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# SIGNLESS LAPLACIAN POLYNOMIAL FOR SPLICE AND LINK OF GRAPHS

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Abstract. The signless Laplacian matrix of a graph $G$ is $Q(G) = A(G) + D(G)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal degree matrix of a graph $G$. The characteristic polynomial of the signless Laplacian matrix is called the signless Laplacian polynomial. The present work is all about the study of signless Laplacian polynomial for the splice of more than two graphs and the link of such graphs. It is noted that such a study is easier when we take into account of the vertex set partition being an equitable partition, because equitable partition of the vertex set reduces the computational steps and also the quotient matrix polynomial is a part of the polynomial of a graph. In this paper we consider the splice and links of complete graphs and of complete bipartite graphs and obtain the signless Laplacian polynomial of these using equitable partition of the vertex set.

Keywords: Signless Laplacian polynomial, equitable partition, splice, link.

AMS Subject Classification: 05C31, 05C50.

## 1. Introduction

The spectra of signless Laplacian matrix perform better when it is compared with the spectra of other commonly used graph matrices (Laplacian, Seidel matrix). Among the generalized adjacency matrices, the signless Laplacian appears to be most convenient in studying graph properties. The study of $Q$-spectra of graphs got additional motivation with advancement in the theory of graphs with least eigenvalue $-2$. Hence, the study of signless Laplacian matrix is the subject of flurry of recent research. The related research can be seen in [1, 3, 5, 6, 7, 8, 12, 18].

In [4], the adjacency polynomial of splice and link of complete graph and star have been obtained. In [13], these results are generalized by taking more copies of the graphs for

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splice and link and using equitable partition on vertex set. Seidel polynomial of splice and link is reported in [14]. Also the adjacency polynomial of the complement of splice and link of certain graphs is reported in the same paper. The distance polynomial of the splice and link is obtained in [15].

The present work is all about the study of signless Laplacian polynomial for the splice of more than two graphs and the link of such graphs (which involves symmetry in the graph structure) by applying the concept of equitable partition on the vertex set.

## 2. Preliminaries

Let $G$ be a simple, connected, undirected, labeled graph on $n$ vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$. Two vertices $v_i$ and $v_j$ are said to be adjacent whenever there is an edge between them. The degree of a vertex $v_i$ is the number of edges incident to it and is denoted by $d_i = deg(v_i)$. The signless Laplacian matrix $Q(G) = A(G) + D(G)$ where $A(G)$ is the adjacency matrix and $D(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in $G$. The signless Laplacian polynomial is defined as, $\phi(Q(G) : x) = \mid xI - Q(G) \mid$, where $I$ is the identity matrix of order $n$. The roots of $\phi\big(Q(G) : x\big) = 0$ are the signless Laplacian eigenvalues constituting its spectrum. If $x_1, x_2, \ldots, x_k$ are the distinct roots of $\phi(Q(G) : x) = 0$ with respective multiplicities $m_1, m_2, \ldots, m_k$ then the signless Laplacian spectrum of $G$ is written as $\{x_1^{(m_1)}, x_2^{(m_2)}, \ldots, x_k^{(m_k)}\}$. Let $K_n$ be the complete graph on $n$ vertices, $K_{r,s}$ be the complete bipartite graph on $r+s$ vertices and $S_n = K_{1,n-1}$ be the star on $n$ vertices. For other graph theoretical notations we follow the book [2].

**Definition 2.1.** [16] *The joined union (or generalized composition or G-Join) denoted by $G[G_1, G_2, \ldots, G_n]$, of $n$ arbitrary graphs $G_1, G_2, \ldots, G_n$ with the vertex set labeling $V(G) = \{v_1, v_2, \ldots, v_n\}$ of a labeled graph $G$, is the graph obtained from the union of graphs $G_1, G_2, \ldots, G_n$ by joining every vertex of $G_i$ to each vertex of $G_j$ whenever $v_i$ and $v_j$ are adjacent in $G$.*

**Definition 2.2.** [16] *A partition $\pi : V_1 \cup V_2 \cup \ldots \cup V_m$ of the vertex set $V(G)$ of a graph $G$ is equitable if the number of neighbors in $V_j$ for a vertex $u$ in $V_i$ is a constant $c_{ij}$, independent of $u$ for all $i, j$ $(1 \le i, j \le m)$.*
*The partition of $V(G)$ into singletons is always equitable. In generalized composition if a graph $G$ is regular then $V(G)$ can be taken as partite set in an equitable partition.*

**Definition 2.3.** [17] *Let $\pi : V_1 \cup V_2 \cup \ldots \cup V_m$ be an equitable partition with parameters $c_{ij}$ and $Q(G/\pi) = [q_{ij}]_{m\times m}$ be the matrix defined as,*

$$q_{ij} = \begin{cases} c_{ij}, & \text{if} \quad i \neq j \\ \sum\limits_{j=1}^{m} c_{ij}, & \text{if} \quad i = j. \end{cases}$$

*The matrix $Q(G/\pi)$ is called the quotient matrix.*

**Theorem 2.1.** [17] *If $\pi : V_1, V_2, \ldots, V_m$ is an equitable partition of a graph $G$, then $\phi(Q(G/\pi) : x)$ divides $\phi(Q(G) : x)$.*

Došlić [10], defined splice and link of two graphs. Ramane et al. [13], gave following definitions of the concept of splice of two graphs for more than two graphs and the link of two copies of such a structure. Moreover, the concept of splice and link for more than two graphs was first time generalized by Došlić and Sharafdini [11].

**Definition 2.4.** [13] *Let $G_1, G_2, \ldots, G_p$ be $p$ disjoint graphs and let us label $p$ vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by $v$. The vertex joining graph at $v$ or the splice of these graphs, denoted as $\vee_v[G_1, G_2, \ldots, G_p]$, is obtained by identifying the vertices $v$ of the $p$ graphs (see Figure 1).*

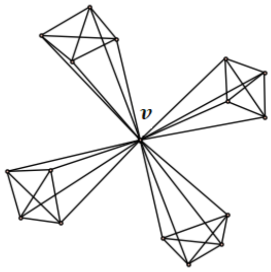


FIGURE 1. $\vee_v[K_5, K_5, K_5, K_5]$

**Definition 2.5.** [13] *Let $G_1, G_2, \ldots, G_{2p}$ be $2p$ graphs and let us label $p$ vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by $v$ and other $p$ vertices, one in each $V(G_i)$ for $i = p+1, p+2, \ldots, 2p$, by $v'$. The edge joining graph at $vv'$ or the link of these graphs be denoted as $\vee^e_{vv'}[G_1, G_2, \ldots, G_{2p}]$ which is obtained by adding a new edge between the identified vertices $v$ and $v'$ of $2p$ graphs (see Figure 2).*

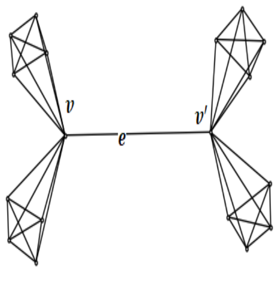


FIGURE 2. $\vee^e_{vv'}[K_5, K_5, K_5, K_5]$

**Definition 2.6.** [13] *Let $G_1, G_2, \ldots, G_p$ be $p$ graphs and let us label $p$ vertices, one in each of $V(G_i)$ for $i = 1, 2, \ldots, p$, by $v$. The edge joining graphs at $v$ or the link of these graphs be denoted as $\vee^e_v[G_1, G_2, \ldots, G_p]$ which is obtained by adding new edges between the vertices labeled by $v$ of $p$ graphs (see Figure 3).*

**Lemma 2.1.** (*Schur Complement* [2]) *Suppose that the order of all four matrices $B_{11}, B_{12}$, $B_{21}$ and $B_{22}$ satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = \begin{cases} |B_{22}|\,|B_{11} - B_{12}B_{22}^{-1}B_{21}|, & \text{if } B_{22} \text{ is a non-singular matrix,} \\ |B_{11}|\,|B_{22} - B_{21}B_{11}^{-1}B_{12}|, & \text{if } B_{11} \text{ is a non-singular matrix.} \end{cases}$$

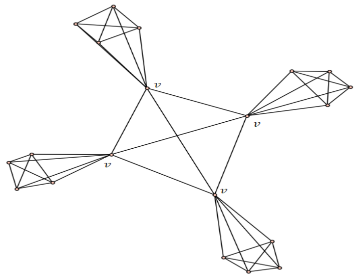


Figure 3. $\vee_v^e[K_5, K_5, K_5, K_5]$

## 3. Signless Laplacian polynomial of splice of graphs

**Theorem 3.1.** *The signless Laplacian polynomial of* $G = \vee_v[\underbrace{K_n, K_n, \ldots, K_n}_{p\text{-copies}}]$ *is*

$$\phi(Q(G):x) = (x-2n+3)^{p-1}(x-n+2)^{p(n-2)}\left(x^2 - (np+2n-p-3)x + 2p(n-1)(n-2)\right).$$

*Proof.* The graph structure of $G = \vee_v[\underbrace{K_n, K_n, \ldots, K_n}_{p\text{-copies}}]$ involves $p$ copies of $K_n$ identified at $v$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $(np-p+1)$ vertices of $G$ into two partite sets: $V_1 = \{v\}$ and $V_2 = \{u : u \text{ is adjacent to } v\}$, these two partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} p(n-1) & p(n-1) \\ 1 & 2n-3 \end{bmatrix}.$$

The polynomial associated with $Q(G/\pi)$ is

$$\phi\left(Q(G/\pi : x)\right) = x^2 - (np+2n-p-3)x + 2p(n-1)(n-2).$$

The remaining part of the spectrum of $\vee_v[K_n, K_n, \ldots, K_n]$, is due to the partition $V_2$ and is, $\{(2n-3)^{(p-1)}, (n-2)^{(p(n-2))}\}$.
Hence, by Theorem 2.1, result follows. □

**Theorem 3.2.** *The signless Laplacian polynomial of* $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$, *where $v$ is selected among $r$ vertices is*

$$\begin{aligned}\phi(Q(G):x) &= x(x-r)^{p(s-1)}(x-s)^{p(r-2)}\left(x^2 - (ps+r+s)x + s(pr+ps-p+1)\right) \\ &\quad \left(x^2 - (r+s)x + s\right)^{p-1}.\end{aligned}$$

*Proof.* The graph structure $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$ involves $p$ copies of $K_{r,s}$ identified at $v$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $(rp+sp-p+1)$ vertices into $2p+1$ partite sets: $V_1 = \{v\}$, $V_i = \{w : w \text{ is not adjacent to } v \text{ in a copy of } K_{r,s}\}$ for $i = 2, 3, \ldots, p+1$ and $V_j = \{u : u \text{ is adjacent to } v \text{ in a copy of } K_{r,s}\}$ for $j = p+2, p+3, \ldots, 2p+1$. These partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} ps & 0 & 0 & \dots & 0 & s & s & \dots & s \\ 0 & s & 0 & \dots & 0 & s & 0 & \dots & 0 \\ 0 & 0 & s & \dots & 0 & 0 & s & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s & 0 & 0 & \dots & s \\ 1 & r-1 & 0 & \dots & 0 & r & 0 & \dots & 0 \\ 1 & 0 & r-1 & \dots & 0 & 0 & r & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & r-1 & 0 & 0 & \dots & r \end{bmatrix}.$$

By Lemma 2.1, polynomial associated with $Q(G/\pi)$ is

$$\phi(Q(G/\pi):x) = x\left(x^2-(ps+r+s)x+s(pr+ps-p+1)\right)\left(x^2-(r+s)x+s\right)^{p-1}.$$

The remaining part of the spectrum of $G$ is due to the partitions $V_i$, $V_j$ which is: $\{r^{(p(s-1))}, s^{(p(r-2))}\}$.
Hence, by Theorem 2.1, the result follows. □

Interchanging $r$ and $s$ in Theorem 3.2 we get following remark.

**Remark 3.1.** *If $v$ is selected among $s$ vertices in $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$ is*

$$\begin{aligned}\phi(Q(G):x) &= x(x-s)^{p(r-1)}(x-r)^{p(s-2)}\left(x^2-(pr+r+s)x+r(pr+ps-p+1)\right)\\ &\quad \left(x^2-(r+s)x+r\right)^{p-1}.\end{aligned}$$

Taking $s = r$ in Theorem 3.2 we get following remark.

**Remark 3.2.** *If $s = r$ in $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$ is*

$$\phi(Q(G):x) = x(x-r)^{p(2r-3)}\left(x^2-r(p+2)x+r(2pr-p+1)\right)\left(x^2-2rx+r\right)^{p-1}.$$

**Corollary 3.1.** *The signless Laplacian polynomial of $G = \vee_v[\underbrace{S_n, S_n, \dots, S_n}_{p\ \text{copies}}]$ is*

$$\phi(Q(G):x) = \begin{cases} x(x-1)^{pn-p-1}\left(x-(np-p+1)\right), & \\ \qquad\qquad \text{if } v \text{ is the central vertex of the star } S_n & \\ x(x^2-nx+1)(x-1)^{p(n-3)}\left(x^2-(n+p)x+(np-p+1)\right), & \\ \qquad\qquad \text{if } v \text{ is a non-central vertex of the star } S_n. & \end{cases}$$

*Proof.* If $v$ is the central vertex of the star $S_n$, then result directly follows by taking $r = 1$ and $s = n-1$ in Theorem 3.2. If $v$ is a non-central vertex of the star $S_n$, then result is obtained by taking $r = n-1$ and $s = 1$ in Theorem 3.2. □

## 4. Signless Laplacian polynomial of link of graphs

**Theorem 4.1.** *The signless Laplacian polynomial of* $G = \vee_{vv'}^{e}[\underbrace{K_n, K_n, \ldots, K_n}_{2p\text{-copies}}]$ *is*

$\phi(Q(G):x) = (x-2n+3)^{2(p-1)}\ (x-n+2))^{2p(n-2)}\ f(x)$, *where*

$$\begin{aligned} f(x) &= x^4 - 2(n-1)(p+2)x^3 \\ &\quad +(n^2p^2 + 8n^2p - 2np^2 + 4n^2 + p^2 - 20np + 12p - 4n - 3)x^2 \\ &\quad -2[2n^3p(p+2) - np^2(8n-10) - 14n^2p + 4n^2 - 4p^2 + 15np - 12n - 5p + 9]x \\ &\quad +4p(n^4p - 6n^3p + 2n^3 + 13n^2p - 9n^2 - 12np + 13n + 4p - 6). \end{aligned}$$

*Proof.* The graph structure $G = \vee_{vv'}^{e}[\underbrace{K_n, K_n, \ldots, K_n}_{2p\text{-copies}}]$ involves $2p$ copies of $K_n$ among which $p$ copies are identified at $v$ and other $p$ copies at $v'$ to have $e = vv'$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $2(np-p+1)$ vertices of $G$ into four partite sets: $V_1 = \{v\}$, $V_2 = \{v'\}$, $V_3 = \{u : u \text{ is adjacent to } v \text{ with } u \neq v'\}$ and $V_4 = \{u' : u' \text{ is adjacent to } v' \text{ with } u' \neq v\}$, these four partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} p(n-1)+1 & 1 & p(n-1) & 0 \\ 1 & p(n-1)+1 & 0 & p(n-1) \\ 1 & 0 & 2n-3 & 0 \\ 0 & 1 & 0 & 2n-3 \end{bmatrix}.$$

The polynomial associated with $Q(G/\pi)$ is

$$\begin{aligned} \phi(Q(G/\pi):x) &= x^4 - 2(n-1)(p+2)x^3 + (n^2p^2 + 8n^2p - 2np^2 + 4n^2 + p^2 \\ &\quad -20np + 12p - 4n - 3)x^2 - 2(2n^3p^2 + 4n^3p - 8n^2p^2 + 10np^2 \\ &\quad -14n^2p + 4n^2 - 4p^2 + 15np - 12n - 5p + 9)x + 4p(n^4p - 6n^3p \\ &\quad +2n^3 + 13n^2p - 9n^2 - 12np + 13n + 4p - 6). \end{aligned}$$

The remaining part of the spectrum of $G$ is due to the partitions $V_3$, $V_4$ which is: $\{(2n-3)^{(2(p-1))}, (n-2)^{(2p(n-2))}\}$.
Hence, by Theorem 2.1, result follows. □

**Theorem 4.2.** *The signless Laplacian polynomial of* $G = \vee_{vv'}^{e}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$, *where* $v$ *and* $v'$ *are selected among* $r$ *vertices is*

$$\begin{aligned} \phi(Q(G):x) &= x(x-r)^{2p(s-1)}(x-s)^{2p(r-2)}\big(x^2 - (ps+r+s)x + s(pr+ps-p+1)\big) \\ &\quad \big(x^3 - (ps+r+s+2)x^2 + (rps+ps^2-ps+2r+3s)x - 2s\big) \\ &\quad \big(x^2 - (r+s)x + s\big)^{2p-2}. \end{aligned}$$

*Proof.* Let $v$ and $v'$ are selected among the $r$ vertices.
The graph structure $G = \vee_{vv'}^{e}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$ involves $2p$ copies of $K_{r,s}$ among which $p$ copies are identified at $v$ and other $p$ copies at $v'$ to have $e = vv'$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $2(pr+ps-p+1)$ vertices of $G$ into $4p+2$ partite sets: $V_1 = \{v\}$, $V_2 = \{v'\}$,

$V_i = \{u : u$ is adjacent to $v$ in a copy of $K_{r,s}\}$ for $i = 3, 4, \ldots, p + 2$, $V_j = \{u' : u'$ is adjacent to $v'$ in a copy of $K_{r,s}\}$ for $j = p+3, p+4, \ldots, 2p+2$, $V_k = \{w : w$ is not adjacent to $v$ in a copy of $K_{r,s}\}$ for $k = 2p + 3, 2p + 4, \ldots, 3p + 2$ and $V_l = \{w' : w'$ is not adjacent to $v'$ in a copy of $K_{r,s}\}$ for $l = 3p + 3, 3p + 4, \ldots, 4p + 2$. These partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} (1+ps)I_1 & I_1 & sJ_{1\times p} & O_{1\times p} & O_{1\times p} & O_{1\times p} \\ I_1 & (1+ps)I_1 & O_{1\times p} & sJ_{1\times p} & O_{1\times p} & O_{1\times p} \\ J_{p\times 1} & O_{p\times 1} & rI_p & O_p & (r-1)I_p & O_p \\ O_{p\times 1} & J_{p\times 1} & O_p & rI_p & O_p & (r-1)I_p \\ O_{p\times 1} & O_{p\times 1} & sI_p & O_p & sI_p & O_p \\ O_{p\times 1} & O_{p\times 1} & O_p & sI_p & O_p & sI_p \end{bmatrix},$$

where $J$ is the matrix with all entries 1, $O$ is the null matrix and $I$ is the identity matrix. By Lemma 2.1, the polynomial associated with $Q(G/\pi)$ is

$$\begin{aligned} \phi(Q(G/\pi) : x) &= x\left(x^2 - (ps + r + s) + s(pr + ps - p + 1)\right) \\ &\quad \left(x^3 - (ps + r + s + 2)x^2 + (rps + ps^2 - ps + 2r + 3s)x - 2s\right) \\ &\quad \left(x^2 - (r + s)x + s\right)^{2p-2}. \end{aligned}$$

The remaining part of the spectrum of $G$ is due to the partitions $V_i$, $V_j$, $V_k$ and $V_l$ which is: $\{r^{(2p(s-1))}, s^{(2p(r-2))}\}$.
Hence, by Theorem 2.1, result follows. □

Interchanging $r$ and $s$ in Theorem 4.2 we get following remark.

**Remark 4.1.** *If $v$ and $v'$ are selected among $s$ vertices in $G = \vee^e_{vv'}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee^e_{vv'}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$ is*

$$\begin{aligned} \phi(Q(G) : x) &= x(x - s)^{2p(r-1)}(x - r)^{2p(s-2)}\left(x^2 - (pr + r + s)x + r(pr + ps - p + 1)\right) \\ &\quad \left(x^3 - (pr + r + s + 2)x^2 + (rps + pr^2 - pr + 2s + 3r)x - 2r\right) \\ &\quad \left(x^2 - (r + s)x + r\right)^{2p-2}. \end{aligned}$$

Taking $s = r$ in Theorem 4.2 we get following remark.

**Remark 4.2.** *If $s = r$ in $G = \vee^e_{vv'}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee^e_{vv'}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p\text{-copies}}]$ is*

$$\begin{aligned} \phi(Q(G) : x) &= x(x - r)^{2p(2r-3)}\left(x^2 - r(p + 2)x + r(2pr - p + 1)\right) \\ &\quad \left(x^3 - (pr + 2r + 2)x^2 + r(2pr - p + 5)x - 2r\right) \\ &\quad (x^2 - 2rx + r)^{2p-2}. \end{aligned}$$

**Theorem 4.3.** *The signless Laplacian polynomial of $G = \vee^e_{v}[\underbrace{K_n, K_n, \ldots, K_n}_{p\text{-copies}}]$ is*

$$\begin{aligned} \phi(Q(G) : x) &= (x - n + 2)^{p(n-2)}\left(x^2 - (3n + 2p - 6)x + 2n^2 + 4np - 10n - 6p + 10\right) \\ &\quad \left(x^2 - (3n + p - 6)x + 2n^2 + 2np - 10n - 3p + 10\right)^{p-1}. \end{aligned}$$

*Proof.* The graph structure $G = \vee_v^e[K_n, K_n, \dots, K_n]$ involves $p$ copies of $K_n$ each having a labeled vertex $v$. These $v$'s are joined among them, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $2n$ vertices of $G$ into $2p$ partite sets: $V_i = \{v\}$ and $V_j = \{u : u \text{ is adjacent to } v \text{ in a copy of } K_n\}$ for $i = 1, 2, 3, \dots, p$ and $j = p+1, p+2, p+3, \dots, 2p$ ( here $V_i$ is a singleton set for the vertex $v$ in $K_n$ ). These $2p$ partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} (n+p-2)I_p + (J-I)_p & (n-1)I_p \\ I_p & (2n-3)I_p \end{bmatrix},$$

where $J$ is the matrix with all entries 1 and $I_p$ is the identity matrix of order $p$.
By Lemma 2.1, the polynomial associated with $Q(G/\pi)$ is

$$\begin{aligned} \phi(Q(G/\pi) : x) &= \big(x^2 - (3n+2p-6)x + 2n^2 + 4np - 10n - 6p + 10\big) \\ &\quad \big(x^2 - (3n+p-6)x + 2n^2 + 2np - 10n - 3p + 10\big)^{p-1}. \end{aligned}$$

The remaining part of the spectrum of $G$ is due to the partitions $V_j$ which is:
$\{(n-2)^{(p(n-2))}\}$.
Hence, by Theorem 2.1, result follows. $\square$

**Theorem 4.4.** *The signless Laplacian polynomial of* $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$, *where $v$ is selected among $r$ vertices is*
$\phi(Q(G) : x) =$

$$\begin{aligned} &\big(x^3 - (2s+2p+r-2)x^2 + (s^2 + 2sp + 2rp + rs - 2r - 2s)x - 2s(p-1)\big) \\ &\big(x^3 - (2s+p+r-2)x^2 + (s^2 + rs + rp + ps - 2r - 2s)x - s(p-2)\big)^{p-1} \\ &(x-r)^{p(s-1)}(x-s)^{p(r-2)}. \end{aligned}$$

*Proof.* Let $v$ be selected among the $r$ vertices.
The graph structure $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \dots, K_{r,s}}_{p\text{-copies}}]$ involves $p$ copies of $K_{r,s}$ each having a labeled vertex $v$. These $v$'s are joined among them, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making the $rp+sp$ vertices of $G$ into $3p$ partite sets: $V_i = \{u : u \text{ is not adjacent to } v \text{ in a copy of } K_{r,s}\}$, $V_j = \{v\}$, $V_t = \{w : w \text{ is adjacent to } v \text{ in a copy of } K_{r,s}\}$ for $i = 1, 2, 3, \dots, p$, $j = p+1, p+2, \dots, 2p$ and $t = 2p+1, 2p+2, \dots, 3p$. These $3p$ partite sets lead to the quotient matrix

$$Q(G/\pi) = \begin{bmatrix} sI_p & O_p & sI_p \\ O_p & (s+p-1)I_p + (J-I)_p & sI_p \\ (r-1)I_p & I_p & rI_p \end{bmatrix},$$

where $J$ is the matrix with all entries 1, $O$ is the null matrix and $I_p$ is the identity matrix of order $p$.
By Lemma 2.1, the polynomial associated with $Q(G/\pi)$ is
$\phi(Q(G/\pi) : x) =$

$$\begin{aligned} &\big(x^3 - (2s+2p+r-2)x^2 + (s^2 + 2sp + 2rp + rs - 2r - 2s)x - 2s(p-1)\big) \\ &\big(x^3 - (2s+p+r-2)x^2 + (s^2 + rs + rp + ps - 2r - 2s)x - s(p-2)\big)^{p-1}. \end{aligned}$$

The remaining part of the spectrum of $G$ is due to the partitions $V_i$ and $V_t$ which is: $\{r^{(p(s-1))}, s^{(p(r-2))}\}$.
Hence, by Theorem 2.1, result follows. $\square$

Interchanging $r$ and $s$ in Theorem 4.4 we get following remark.

**Remark 4.3.** *If $v$ is selected among $s$ vertices in $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$ is*

$\phi(Q(G) : x) =$

$$\big(x^3 - (2r + 2p + s - 2)x^2 + (r^2 + 2rp + 2sp + rs - 2r - 2s)x - 2r(p - 1)\big)$$
$$\big(x^3 - (2r + p + s - 2)x^2 + (r^2 + rs + sp + rp - 2r - 2s)x - r(p - 2)\big)^{p-1}$$
$$(x - s)^{p(r-1)}(x - r)^{p(s-2)}.$$

Taking $s = r$ in Theorem 4.4 we get following remark.

**Remark 4.4.** *If $s = r$ in $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$, then the signless Laplacian polynomial of $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\text{-copies}}]$ is*

$$\begin{aligned} \phi(Q(G) : x) &= (x - r)^{p(2r-3)} \big(x^3 - (3r + p - 2)x^2 + 2r(r + p - 2)x - r(p - 2)\big)^{p-1} \\ &\quad \big(x^3 - (3r + 2p - 2)x^2 + 2r(r + 2p - 2)x - 2r(p - 1)\big). \end{aligned}$$

**Remark 4.5.** *When we put, $r = 1$ and $s = n - 1$ in Theorem 4.4, we get $G = \vee_v^e[\underbrace{K_{1,n-1}, K_{1,n-1}, \ldots, K_{1,n-1}}_{p\text{-copies}}]$ which is same as, $\vee_v^e[\underbrace{S_n, S_n, \ldots, S_n}_{p\text{-copies}}]$, where $v$ is a central vertex. Hence,*
$\phi(Q(G) : x) =$

$$(x - 1)^{p(n-2)} \big(x^2 - (n + 2p - 2)x + 2(p - 1)\big) \big(x^2 - (n + p - 2)x + (p - 2)\big)^{p-1}.$$

## 5. Conclusions

The study of signless Laplacian polynomial for the splice of more than two graphs and the link of such graphs become easier when we take into account of the vertex set partition being an equitable partition, because equitable partition of the vertex set reduces the computational steps and also the quotient matrix polynomial is a part of the polynomial of a graph.

In [9] the relation between adjacency polynomial of subdivision graph and signless Laplacian polynomial of underlying graph is given as

$$\phi(A(S(G)) : x) = x^{m-n}\phi(Q(G) : x^2),$$

where $S(G)$ is the subdivision graph of $G$, $A(G)$ is the adjacency matrix of $G$, $n$ is the number of vertices of $G$ and $m$ is the number of edges of $G$. Using this result and the results of this paper, it is easy to obtain adjaceny polynomial of the subdivison graphs of splice and link considered in this paper.

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