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ON CONNECTIVE ECCENTRIC MATRIX OF A GRAPH

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Abstract. In the present paper, the connective eccentric matrix $CE(\mathcal{G})$ for a simple connected graph $\mathcal{G}$ is introduced and bounds of spectral radius of $CE(\mathcal{G})$ are obtained. The notion of connective eccentric energy $\vartheta(\mathcal{G})$ is also introduced and some upper and lower bounds of $\vartheta(\mathcal{G})$ are obtained here.

Keywords: Topological index, graph eigenvalues, connective eccentric index, connective eccentric matrix, connective eccentric energy.

AMS Subject Classification: 05C50, 05C35.

1. Introduction

Let $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ be a non-null connected graph with no self-loops and parallel edges. Suppose $\mathcal{V} = \mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and $\mathcal{E} = \mathcal{E}(\mathcal{G}) = \{e_{ij} | v_i$ and v_j are adjacent in $\mathcal{G}\}$ is the set of edges. The number of edges incident to v_i is called the degree of v_i, and is denoted by d_i. Let δ and Δ be the lowest and highest degree among the degrees of vertices of $\mathcal{G}$ respectively. Let $d(v_i, v_j)$ be the shortest distance between the vertices v_i and v_j. The maximum distance from v_i to any other vertex of $\mathcal{G}$ is called the eccentricity of v_i, and is denoted by $\varepsilon(v_i)$. A connected graph $\mathcal{G}$ is called l-eccentric graph if all the vertices of $\mathcal{G}$ have the same eccentricity l. The minimum and maximum eccentricity among all the vertices of $\mathcal{G}$ are called radius and diameter of $\mathcal{G}$ respectively, and are denoted by $r = r(\mathcal{G})$ and $D = D(\mathcal{G})$ respectively. The adjacency matrix $A(\mathcal{G}) = (\alpha_{ij})$ is a 0-1 matrix of order n with $\alpha_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise.

A topological index or connectivity index of a graph is an invariant of the form $\tau(\mathcal{G}) = \sum_{v_i \sim v_j} \Gamma(v_i, v_j)$ where $\Gamma : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a symmetric function, i.e., $\Gamma(v_i, v_j) = \Gamma(v_j, v_i)$. The study of topological indices are mainly related to the subjects of chemical graph theory, mathematical chemistry and molecular topology. Some of the popular topological indices

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are the first Zagreb index [6] where $\Gamma(v_i, v_j) = d_i + d_j$, second Zagreb index [6] where $\Gamma(v_i, v_j) = d_i d_j$, Randić connectivity index [9] where $\Gamma(v_i, v_j) = \frac{1}{\sqrt{d_i d_j}}$, harmonic index [2] where $\Gamma(v_i, v_j) = \frac{1}{d_i + d_j}$ etc.

The connective eccentric index was introduced by Gupta et al. [4] in 2000. It is denoted by $C^{\xi}(\mathcal{G})$ and is defined by $C^{\xi}(\mathcal{G}) = \sum\limits_{v_i \sim v_j} \left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)$, i.e., $C^{\xi}(\mathcal{G}) = \sum\limits_{v_i \in V} \frac{d_i}{\varepsilon(v_i)}$. In 2011, Ghorbani [3] derived some bounds of connective eccentric index and computed it for two infinite classes of fullerenes. In 2014, De et al. [1] obtainted some other bounds of connective eccentric index and got some exact formulae for graphs under some basic graph operations.

In 2017, Revankar et al. [10] have introduced the concept of eccentricity sum matrix and energy of that matrix for a graph. The eccentricity sum matrix for a graph $\mathcal{G}$ is defined by, $ES(\mathcal{G}) = (p_{ij})_{n \times n}$ where

$$p_{ij} = \begin{cases} \varepsilon(v_i) + \varepsilon(v_j), & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

Motivated by the aforesaid work of Revankar et al. [10], here we define connective eccentric matrix of a graph $\mathcal{G}$ as $CE(\mathcal{G}) = (a_{ij})_{n \times n}$ where

$$a_{ij} = \begin{cases} \frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}, & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\varepsilon(v_i) \neq 0$ for all $v_i \in \mathcal{V}, CE(\mathcal{G})$ is a well defined matrix. Since it is a real symmetric matrix, all of its eigenvalues are real. Let $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ be the eigenvalues of $CE(\mathcal{G})$. Note that

$$tr(CE(\mathcal{G})) = \sum_{i=1}^{n} \zeta_i = 0. \tag{1}$$

Energy of a graph [5] is one of the most studied graph parameters in recent years. It is defined by $\sum\limits_{i=1}^{n} |\lambda_i|$ where $\lambda_i, i = 1, 2, \ldots, n$ are the adjacency eigenvalues of $\mathcal{G}$. In an analogy, the energy of $CE(\mathcal{G})$, denoted by $\vartheta(CE(\mathcal{G}))$ is defined as

$$\vartheta(CE(\mathcal{G})) = \sum_{i=1}^{n} |\zeta_i|.$$

It is easy to follow that –

$$tr\left([CE(\mathcal{G})]^2\right) = \sum_{i=1}^{n} \zeta_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = 2\sum_{i<j}\left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)^2 = 2Q \text{ (say)}, \tag{2}$$

where $Q = \sum\limits_{i<j} \left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)^2$.

2. Preliminaries

Lemma 2.1 (Arithmetic mean-geometric mean inequality [11]). *If $x_1, x_2, \ldots, x_n$ are n positive real numbers, then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{(x_1 x_2 \ldots x_n)}.$$

The equality holds when $x_1 = x_2 = \cdots = x_n$.

Lemma 2.2 (Cauchy-Schwarz inequality [11]). *If $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are two sets of positive reals, then*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Lemma 2.3. [8] *If $\{x_1, x_2, \ldots, x_n\}$ is a set of non negative reals, then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} x_i - \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n}\sqrt{x_i}\right)^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} x_i - \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right].$$

Lemma 2.4. [7] *If H is a Hermitian matrix of order n and ζ_1 is the largest eigenvalue of H, then $\zeta_1 = \max \dfrac{q^T H q}{q^T q}$, where q is a non-zero column vector of order n.*

3. Exact values and bounds of ζ_i

Theorem 3.1. *Let $\mathcal{G}$ be an l-eccentric graph. Then the eigenvalues of $CE(\mathcal{G})$ are $\frac{2(n-1)}{l}$ with multiplicity 1 and $-\frac{2}{l}$ with multiplicity $n-1$.*

Proof. Since $\mathcal{G}$ is an l-eccentric graph, $CE(\mathcal{G})$ is given by

$$a_{ij} = \begin{cases} \frac{2}{l}, & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic equation of $CE(\mathcal{G})$ is $\det(\zeta I_n - CE(\mathcal{G})) = 0$.

Let $A(K_n)$ be the adjacency matrix of K_n, the complete graph of order n. Then

$$\begin{aligned}
\det(\zeta I_n - CE(\mathcal{G})) &= \det\left(\zeta I_n - \frac{2A(K_n)}{l}\right) \\
&= \left(\frac{2}{l}\right)^n \det\left(\frac{\zeta l}{2} I_n - A(K_n)\right) \\
&= \left(\frac{2}{l}\right)^n \left(\frac{\zeta l}{2} - (n-1)\right)\left(\frac{\zeta l}{2} + 1\right)^{n-1}.
\end{aligned}$$

Therefore the eigenvalues of $CE(\mathcal{G})$ are $\frac{2(n-1)}{l}$ with multiplicity 1 and $-\frac{2}{l}$ with multiplicity $n-1$. □

Example 3.1.

(i) Complete graph K_n of order n is 1-eccentric graph. So the eigenvalues of $CE(K_n)$ are $2(n-1)$ with multiplicity 1 and -2 with multiplicity $n-1$.

(ii) Complete bipartite graph $K_{a,b}$ of order $n = a+b, a, b > 1$ is 2-eccentric graph. Thus the eigenvalues of $CE(K_{a,b})$ are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

(iii) If $\mathcal{G} = C_n$ is a cycle of even order then it is $\frac{n}{2}$-eccentric graph. Hence its connective eccentric eigenvalues are $\frac{4(n-1)}{n}$ with multiplicity 1 and $-\frac{4}{n}$ with multiplicity $n-1$.

(iv) If $\mathcal{G} = C_n$ is a cycle of odd order then it is $\frac{n-1}{2}$-eccentric graph. The eigenvalues of $CE(\mathcal{G})$ are 4 with multiplicity 1 and $-\frac{4}{n-1}$ with multiplicity $n-1$.

Next, we consider a class of graphs which are almost l-eccentric – only one vertex has eccentricity 1, and each of the remaining vertices has eccentricity 2. It is evident that all such graphs are not isomorphic. As for example, for $n \geq 5$, the wheel graph W_n, the star graph $K_{1,n-1}$, and the graph C_n', obtained from the cycle graph C_n, by joining one

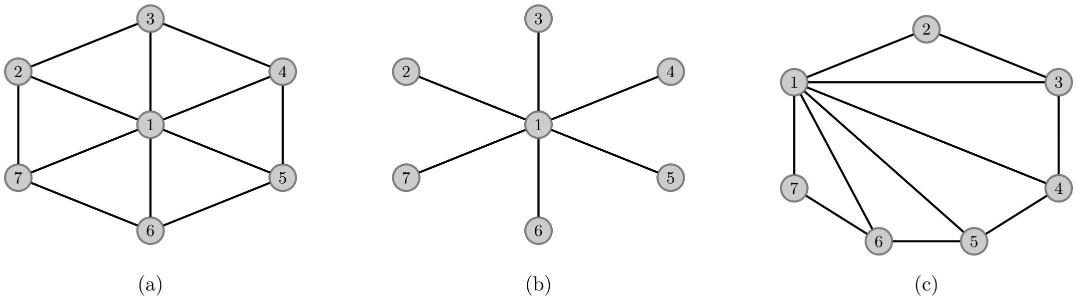

FIGURE 1. (a) The wheel graph W_7, (b) The star graph $K_{1,6}$, (c) The graph C_7'.

particular vertex to all the other vertices, as shown in Figure 1. All these three graphs have the same connective eccentric matrix

$$CE(\mathcal{G}) = \begin{pmatrix} 0 & \frac{3}{2} & \frac{3}{2} & \cdots & \frac{3}{2} \\ \frac{3}{2} & 0 & 1 & \cdots & 1 \\ \frac{3}{2} & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{3}{2} & 1 & \cdots & 1 & 0 \end{pmatrix}_{n\times n}. \tag{3}$$

Theorem 3.2. *Let $\mathcal{G}$ be a graph whose exactly one vertex has eccentricity 1, and each of the other vertices has eccentricity 2. Then the connective eccentric eigenvalues of $\mathcal{G}$ are -1 with multiplicity $n-2$ and $\frac{(n-2)\pm\sqrt{n^2+5n-5}}{2}$.*

Proof. Observe that $\zeta_i = -1$ is an eigenvalue with corresponding eigenvector $q_i = (0,1,0,\ldots,0,-1,0,\ldots,0)^T$ (with -1 at i-th position), $i = 3,4,\ldots,n$ for the connective eccentric matrix $CE(\mathcal{G})$ given by (3). Clearly, the multiplicity of the eigenvalue -1 is $(n-2)$.

Let, the characteristic polynomial of $CE(\mathcal{G})$ be

$$\begin{aligned} \det(\zeta I_n - CE(\mathcal{G})) &= (\zeta+1)^{(n-2)}(\zeta^2+r\zeta+s) \\ &= \left[\zeta^{(n-2)} + (n-2)\zeta^{(n-3)} + \cdots + 1\right](\zeta^2+r\zeta+s) \\ &= \zeta^n + [r+(n-2)]\zeta^{(n-1)} + \cdots + s. \end{aligned} \tag{4}$$

As the trace of $CE(\mathcal{G})$ is 0, it follows that $r = -(n-2)$ and $s = \det(CE(\mathcal{G})) = \left(\frac{3}{2}\right)^2(-1)^n\det(A(K_n)) = -\frac{9}{4}(n-1)$. So, the remaining two eigenvalues of $CE(\mathcal{G})$ are the roots of $\zeta^2-(n-2)\zeta-\frac{9}{4}(n-1)$, i.e., $\frac{(n-2)\pm\sqrt{n^2+5n-5}}{2}$.

□

Theorem 3.3. *For any connected graph $\mathcal{G}$,*

$$\zeta_1 \leq \sqrt{\frac{2Q(n-1)}{n}},$$

where $Q = \sum_{i<j}\left(\frac{1}{\varepsilon(v_i)}+\frac{1}{\varepsilon(v_j)}\right)^2$ and equality holds if and only if $\mathcal{G}$ is an l-eccentric graph.

Proof. Applying Cauchy-Schwarz inequality [11] on $(n-1)$-tuples $1,1,\ldots,1$ and $|\zeta_i|$, $i = 2,3,\ldots,n$, one gets

$$\left(\sum_{i=2}^{n}|\zeta_i|\right)^2 \leq (n-1)\left(\sum_{i=2}^{n}\zeta_i^2\right). \tag{5}$$

Again,

$$\left(\sum_{i=2}^{n} \zeta_i\right)^2 \leq \left(\sum_{i=2}^{n} |\zeta_i|\right)^2. \tag{6}$$

From (5) and (6), it follows that

$$\left(\sum_{i=2}^{n} \zeta_i\right)^2 \leq (n-1)\left(\sum_{i=2}^{n} \zeta_i^2\right). \tag{7}$$

Since $\sum_{i=1}^{n} \zeta_i = 0$, $\sum_{i=2}^{n} \zeta_i = -\zeta_1$ and from (2), $\sum_{i=2}^{n} \zeta_i^2 = 2Q - \zeta_1^2$.

Then from (7), $(-\zeta_1)^2 \leq (n-1)\left(2Q - \zeta_1^2\right)$, i.e., $\zeta_1 \leq \sqrt{\frac{2Q(n-1)}{n}}$.

It is easy to verify that the equality occurs in the above inequality when $\mathcal{G}$ is an l-eccentric graph. □

Theorem 3.4. *Let $\mathcal{G}$ be a simple connected graph. Then*

$$\zeta_1 \geq \frac{2C^{\xi}(\mathcal{G})}{n}.$$

The equality holds if and only if $\mathcal{G}$ is a complete graph.

Proof. For $q = (1, 1, \ldots, 1)^T$, it is seen from Lemma 2.4 that

$$\zeta_1 \geq \frac{q^T CE(\mathcal{G}) q}{q^T q} = \frac{2\sum_{i<j}\left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)}{n} = \frac{2\sum_{i=1}^{n} \frac{(n-1)}{\varepsilon(v_i)}}{n}.$$

Since $d_i \leq (n-1)$ for all $i = 1, 2, \ldots, n$,

$$\zeta_1 \geq \frac{2}{n}\sum_{i=1}^{n} \frac{d_i}{\varepsilon(v_i)} = \frac{2C^{\xi}(\mathcal{G})}{n}. \tag{8}$$

A simple connected graph $\mathcal{G}$ with n vertices is complete if and only if the degree of each vertex of $\mathcal{G}$ is $n-1$. So, if $\mathcal{G}$ is not a complete graph, then the inequality in (8) becomes strict.

Again, for $\mathcal{G} = K_n, \zeta_1 = 2(n-1)$ and $C^{\xi}(\mathcal{G}) = n(n-1)$. Thus equality holds for $\mathcal{G} = K_n$. Hence the proof is complete. □

The above theorem is verified with the help of the following examples.

Example 3.2. *(i) For $\mathcal{G} = C_6$, $\zeta_1 = \frac{10}{3}$ and $C^{\xi}(C_6) = \sum_{i=1}^{n} \frac{d_i}{\varepsilon(v_i)} = \sum_{i=1}^{6} \frac{2}{3} = 4$. It is evident that,*

$$\zeta_1 = \frac{10}{3} > \frac{2C^{\xi}(C_6)}{6} = \frac{4}{3}.$$

(ii) For $\mathcal{G} = K_6$, $\zeta_1 = 10$ and $C^{\xi}(K_6) = \sum_{i=1}^{n} \frac{d_i}{\varepsilon(v_i)} = \sum_{i=1}^{6} 5 = 30$. Also it is evident that,

$$\zeta_1 = \frac{2C^{\xi}(K_6)}{6}.$$

4. Properties and bounds of $\vartheta(CE(\mathcal{G}))$

Theorem 4.1. *Let $\mathcal{G}$ be an l-eccentric graph. Then* $\vartheta(CE(\mathcal{G})) = \frac{4(n-1)}{l}\cdot$

Proof. The complete list of eigenvalues of $\mathcal{G}$ is given by Theorem 3.1.
Therefore,

$$\vartheta(CE(\mathcal{G})) = \sum_{i=1}^{n} |\zeta_i| = 1 \times \frac{2(n-1)}{l} + (n-1) \times \left|-\frac{2}{l}\right| = \frac{4(n-1)}{l}\cdot$$

□

Example 4.1. *(i)* $\vartheta(CE(K_n)) = 4(n-1)$.
(ii) $\vartheta(CE(K_{a,b})) = 2(n-1)$ *where* $n = a+b$ *and* $a, b > 1$.
(iii) $\vartheta(CE(C_n)) = \frac{8(n-1)}{n}$ *when n is even and* $\vartheta(CE(C_n)) = 8$ *when n is odd.*
(iv) $\vartheta(\mathcal{G}) = (n-2) + \sqrt{n^2+5n-5}$, *where the graph $\mathcal{G}$ contains exactly one vertex with eccentricity* 1, *and every other vertex with eccentricity* 2.

Theorem 4.2. *For any connected graph $\mathcal{G}$,*

$$\sqrt{2Q} \leq \vartheta(CE(\mathcal{G})) \leq \sqrt{2nQ}$$

where $Q = \sum\limits_{i<j} \left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)^2$.

Proof. Cauchy-Schwarz inequality [11] on n-tuples $1, 1, \ldots, 1$ and $|\zeta_i|$, $i = 1, 2, \ldots, n$ can be applied to get

$$\begin{aligned} \left(\sum_{i=1}^{n} |\zeta_i|\right)^2 &\leq n\left(\sum_{i=1}^{n} \zeta_i^2\right) \\ \text{i.e., } [\vartheta(CE(\mathcal{G}))]^2 &\leq 2nQ \\ \text{or, } \vartheta(CE(\mathcal{G})) &\leq \sqrt{2nQ}. \end{aligned} \tag{9}$$

Again,

$$\begin{aligned} [\vartheta(CE(\mathcal{G}))]^2 &= \left(\sum_{i=1}^{n} |\zeta_i|\right)^2 \geq \sum_{i=1}^{n} |\zeta_i|^2 = 2Q, \\ \text{or, } [\vartheta(CE(\mathcal{G}))] &\geq \sqrt{2Q}. \end{aligned} \tag{10}$$

From (9) and (10) the required result follows. □

Theorem 4.3. *For any graph $\mathcal{G}$,*

$$\sqrt{2Q + n(n-1)(\det CE(\mathcal{G}))^{2/n}} \leq \vartheta(CE(\mathcal{G})) \leq \sqrt{2(n-1)Q + n(\det CE(\mathcal{G}))^{2/n}} \tag{11}$$

where $Q = \sum\limits_{i<j} \left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)^2$.

Proof.

$$[\vartheta(CE(\mathcal{G}))]^2 = \left(\sum_{i=1}^{n} |\zeta_i|\right)^2 = \sum_{i=1}^{n} |\zeta_i|^2 + 2\sum_{i<j} |\zeta_i||\zeta_j| = 2Q + \sum_{i\neq j} |\zeta_i||\zeta_j|. \tag{12}$$

Applying arithmetic mean-geometric mean inequality [11] on $n(n-1)$ non-negative numbers,

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\zeta_i||\zeta_j| \geq \left(\prod_{i\neq j}|\zeta_i||\zeta_j|\right)^{1/n(n-1)} = \left(\prod_{i=1}^{n}|\zeta_i|^{2(n-1)}\right)^{1/n(n-1)}$$
$$= \prod_{i=1}^{n}|\zeta_i|^{\frac{2}{n}} = (\det CE(\mathcal{G}))^{2/n}$$
$$\text{or, } \sum_{i\neq j}|\zeta_i||\zeta_j| \geq n(n-1)(\det CE(\mathcal{G}))^{2/n}. \tag{13}$$

From (12) and (13),

$$\sqrt{2Q + n(n-1)(\det CE(\mathcal{G}))^{2/n}} \leq \vartheta(CE(\mathcal{G})). \tag{14}$$

Now, application of Lemma 2.3 on $|\zeta_i|^2$, $i = 1, 2, \ldots, n$ gives

$$n\left[\frac{1}{n}\sum_{i=1}^{n}|\zeta_i|^2 - \left(\prod_{i=1}^{n}|\zeta_i|^2\right)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n}|\zeta_i|^2 - \left(\sum_{i=1}^{n}|\zeta_i|\right)^2$$
$$\leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}|\zeta_i|^2 - \left(\prod_{i=1}^{n}|\zeta_i|^2\right)^{\frac{1}{n}}\right].$$

So,

$$\begin{aligned} 2Q - n(\det CE(\mathcal{G}))^{2/n} &\leq 2nQ - \vartheta(CE(\mathcal{G}))^2 \\ \text{or, } \vartheta(CE(\mathcal{G}))^2 &\leq 2(n-1)Q + n(\det CE(\mathcal{G}))^{2/n} \\ \text{or, } \vartheta(CE(\mathcal{G})) &\leq \sqrt{2(n-1)Q + n(\det CE(\mathcal{G}))^{2/n}}. \end{aligned} \tag{15}$$

From (14) and (15) the theorem follows. □

Theorem 4.4. *Let $\mathcal{G}$ be a graph with radius r and diameter D. Then*

$$\sqrt{\frac{8n(n-1)}{D^2} + n(n-1)(\det CE(\mathcal{G}))^{2/n}} \leq \vartheta(CE(\mathcal{G}))$$
$$\leq \sqrt{\frac{8n(n-1)^2}{r^2} + n(\det CE(\mathcal{G}))^{2/n}}. \tag{16}$$

Proof. As r and D are the minimum and maximum eccentricities of the vertices,

$$r \leq \varepsilon(v_i) \leq D$$
$$\text{or, } \frac{1}{D} \leq \frac{1}{\varepsilon(v_i)} \leq \frac{1}{r}$$
$$\text{or, } \sum_{i<j}\left(\frac{2}{D}\right)^2 \leq Q = \sum_{i<j}\left(\frac{1}{\varepsilon(v_i)} + \frac{1}{\varepsilon(v_j)}\right)^2 \leq \sum_{i<j}\left(\frac{2}{r}\right)^2$$
$$\text{or, } \frac{4n(n-1)}{D^2} \leq Q \leq \frac{4n(n-1)}{r^2}. \tag{17}$$

Then from (11) and (17),

$$\sqrt{\frac{8n(n-1)}{D^2} + n(n-1)(\det CE(\mathcal{G}))^{2/n}} \leq \vartheta(CE(\mathcal{G}))$$
$$\leq \sqrt{\frac{8n(n-1)^2}{r^2} + n(\det CE(\mathcal{G}))^{2/n}}.$$

□

Theorem 4.5. *Let $\mathcal{G}$ be a simple connected graph. If $\vartheta(CE(\mathcal{G}))$ is an integer, then it will be an even integer.*

Proof. Let $\zeta_1, \zeta_2, \dots, \zeta_r$ be the complete list of non negative eigenvalues of $CE(\mathcal{G})$. As already given in (1),

$$\mathrm{tr}(CE(\mathcal{G})) = \sum_{i=1}^{n} \zeta_i = \sum_{i=1}^{r} \zeta_i + \sum_{i=r+1}^{n} \zeta_i = 0.$$

Therefore,

$$\sum_{i=1}^{r} \zeta_i = -\left(\sum_{i=r+1}^{n} \zeta_i\right).$$

Thus,

$$\begin{aligned} \vartheta(CE(\mathcal{G})) &= \sum_{i=1}^{n} |\zeta_i| \\ &= \sum_{i=1}^{r} \zeta_i - \left(\sum_{i=r+1}^{n} \zeta_i\right) \\ &= 2\sum_{i=1}^{r} \zeta_i. \end{aligned}$$

This completes the proof. □

5. Conclusions

The connective eccentric matrix $CE(\mathcal{G})$ and connective eccentric energy $\vartheta(CE(\mathcal{G}))$ of a simple connected graph $\mathcal{G}$ are introduced. If the eccentricity of all the vertices are same, then the complete connective eccentric spectrum and hence the connective eccentric energy can be found easily. For a genral graph. an upper bound and a lower bound of spectral radius of connective eccentric matrix are obtained. Upper and lower bounds of connective eccentric energy are also obtained here.

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