, H. and Prakash, O., (2019), A class of constacyclic codes over the ring $\mathbb{Z}_4[u,v]/\langle u^2,v^2,uv-vu\rangle$ and their Gray images, Filomat, 33(8), pp. 2237-2248.

[11] Islam, H. and Prakash, O., (2021), New $\mathbb{Z}_4$ codes from constacyclic codes over a non-chain ring, Comp. Appl. Math., 40(12), https://doi.org/10.1007/s40314-020-01398-y.

[12] Islam, H., Prakash, O. and Solé, P., (2020), $\mathbb{Z}_4\mathbb{Z}_4[u]$-additive cyclic and constacyclic codes, Adv. Math. Commun., https://doi.org/10.3934/amc.2020094.

[13] Kom, S. T., Devi. O. R. and Chanu, T. R., (2021), A note on constacyclic codes over the ring $\mathbb{Z}_3[u,v]/\langle u^2-u,v^2,uv,vu\rangle$, J. Math. Comput. Sci., 11(2), pp. 1437-1454.

[14] Özen, M., Özzaim, N. T. and Aydin, N., (2016), Cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$, Turkish J. Math., 42, pp. 1235-1247.

[15] Özen, M., Uzekmek, F. Z., Aydin, N. and Özzaim, N. T., (2016), Cyclic and some constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$, Finite Fields Appl., 38, pp. 27-39.

[16] Özen, M., Uzekmek, F. Z. and Öztas, E. S., (2021), Cyclic and some constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^3 - u^2\rangle$ and their Gray images, Turk J. Math., 45, pp. 579-596.

[17] Özkan, M., Dertli, A. and Cengellenmis, Y., (2019), On Gray images of constacyclic codes over the finite ring $\mathbb{F}_2 + u_1\mathbb{F}_2 + u_2\mathbb{F}_2$, TWMS J. App. Eng. Math., 9(4), pp. 876-881.

[18] Qian, J. F., Zhang, L. N. and Zhu, S. X., (2006), $(1 + u)$-Constacyclic and cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$, Appl. Math. Lett., 19, pp. 820-823.

[19] Shi, M., Alahmadi, A. and Solé, P., (2017), Codes and Rings: Theory and Practice, Academic Press.

[20] Shi, M., Qian, L., Sok, L., Aydin, N. and Solé, P., (2017), On constayclic codes over $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ and their Gray images, Finite Fields Appl., 45, pp. 86-95.

[21] Wan, Z. X., (1997), Quaternary Codes, World Scientific, Singapore.

[22] Yildiz, B. and Aydin, N., (2014), On cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their $\mathbb{Z}_4$-images, Int. J. Inf. Coding Theory, 2, pp. 226-237.

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