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STABLE ALGORITHM FOR THE WAVE EQUATION USING FINITE DIFFERENCE METHOD

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Abstract. The numerical solution of hyperbolic equations has shown instabilities in some studies (see [2]). In this paper, we use a trick to remedy the reasons for instabilities of particular example of hyperbolic equation, namely the wave equation. We propose a simple method based on the finite difference method and the projection on a manifold that conserves energy.

Keywords: wave equation, projection, finite difference method.

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1. Introduction

Most of the problems in science and technology can be expressed mathematically in the form of partial differential equations. Three types of partial differential equations can be formed, namely: the elliptical equation, the parabolic equation and the hyperbolic equation, in which each of the equations has a certain character and a specific use. Especially, the hyperbolic equations are usually associated with vibrations, wave's equation, diffusion, etc.

A plethora of numerical method, for solving hyperbolic partial differential equations has been developed (see [4]) some of them have shown instabilities. Difficulty in solving this type of equations can occur for problems where the solution lack regularity, either initially or during evolution.

Solutions exist to remedy the instability by using Spectral Fourier method and the projection on a manifold that conserves energy [1] and [2], these latter methods have successfully been applied to problems with periodic boundary conditions.

In this paper, we propose an extension of the ideas proposed in [1] and [2] to correct the instability which can arise during the solution of a particular case of hyperbolic equation, namely the wave equation with Dirichlet boundary conditions. This new method allows us

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to apply a different strategy, based on a new correction procedure, even if the boundary conditions are not periodic using a scheme that discretize the energy. The proposed method seems much simpler to implement and more general. Moreover the said projection operator is a simple operator that suffices in achieving our main idea.

This paper is organized as follows. In section two, we present the wave equation with Dirichlet boundary conditions. After that, the projection Algorithm is discussed in section three. Section four is dedicated for the numerical results and discussion. Finally, we conclude in section five.

2. Wave equation

We consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad x \in]a,b[\ , \quad t > 0, \tag{1}$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad x \in]a,b[. \tag{2}$$

Equation (1) with general boundary conditions will not be considered in the present work. We other confine on selves to the specific case of the homogeneous Dirichlet boundary conditions, defined as:

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t > 0. \tag{3}$$

The final time T is compatible with these equations: if f is zero in the vicinity of a and b, then $u(.,t)$ is zero in the vicinity of a and b $\forall t \leq T$.

We multiply (1) by $\dfrac{\partial u}{\partial t}$ and integrate by parts, to obtain:

$$\int_0^t \int_a^b \frac{\partial^2 u}{\partial t^2}(x,t) \frac{\partial u}{\partial t}(x,t) dx dt - c^2 \int_0^t \int_a^b \frac{\partial^2 u}{\partial x^2}(x,t) \frac{\partial u}{\partial t}(x,t) dx dt = 0. \tag{4}$$

In the first parts, integrating out the time variable, we get;

$$\frac{1}{2} \int_a^b \left[\left(\frac{\partial u}{\partial t}(x,t) \right)^2 \right]_0^t dx, \tag{5}$$

the second integral can be done by firstly integrating out time variable

$$-c^2 \int_0^t \left[\frac{\partial u}{\partial x}(x,t) \frac{\partial u}{\partial t}(x,t) \right]_a^b dt + \frac{c^2}{2} \int_a^b \left[\left(\frac{\partial u}{\partial x}(x,t) \right)^2 \right]_0^t dx. \tag{6}$$

We denote by $\langle u, v \rangle = \int_a^b uv dx$ and $\|.\|$ the associate norm. The term (5) may be rewritten as:

$$\frac{1}{2} \left\langle \frac{\partial u}{\partial t}(x,t), \frac{\partial u}{\partial t}(x,t) \right\rangle - \frac{1}{2} \left\langle \frac{\partial u}{\partial t}(x,0), \frac{\partial u}{\partial t}(x,0) \right\rangle. \tag{7}$$

Moreover, the term (6) may be rewritten using the boundary conditions (3)), as:

$$+\frac{c^2}{2} \left\langle \frac{\partial u}{\partial x}(x,t), \frac{\partial u}{\partial x}(x,t) \right\rangle - \frac{c^2}{2} \left\langle \frac{\partial u}{\partial x}(x,0), \frac{\partial u}{\partial x}(x,0) \right\rangle, \tag{8}$$

Equations (7) and (8) is also true for periodic problems.

We obtain from (7) and (8)

$$\left\| \frac{\partial u}{\partial t}(x,t) \right\|^2 + c^2 \left\| \frac{\partial u}{\partial x}(x,t) \right\|^2 = \left\| \frac{\partial u}{\partial t}(x,0) \right\|^2 + c^2 \left\| \frac{\partial u}{\partial x}(x,0) \right\|^2, \ \forall t > 0. \tag{9}$$

Now, we introduce the following energy

$$H(u)(t) = \left\| \frac{\partial u}{\partial t}(x,t) \right\|^2 + c^2 \left\| \frac{\partial u}{\partial x}(x,t) \right\|^2. \tag{10}$$

Equation (9) shows that this energy is conserved

$$H(u)(t) = H(u)(0), \text{ for each } t > 0. \tag{11}$$

Hence, and given that H is conserved, any numerical solution of our problem should maintain this propriety of H. A discrete energy, an approximation of H, should therefore be conserved at each time step for any numerical scheme that shell be adopted.

2.1. **Discretization.** We will solve the system (1), on the interval $]0, 1[$, with the initial conditions and the boundary conditions well chosen, we apply the finite difference method. For this, we choose, time step Δt, space step Δx and grid points $(x_i = i\Delta x, t^n = n\Delta t)$, $i = 1, ..., M, n = 1, ..., N$. We done the values $u_i^n = u(x_i, t^n)$ to the exact solution for $i = 1, ..., M, n = 1, ..., N$. In general, in the treatment of this type of second order hyperbolic equation, it is usual to apply explicit schemes with restriction on the value of $\lambda = \frac{\Delta t}{\Delta x}$ to get the convergence and the stability, so that, for an explicit stable scheme of a wave equation, it's necessary that the discretization in space and time satisfy the following CFL condition $|c\lambda| \leq 1$. However, this is not always the case for implicit scheme. By applying the leap-frog scheme to the equation (1), which is an explicit scheme, we simply obtain the following equation

$$u_i^{n+1} = (2 - 2(c\lambda)^2)u_i^n + (c\lambda)^2(u_{i+1}^n + u_{i-1}^n) + u_i^{n-1}, i = 1, ..., M, n = 1, ..., N. \tag{12}$$

This scheme is stable if the CFL condition is satisfied, for the application of this scheme, we need the values of u_i^n and u_i^{n-1} for each time step to have the value of u_i^{n+1}.

The Newmark scheme allows us to do. Moreover, it gives a more stable formula for computing u_i^{n+1}, that is to say is stable for a larger CFL condition. The scheme is given as follows

$$\begin{aligned} u_i^{n+1} &= u_i^n + \Delta t v_i^n + (c\lambda)^2(\beta w_i^{n+1} + (\tfrac{1}{2} - \beta)w_i^n), \\ v_i^{n+1} &= v_i^n + \frac{(c\lambda)^2}{\Delta t}(\theta w_i^{n+1} + (1-\theta)w_i^n), \end{aligned} \quad i = 1, ..., M, n = 1, ..., N, \tag{13}$$

with $w_i = u_{i+1} + u_i + u_{i-1}$, the parameters β and θ satisfy $0 \leq \beta \leq \frac{1}{2}, 0 \leq \theta \leq 1$. The values taken for β and θ will define the properties of the scheme. This method is implicit, except for $\beta = 0$. It is a second order scheme for $\theta = \frac{1}{2}$ and a first order one for $\theta \neq \frac{1}{2}$. For $\theta = \frac{1}{2}$ and $\beta = 0$, we find the centered difference scheme which is implicit. It is adapted for the treatment of non-linear problems since the matrix to reverse is diagonal, making it less costly problem. However this scheme is conditionally stable, it must satisfy the condition CFL. For $\theta = \frac{1}{2}$ and $\beta = \frac{1}{4}$, we find an average acceleration scheme, which is unconditionally stable and widely used. However, this method is not suitable for the simulation of larger time intervals as it leads to parasitic oscillations. For this type of simulation, it's better to use $\theta > \frac{1}{2}$ and $\beta > \frac{(\frac{1}{2}+\theta)^2}{4}$, we get then a first order scheme, (for more details, see for example [4]).

3. Projection algorithm

The idea of [1] and [2] is to project the trajectory onto the constant energy manifold. First, let us define the manifold $\mathfrak{M}$ corresponding to the set of all functions u such that $H(u) = H(u(0))$. As in [1] and [2]. Following ([3], Chapter 4), we consider the definition

of the projection $\mathcal{P}_{\mathfrak{M}}$ on the manifold $\mathfrak{M}$. For each $\tilde{u} = (\tilde{q}, \tilde{p}) \in \mathbb{R}^{2d}$, we define $u = (q, p) = \mathcal{P}_{\mathfrak{M}}(\tilde{u}) \in \mathbb{R}^{2d}$ such that

$$u = \mathcal{P}_{\mathfrak{M}}(\tilde{u}) = \tilde{u} + \lambda \nabla H(\tilde{u}), \tag{14}$$

where $\lambda \in \mathbb{R}$ is such that $H(u) = H_0$. That is,

$$H(\tilde{u} + \lambda \nabla H(\tilde{u})) = H_0. \tag{15}$$

The new idea in this work, is to discretise the energy H defined in (10) by finite difference method. We obtain at each time step a discrete energy H_n (approximation of H), given by:

$$\begin{aligned} H_n(u) &= \frac{1}{\Delta t^2} \|u - u_n\|_2 + \frac{c^2}{\Delta x^2} \|Du\|_2 , \\ &= \frac{1}{\Delta t^2} \langle u - u_n, u - u_n \rangle + \frac{c^2}{\Delta x^2} \left\langle D^T D u, u \right\rangle , \end{aligned} \tag{16}$$

with $n = 1, ..., N-1$ and D is the discrete differentiation matrix

$$D = \begin{pmatrix} -1 & 1 & 0 & \cdots \\ 0 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \in \mathcal{M}_N.$$

The property of conservation of energy (11) for the discretise case (16), reads

$$H_n(u_{n+1}) = H_{n+1}(u_{n+2}), \;\; n = 1, ..., N-2. \tag{17}$$

We define the manifold $\mathfrak{M}_n$ corresponding to the set of all vectors v which preserve the discrete energy:

$$\mathfrak{M}_n = \left\{ v \in \mathbb{R}^{2d}; H_n(v) = H_{n-1}(u_n), \; n = 2, ..., N \right\}. \tag{18}$$

Let $\mathcal{P}_{\mathfrak{M}_n}$ denotes the projection onto the manifold $\mathfrak{M}_n$: for $w \in \mathbb{R}^{2d}$ we associate $v = \mathcal{P}_{\mathfrak{M}_n}(w)$, defined by:

$$v = \mathcal{P}_{\mathfrak{M}_n}(w) = w + \lambda \nabla H_n(w), \tag{19}$$

where λ is chosen such that $v \in \mathfrak{M}_n$.

We propose a new Algorithm to solve equation (1), which consists of projecting (at each step) the solution obtained by our numerical scheme (here Newmark) onto the manifold $\mathfrak{M}_n$. To above mentioned Algorithm work as follows: So we provide to amend the Newmark scheme, as well

- Calculate $\widetilde{u}_{n+1}$ by the Newmark scheme.
- Define

$$u_{n+1} = \mathcal{P}_{\mathfrak{M}_n}(\widetilde{u}_{n+1}) = \widetilde{u}_{n+1} + \lambda_n \nabla H_n(\widetilde{u}_{n+1}), \tag{20}$$

  where λ_n are chosen such that $u_{n+1} \in \mathfrak{M}_n$.

  Note that, conservation of energy is presented by

$$H_n(u_{n+1}) = H_{n-1}(u_n) = ... = H_1(u_2). \tag{21}$$

To search the λ_n, $n = 1, ..., N-1$, we should take into account the fact that every time equation (20) is solved, we obtain a second order degree equation with variable λ_n, which may be easily solved. This equations can have at most two solutions. If it have two solutions, then we have two values for λ_n, which represents two direction of projection of the gradient, we selected the good direction. On the other hand, If has only one solution, then that is the good solution. Finally, if it doesn't have any solution, the projection is not performed.

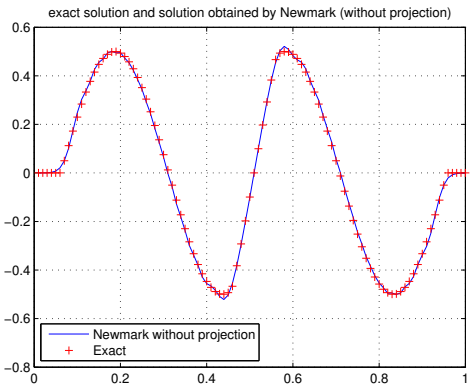

FIGURE 1. Solution of the wave equation with initial data (23) with respect to x without projection at t=1, $T = 1$, $x = 1$, $c = 0.2$, $\Delta t = 10^{-2}$, $\Delta x = 10^{-2}$, $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$.

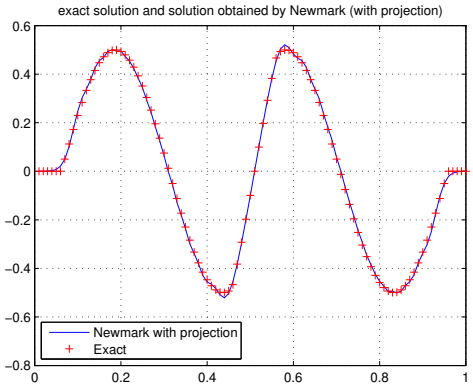

FIGURE 2. Solution of the wave equation with initial data (23) with respect to x with projection at t=1, $T = 1$, $x = 1$, $c = 0.2$, $\Delta t = 10^{-2}$, $\Delta x = 10^{-2}$, $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$.

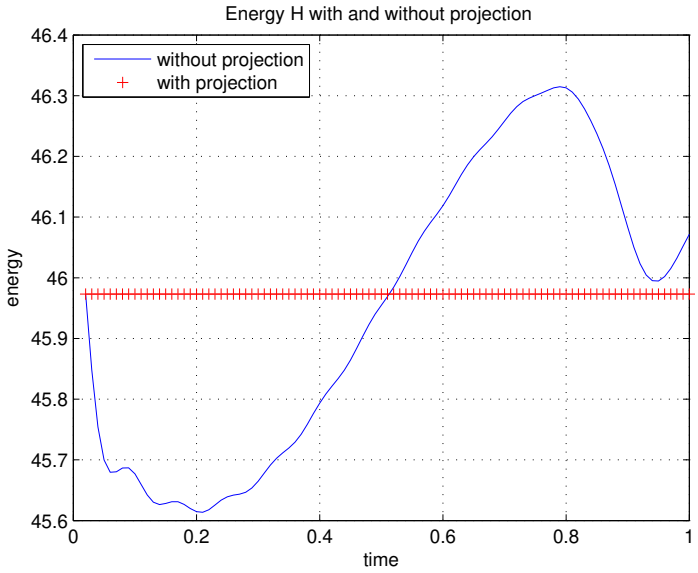

FIGURE 3. Energy H of solution of the wave equation with initial data (23) with and without projection, $T = 1$, $x = 1$, $c = 0.2$, $\Delta t = 10^{-2}$, $\Delta x = 10^{-2}$, $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$.

4. NUMERICAL RESULTS

Now we present some numerical results of the solution of the equation (1) associated with the boundary conditions (3) with and without projection. In example (23), we seek for an approximate solution for the exact solution (1) with initial condition (23), given by

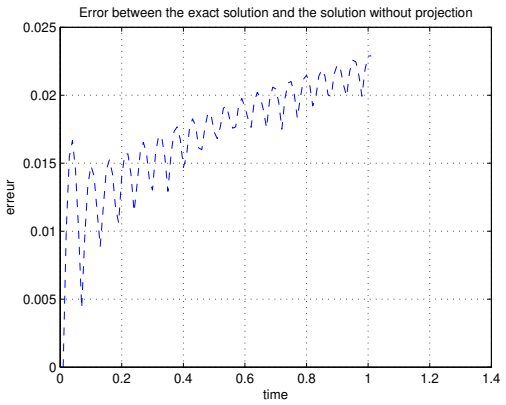

FIGURE 4. Error between the exact solution and the numerical solution without projection of the wave equation with initial data (23), $T = 1$, $x = 1$, $c = 0.2$, $\Delta t = 10^{-2}$, $\Delta x = 10^{-2}$, $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$.

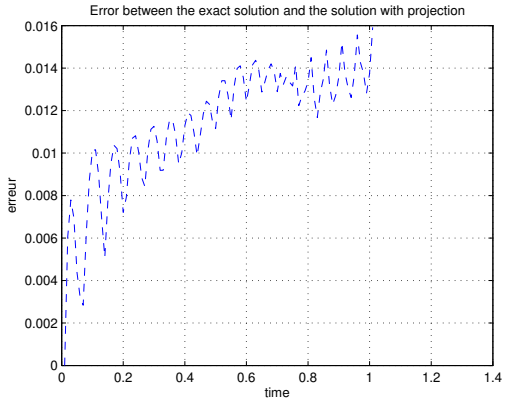

FIGURE 5. Error between the exact solution and the numerical solution with projection of the wave equation with initial data (23), $T = 1$, $x = 1$, $c = 0.2$, $\Delta t = 10^{-2}$, $\Delta x = 10^{-2}$, $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$.

d'Alembert's formula (22)

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{22}$$

The initial conditions are

$$u(x,0) = \begin{cases} -\sin(4\pi x) & si \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 0 & si \quad x \in \left[0, \frac{1}{4}\right[\cup \left]\frac{3}{4}, 1\right] \end{cases}, \frac{\partial u}{\partial t}(x,0) = 0. \tag{23}$$

All numerical results of the approximate solution that we presented are given by (1) with boundary condition (3) and initial data (23) are obtained using the Newmark scheme with $\theta = \frac{1}{2}$ and $\beta = \frac{(\frac{1}{2}+\theta)^2}{4}$. In fact, it is only a simulation that illustrates the bad behavior of this scheme for the hyperbolic equation, among many other choices for different discretization parameters (we respect CFL condition) and with the same initials conditions, the simulation is stable and convergent. The instability is not systematic, and by changing the discretization parameters, we can make the system work. What we want to correct using the projection? This is to give a correction where the solutions are not stable with the same initial conditions.

In Figure 1, we represent the approximate solution and the exact solution given by (22). without using the projection, and in Figure 3, we represent the approximate solution and the exact solution given by (22) using the projection onto the manifold $\mathfrak{M}_n$.

We note that we have the same solution if we use the projection or not, but there is a difference if we calculate the error (see Figure 5), where the error using the projection is smaller than that calculated without projection. Moreover, if we compare the energy using the projection or not, we see that the energy is more stable when we use the projection (see Figure 2), where we represent the energy of (16) under the same conditions (1) with

(3) and (23) with and without using the projection. We observe that the energy is best conserved by using the projection, which confirms the theoretical results obtained above.

Conclusion and perspective

In this article, we solved the hyperbolic equation (the wave equation). We developed the projection using a new method, that discretized energy, again we used the finite differences method instead of the spectral discretization by Fourier series (used in most of works already done), which requires us to take periodic boundary conditions. But, in this work, we used well chosen boundary conditions to facilitate the calculation to keep the energy conservation, which is the case for the periodic boundary conditions. Improvements in the choice of boundary conditions, in a future work, we will consider more general boundary conditions and generalize the projection, to project onto several manifolds, where the energy is conserved.

References

[1] X. Dai, C. Le Bris, F. Legoll, Y. Maday, (2013), Symmetric parareal algorithms for Hamiltonian systems, Mathematical Modelling and Numerical Analysis, vol. 47 (3), pp 717-742.
[2] X. Dai, Y. Maday, (2013), Stable parareal in time method for first-and second-order hyperbolic systems, SIAM J. Sci. Comput, 35 (1), A52–A78.
[3] E. Hairer, C. Lubich, G. Wanner, (2002), Geometric numerical integration: Structure preserving algorithms for ordinary differential equations, Springer Series in Computational Mathematics 31, Springer-Verlag, Berlin, Heidelberg.
[4] A. Quarteroni, F. Saleri, P. Gervasio, (2014), Scientific Computing with MATLAB and Octave, Springer-Verlag Berlin Heidelberg.

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