TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 310-321

TOTAL EDGE IRREGULARITY STRENGTH OF JOIN OF PATH AND COMPLEMENT OF A COMPLETE GRAPH

A. S. SARANYA[1*], K. R. SANTHOSH KUMAR[1], §

Abstract. An edge irregular total k-labeling of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, ..., k\}$ in such a way that any two different edges have distinct weights. The weight of an edge uv is the sum of the label of uv and the labels of vertices u and v. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G. In this paper, we determine the exact value of the total edge irregularity strength of $P_n + \overline{K_m}$.

Keywords: Edge Irregularity strength, Total edge irregularity strength, Join of two graphs.

AMS Subject Classification: 05C78.

1. Introduction

Let G be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A *labeling* (or valuation) of a graph is a map that relates the graph element to some numbers (usually to the positive or nonnegative integers). As the graph element, one can take edge set, vertex set or union of vertex set and edge set. In the case that the graph element is the union of vertex set and edge set, then the labeling is called *total labeling*. Various kinds of graph labelings can be found on [5].

Chartrand, Jacobson, Lehel, Oellerman, Ruiz and Saba in [3] proposed a graph labeling problem as the following: Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e, the weights (label sums of edges incident with the vertex) of vertices are distinct. What is the minimum value of the largest label over all such irregular assignments? This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$. Some interesting results on the irregularity strength can be found in [1], [9], [10] and [12]. Bača et al.[2]

[1] Department of Mathematics, University College, (Affiliated to University of Kerala), Thiruvananthapuram, Kerala, India.
e-mail: saranya3290@gmail.com; ORCID: https://orcid.org/0000-0002-2807-789X.
[*] Corresponding author.
e-mail:santhoshkumargwc@gmail.com; ORCID: https://orcid.org/0000-0002-1678-2567.
§ Manuscript received: December 19, 2021; accepted: May 06, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.1

defined an *edge irregular total labeling* $f : V(G) \cup E(G) \to \{1, 2, ..., k\}$ of a graph G as the labeling of vertices and edges of G in such a way that for any different edges e and f, weights of e and f are distinct. The *weight* of an edge $e = xy$ is $wt(xy) = f(x) + f(xy) + f(y)$. The minimum k for which the graph G has an edge irregular total k- labeling is called the *total edge irregularity strength* of the graph G, $tes(G)$.

The lower bound and upper bound of the total edge irregularity strength of any graph was given by Bača et al. [2] as:

Theorem 1.1. [2] *Let G be a graph with vertex set $V(G)$ and a non-empty edge set $E(G)$. Then $\lceil \frac{|E(G)|+2}{3} \rceil \leq tes(G) \leq |E(G)|$.*

Theorem 1.2. [2] *For any graph G with maximum degree $\Delta = \Delta(G)$, $tes(G) \geq max\{\lceil \frac{|E(G)|+2}{3} \rceil, \lceil \frac{\Delta+1}{2} \rceil\}$.*

They also determined the total edge irregularity strength of path P_n, cycle C_n, star S_n, wheel W_n and friendship graph F_n as: $tes(P_n) = tes(C_n) = \lceil \frac{n+2}{3} \rceil$; $tes(S_n) = \lceil \frac{n+1}{2} \rceil$; $tes(W_n) = \lceil \frac{2n+2}{3} \rceil$; and $tes(F_n) = \lceil \frac{3n+2}{3} \rceil$. The definitions of various types of graphs mentioned here can be found on [4].
The following conjecture presented by Ivancŏ and Jendrol' gave the exact value of the total edge irregularity strength for arbitrary graph.

Conjecture 1.1. [7] *Let G be an arbitrary graph different from K_5. Then $tes(G) = max\{\lceil \frac{|E(G)|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$.*

The conjecture has been proved to be true for all trees by Ivancŏ and Jendrol' [7]. While Jendrol', Miškuf, and Soták [8] proved: $tes(K_5) = 5$; $tes(K_n) = \lceil \frac{n^2-n-4}{6} \rceil$, for $n \geq 6$ and $tes(K_{m,n}) = \lceil \frac{mn+2}{3} \rceil$ for $n, m \geq 2$. Indriati et al. [6] determined the total edge irregularity strength of generalized helm, that is $tes({H_n}^1) = \lceil \frac{4n+2}{3} \rceil$, $tes({H_n}^2) = \lceil \frac{5n+2}{3} \rceil$, and $tes({H_n}^m) = \lceil \frac{(m+3)n+2}{3} \rceil$ for $n \geq 3$ and $m \equiv 0 \pmod 3$. Muthu Guru Packiam, Manimaran, and Thuraiswamy [11] investigated how the addition of a new edge affects the total edge irregularity strength of a graph.

Definition 1.1. [4] *Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. The sum (join) of G and H denoted by $G + H$ is defined as a graph with vertex set $V(G) \cup V(H)$ and edge set which contains all edges of G and H together with every vertex of G is joined to every vertex of H and viceversa.*

In this paper, we determine the total edge irregularity strength of join of a path and the complement of complete graph. i.e, $tes(P_n + \overline{K_m})$. This paper also supports Conjecture 1.1 by finding that $tes(P_n + \overline{K_m}) = max\{\lceil \frac{|E(P_n+\overline{K_m})|+2}{3} \rceil, \lceil \frac{\Delta(P_n+\overline{K_m})+1}{2} \rceil\}$.

2. Total Edge Irregularity Strength of $P_n + \overline{K_m}$

$P_n + \overline{K_m}$, where $n \geq 1$ and $m \geq 1$ is the join of a path P_n and complement of a complete graph $\overline{K_m}$ with $n + m$ vertices and $nm + n - 1$ edges. When $n = 1$ and $m > 1$ $P_n + \overline{K_m}$ is a star, $S_m = K_{1,m}$. Bača et al. in [2] determined the total edge irregularity strength of a star graph $S_m = K_{1,m}$ on $m + 1$ vertices, $m > 1$ as $tes(K_{1,m}) = \lceil \frac{m+1}{2} \rceil$. Here we determine $tes(P_n + \overline{K_m})$ for $n \geq 2$ and $m \geq 1$. By Theorem 1.2,
$tes(P_n + \overline{K_m}) \geq max\{\lceil \frac{|E(P_n+\overline{K_m})|+2}{3} \rceil, \lceil \frac{\Delta(P_n+\overline{K_m})+1}{2} \rceil\}$.

As the maximum degree $\Delta(P_n+\overline{K_m}) = \begin{cases} m+1; & n=2 \\ m+2; & n>2 \end{cases}$,
this implies that $tes(P_n+\overline{K_m})\geq\lceil\frac{n+nm+1}{3}\rceil$. To show that $\lceil\frac{n+nm+1}{3}\rceil$ is an upperbound for the $tes(P_n+\overline{K_m})$, we describe an edge irregular total $\lceil\frac{n+nm+1}{3}\rceil$- labeling for $P_n+\overline{K_m}$.

Theorem 2.1. *Let $n\geq2$, then $tes(P_n+\overline{K_1}) = \lceil\frac{2n+1}{3}\rceil$.*

Proof. Let $V(P_n+\overline{K_1}) = \{u_1,u_2,...,u_n,v\}$ and
$E(P_n+\overline{K_1}) = \{u_iu_{i+1} : 1\leq i\leq n-1\}\cup\{u_iv : 1\leq i\leq n\}$.
We have $tes(P_n+\overline{K_1})\geq\lceil\frac{2n+1}{3}\rceil$. Take $k=\lceil\frac{2n+1}{3}\rceil$.
Then $tes(P_n+\overline{K_1})\geq k$. Now to prove the reverse inequality we show that there exists a total edge irregular k- labeling f from $V(P_n+\overline{K_1})\cup E(P_n+\overline{K_1})$ to $\{1,2,3,...,k\}$ by defining f as:
Case(1): $n\equiv 0 \pmod 3$

$$f(u_i) = \begin{cases} k-\lceil\frac{i+1}{2}\rceil+1; & 1\leq i\leq\lceil\frac{2n-1}{3}\rceil \\ n-i+1; & \lceil\frac{2n-1}{3}\rceil<i\leq n. \end{cases}$$
$$f(v)=1.$$
$$f(u_iu_{i+1}) = \begin{cases} \lceil\frac{2n-1}{3}\rceil; & 1\leq i<\lceil\frac{2n-1}{3}\rceil \\ \lfloor\frac{k}{2}\rfloor-n+i+1; & \lceil\frac{2n-1}{3}\rceil\leq i\leq n-1. \end{cases}$$
$$f(u_iv) = \begin{cases} k-\lceil\frac{i}{2}\rceil; & 1\leq i\leq\lceil\frac{2n-1}{3}\rceil \\ 1; & \lceil\frac{2n-1}{3}\rceil<i\leq n. \end{cases}$$

Under this assignment the weights of edges are:
$wt(u_iu_{i+1}) = 2n+2-i$ for $1\leq i<\lceil\frac{2n-1}{3}\rceil$. These weights vary as $\{2n+1,2n,...,\frac{4n}{3}+3\}$.
$wt(u_iv) = \frac{4n}{3}+3-i$ for $1\leq i\leq\lceil\frac{2n-1}{3}\rceil$. These weights vary as $\{\frac{4n}{3}+2,...,\frac{2n}{3}+3\}$.
$wt(u_iu_{i+1}) = \frac{4n}{3}+2-i$ for $\lceil\frac{2n-1}{3}\rceil\leq i\leq n-1$. These weights vary as $\{\frac{2n}{3}+2,...,\frac{n}{3}+3\}$.
$wt(u_iv) = n+3-i$ for $\lceil\frac{2n-1}{3}\rceil<i\leq n$. These weights vary as $\{\frac{n}{3}+2,...,3\}$.

Case(2): $n\equiv 1 \pmod 3$ or $n\equiv 2 \pmod 3$

$$f(u_i) = \begin{cases} k-\lceil\frac{i}{2}\rceil+1; & 1\leq i\leq\lceil\frac{2n-1}{3}\rceil \\ n-i+1; & \lceil\frac{2n-1}{3}\rceil<i\leq n. \end{cases}$$
$$f(v)=1.$$
$$f(u_iu_{i+1}) = \begin{cases} \lceil\frac{2n-1}{3}\rceil; & 1\leq i<\lceil\frac{2n-1}{3}\rceil \\ \lfloor\frac{k}{2}\rfloor-n+i+1; & \lceil\frac{2n-1}{3}\rceil\leq i\leq n-1. \end{cases}$$
$$f(u_iv) = \begin{cases} k-\lfloor\frac{i}{2}\rfloor; & 1\leq i\leq\lceil\frac{2n-1}{3}\rceil \\ 1; & \lceil\frac{2n-1}{3}\rceil<i\leq n. \end{cases}$$

Under this mapping the weights of edges are:
When $n\equiv 1 \pmod 3$:
$wt(u_iu_{i+1}) = 2n+2-i$ for $1\leq i<\lceil\frac{2n-1}{3}\rceil$. These weights vary as $\{2n+1,2n,...,\frac{4n}{3}+\frac{8}{3}\}$.
$wt(u_iv) = \frac{4n}{3}+\frac{8}{3}-i$ for $1\leq i\leq\lceil\frac{2n-1}{3}\rceil$. These weights vary as $\{\frac{4n}{3}+\frac{5}{3},...,\frac{2n}{3}+\frac{7}{3}\}$.
$wt(u_iu_{i+1}) = \frac{4n}{3}+\frac{5}{3}-i$ for $\lceil\frac{2n-1}{3}\rceil\leq i\leq n-1$. These weights vary as $\{\frac{2n}{3}+\frac{4}{3},...,\frac{n}{3}+\frac{8}{3}\}$.
$wt(u_iv) = n+3-i$ for $\lceil\frac{2n-1}{3}\rceil<i\leq n$. These weights vary as $\{\frac{n}{3}+\frac{5}{3},...,3\}$.

When $n \equiv 2 \pmod 3$:
$wt(u_iu_{i+1}) = 2n+2-i$ for $1 \leq i < \lceil \frac{2n-1}{3} \rceil$. These weights vary as $\{2n+1, 2n, ..., \frac{4n}{3} + \frac{10}{3}\}$.
$wt(u_iv) = \frac{4n}{3} + \frac{10}{3} - i$ for $1 \leq i \leq \lceil \frac{2n-1}{3} \rceil$. These weights vary as $\{\frac{4n}{3} + \frac{7}{3}, ..., \frac{2n}{3} + \frac{11}{3}\}$.
$wt(u_iu_{i+1}) = \frac{4n}{3} + \frac{7}{3} - i$ for $\lceil \frac{2n-1}{3} \rceil \leq i \leq n-1$. These weights vary as $\{\frac{2n}{3} + \frac{8}{3}, ..., \frac{n}{3} + \frac{10}{3}\}$.
$wt(u_iv) = n+3-i$ for $\lceil \frac{2n-1}{3} \rceil < i \leq n$. These weights vary as $\{\frac{n}{3} + \frac{7}{3}, ..., 3\}$.

The weights of the $2n-1$ edges of $P_n + \overline{K_1}$ under the labeling f constitute the set $\{3, 4, 5, ..., 2n+1\}$ and the function f is a mapping from $V(P_n + \overline{K_1}) \cup E(P_n + \overline{K_1})$ into $\{1, 2, ...k\}$. So we have $tes(P_n + \overline{K_1}) \leq k = \lceil \frac{2n+1}{3} \rceil$. Combining this with the lower bound, we conclude that $tes(P_n + \overline{K_1}) = k = \lceil \frac{2n+1}{3} \rceil$. □

Next we determine the total edge irregularity strength of the graph $P_n + \overline{K_2}$ for $n \geq 2$.

Theorem 2.2. *Let $n \geq 2$, then $tes(P_n + \overline{K_2}) = \lceil \frac{3n+1}{3} \rceil$.*

Proof. Let $V(P_n + \overline{K_2}) = \{u_1, u_2, ..., u_n, v_1, v_2\}$ and
$E(P_n + \overline{K_2}) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_j : 1 \leq i \leq n \text{ and } j = 1, 2\}$.
We have $tes(P_n + \overline{K_2}) \geq \lceil \frac{3n+1}{3} \rceil$. Take $k = \lceil \frac{3n+1}{3} \rceil$. Thus $tes(P_n + \overline{K_2}) \geq k$. Now to prove the reverse inequality we define a k - labeling $f : V(P_n + \overline{K_2}) \cup E(P_n + \overline{K_2}) \to \{1, 2, ...k\}$ as:

$$f(u_i) = i; \quad 1 \leq i \leq n.$$
$$f(v_1) = 1, f(v_2) = k.$$
$$f(u_iv_1) = i, f(u_iv_2) = k-1; \quad 1 \leq i \leq n.$$
$$f(u_iu_{i+1}) = 1; \quad 1 \leq i \leq n-1.$$

Under this assignment the weight of edges are:
For $1 \leq i \leq n$, $wt(u_iv_1) = 2i+1$.
For $1 \leq i \leq n-1$, $wt(u_iu_{i+1}) = 2i+2$.
$wt(u_1v_2) = 2n+2$ and $wt(u_nv_2) = 3n+1$.
The weights of the $3n-1$ edges of $P_n + \overline{K_2}$ under the k- labeling f constitute the set $\{3, 4, ..., 3n+1\}$ and are distinct. Thus f is a total edge irregular k- labeling and $tes(P_n + \overline{K_2}) \leq k$. This completes the proof. □

Theorem 2.3. *For $n \geq 4$, $tes(P_n + \overline{K_3}) = \lceil \frac{4n+1}{3} \rceil$.*

Proof. Let $V(P_n + \overline{K_3}) = \{u_1, u_2, ..., u_n, v_1, v_2, v_3\}$ and
$E(P_n + \overline{K_3}) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_j : 1 \leq i \leq n \text{ and } j = 1, 2, 3\}$. We have $tes(P_n + \overline{K_3}) \geq \lceil \frac{4n+1}{3} \rceil$.
Take $k = \lceil \frac{4n+1}{3} \rceil$.
Thus $tes(P_n + \overline{K_3}) \geq k$. Now to prove the reverse inequality we define a k - labeling $f : V(P_n + \overline{K_3}) \cup E(P_n + \overline{K_3}) \to \{1, 2, ...k\}$ as:
Case(1): When n is even and $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$

$$f(u_i) = \begin{cases} k-i+1; & 1 \leq i \leq \frac{n}{2} \\ \frac{n+2}{2} - \lceil \frac{i}{2} \rceil; & \frac{n}{2} < i \leq n. \end{cases}$$

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor \frac{n}{2}; & j = 1, 3 \\ 1; & j = 2. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} n - \lceil \frac{k}{2} \rceil + i; & 1 \le i < \frac{n}{2} \\ \lfloor \frac{k}{2} \rfloor - \lfloor \frac{n+2}{4} \rfloor; & i = \frac{n}{2} \\ \frac{n+2}{2}; & \frac{n}{2} < i \le n-1. \end{cases}$$

$$f(u_iv_j) = \begin{cases} \lfloor \frac{|E|}{3} \rfloor; & 1 \le i \le \frac{n}{2};\ \ j = 1,3 \\ \lfloor \frac{k}{2} \rfloor + \frac{n}{2}; & 1 \le i \le \frac{n}{2};\ \ j = 2 \\ \lfloor \frac{k}{2} \rfloor - \lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor + 1; & \frac{n}{2} < i \le n;\ \ j = 1,3 \\ \frac{n+2}{2} - \lfloor \frac{i}{2} \rfloor; & \frac{n}{2} < i \le n;\ \ j = 2. \end{cases}$$

Case(2): When n is even and $n \equiv 2 \pmod 3$

$$f(u_i) = \begin{cases} k - i + 1; & 1 \le i \le \frac{n}{2} \\ \frac{n+2}{2} - \lceil \frac{i}{2} \rceil; & \frac{n}{2} < i \le n. \end{cases}$$

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor \frac{n}{2}; & j = 1,3 \\ 1; & j = 2. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} n - \lfloor \frac{k}{2} \rfloor + i; & 1 \le i < \frac{n}{2} \\ \lceil \frac{k}{2} \rceil - \lfloor \frac{n+2}{4} \rfloor; & i = \frac{n}{2} \\ \frac{n+2}{2}; & \frac{n}{2} < i \le n-1. \end{cases}$$

$$f(u_iv_j) = \begin{cases} \lceil \frac{|E|}{3} \rceil; & 1 \le i \le \frac{n}{2};\ \ j = 1,3 \\ \lceil \frac{k}{2} \rceil + \frac{n}{2}; & 1 \le i \le \frac{n}{2};\ \ j = 2 \\ \lceil \frac{k}{2} \rceil - \lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor + 1; & \frac{n}{2} < i \le n;\ \ j = 1,3 \\ \frac{n+2}{2} - \lfloor \frac{i}{2} \rfloor; & \frac{n}{2} < i \le n;\ \ j = 2. \end{cases}$$

Under this assignment the weight of edges of $P_n + \overline{K_3}$, when n is even are:
For $1 \le i \le \frac{n}{2}$, $\ j = 1,3$, $wt(u_iv_j) = \frac{17n}{4} - i - \frac{nj}{4} + 2$.
The weights corresponds to the elements of $\{4n+1, ..., 3n+2\}$.
For $1 \le i < \frac{n}{2}$, $wt(u_iu_{i+1}) = 3n + 2 - i$.
The weights corresponds to the elements of $\{3n+1, ..., \frac{5n}{2} + 3\}$.
For $1 \le i \le \frac{n}{2}$; $\ j = 2$, $wt(u_iv_j) = \frac{5n}{2} + 3 - i$.
The weights corresponds to the elements of $\{\frac{5n}{2} + 2, ..., 2n + 3\}$.
For $\frac{n}{2} < i \le n$; $\ j = 1,3$, $wt(u_iv_j) = \frac{11n}{4} + \frac{7}{2} - i - \frac{j}{2}(\frac{n}{2} + 1)$.
The weights corresponds to the elements of $\{2n+2, ..., \frac{3n}{2} + 3, \frac{3n}{2} + 1, ...n + 2\}$.
For $i = \frac{n}{2}$, $wt(u_iu_{i+1}) = \frac{3n}{2} + 2$.
For $\frac{n}{2} < i \le n - 1$, $wt(u_iu_{i+1}) = \frac{3n}{2} + 2 - i$.
The weights corresponds to the elements of $\{n+1, ..., \frac{n}{2} + 3\}$.
For $\frac{n}{2} < i \le n$; $\ j = 2$, $wt(u_iv_j) = n + 3 - i$.
The weights corresponds to the elements of $\{\frac{n}{2} + 2, ..., 3\}$.

Case(3): When n is odd and $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$

$$f(u_i) = \begin{cases} k - i + 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{n+1}{2} - \lfloor \frac{i}{2} \rfloor; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor \lceil \frac{n}{2} \rceil; & j = 1,3 \\ 1 & j = 2. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} n - \lceil \frac{k}{2} \rceil + i - 1; & 1 \leq i < \lceil \frac{n}{2} \rceil \\ \lfloor \frac{k}{2} \rfloor - \lceil \frac{i+1}{2} \rceil; & i = \lceil \frac{n}{2} \rceil \\ \frac{n+1}{2}; & \lceil \frac{n}{2} \rceil < i \leq n-1. \end{cases}$$

$$f(u_iv_j) = \begin{cases} \lfloor \frac{|E|}{3} \rfloor; & 1 \leq i \leq \lceil \frac{n}{2} \rceil;\ j = 1,3 \\ \lfloor \frac{k}{2} \rfloor + \frac{n-3}{2}; & 1 \leq i \leq \lceil \frac{n}{2} \rceil;\ j = 2 \\ \lfloor \frac{k}{2} \rfloor - \lceil \frac{i}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \leq n;\ j = 1,3 \\ \frac{n+3}{2} - \lceil \frac{i}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \leq n;\ j = 2. \end{cases}$$

Case(4): When n is odd and $n \equiv 2 \pmod 3$

$$f(u_i) = \begin{cases} k - i + 1; & 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ \frac{n+1}{2} - \lfloor \frac{i}{2} \rfloor; & \lceil \frac{n}{2} \rceil < i \leq n. \end{cases}$$

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor \lceil \frac{n}{2} \rceil; & j = 1,3 \\ 1; & j = 2. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} n - \lfloor \frac{k}{2} \rfloor + i - 1; & 1 \leq i < \lceil \frac{n}{2} \rceil \\ \lceil \frac{k}{2} \rceil - \lceil \frac{i+1}{2} \rceil; & i = \lceil \frac{n}{2} \rceil \\ \frac{n+1}{2}; & \lceil \frac{n}{2} \rceil < i \leq n-1. \end{cases}$$

$$f(u_iv_j) = \begin{cases} \lceil \frac{|E|}{3} \rceil; & 1 \leq i \leq \lceil \frac{n}{2} \rceil;\ j = 1,3 \\ \lceil \frac{k}{2} \rceil + \frac{n-3}{2}; & 1 \leq i \leq \lceil \frac{n}{2} \rceil;\ j = 2 \\ \lceil \frac{k}{2} \rceil - \lceil \frac{i}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \leq n;\ j = 1,3 \\ \frac{n+3}{2} - \lceil \frac{i}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \leq n;\ j = 2. \end{cases}$$

Under this assignment the weight of edges of $P_n + \overline{K_3}$, when n is odd are:
For $1 \leq i \leq \lceil \frac{n}{2} \rceil;\ j = 1,3,\ wt(u_iv_j) = \frac{17n}{4} - i - \frac{(n+1)j}{4} + \frac{9}{4}$.
The weights corresponds to the elements of $\{4n+1, ..., 3n+1\}$.
For $1 \leq i < \lceil \frac{n}{2} \rceil$, $wt(u_iu_{i+1}) = 3n - i + 1$.
The weights corresponds to the elements of $\{3n, ..., \frac{5n}{2} + \frac{3}{2}\}$.
For $1 \leq i \leq \lceil \frac{n}{2} \rceil;\ \ j = 2,\ wt(u_iv_j) = \frac{5n}{2} + \frac{3}{2} - i$.
The weights corresponds to the elements of $\{\frac{5n}{2} + \frac{1}{2}, ..., 2n+1\}$.
For $\lceil \frac{n}{2} \rceil < i \leq n;\ \ j = 1,3,\ wt(u_iv_j) = \frac{11n}{4} - i - \frac{(n+1)j}{4} + \frac{7}{4}$.
The weights corresponds to the elements of $\{2n, ..., \frac{3n}{2} + \frac{3}{2}, \frac{3n}{2} - \frac{1}{2}, ..., n+1\}$.
For $i = \lceil \frac{n}{2} \rceil$, $wt(u_iu_{i+1}) = \frac{3n}{2} + \frac{1}{2}$.
For $\lceil \frac{n}{2} \rceil < i \leq n-1$, $wt(u_iu_{i+1}) = \frac{3(n+1)}{2} - i$.
The weights corresponds to the elements of $\{n, ..., \frac{n}{2} + \frac{5}{2}\}$.
For $\lceil \frac{n}{2} \rceil < i \leq n; j = 2$, $wt(u_iv_j) = n + 3 - i$.
The weights corresponds to the elements of $\{\frac{n}{2} + \frac{3}{2}, ..., 3\}$.
From the above two cases, the weights of the edges of $P_n + \overline{K_3}$, $n \geq 4$ under the labeling f constitute the set $\{3, 4, ..., 4n+1\}$ and are distinct. So f is a total edge irregular k-labeling. Thus $tes(P_n + \overline{K_3}) \leq k$. Combining this with the lower bound, we conclude that $tes(P_n + \overline{K_3}) = \lceil \frac{4n+1}{3} \rceil$. □

Theorem 2.4. *For $m \geq 3$, $tes(P_2 + \overline{K_m}) = \lceil \frac{2m+3}{3} \rceil$ and $tes(P_3 + \overline{K_m}) = \lceil \frac{3m+4}{3} \rceil$.*

Proof. (1) Let $V(P_2 + \overline{K_m}) = \{u_i : 1{\le}i{\le}2\} \cup \{v_j : 1{\le}j{\le}m\}$ and $E(P_2 + \overline{K_m}) = \{u_1u_2\} \cup \{u_iv_j : 1{\le}i{\le}2 \text{ and } 1{\le}j{\le}m\}$. We have $tes(P_2 + \overline{K_m}){\ge}\lceil \frac{2m+3}{3} \rceil$.
Take $k = \lceil \frac{2m+3}{3} \rceil$.
Thus $tes(P_2 + \overline{K_m}) \ge k$. Now we define a k - labeling $f : V(P_2 + \overline{K_m}){\cup}E(P_2 + \overline{K_m}) \to \{1, 2, ...k\}$ as : $f(u_1) = k$, $f(u_2) = 1$.

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor; & 1{\le}j{\le}m \text{ and } j \text{ odd} \\ \frac{j}{2}; & 1{\le}j{\le}m \text{ and } j \text{ even.} \end{cases}$$

Case(1): When $m \equiv 0 \pmod 3$ or $m \equiv 1 \pmod 3$

$$f(u_1u_2) = \lceil \frac{k}{2} \rceil.$$

$$f(u_1v_j) = \begin{cases} \lceil \frac{|E|}{3} \rceil; & 1{\le}j{\le}m \text{ and } j \text{ odd} \\ \lceil \frac{k}{2} \rceil + 1; & 1{\le}j{\le}m \text{ and } j \text{ even.} \end{cases}$$

$$f(u_2v_j) = \begin{cases} \lceil \frac{k}{2} \rceil - 1; & 1{\le}j{\le}m \text{ and } j \text{ odd} \\ 1; & 1{\le}j{\le}m \text{ and } j \text{ even.} \end{cases}$$

Case(2): When $m \equiv 2 \pmod 3$

$$f(u_1u_2) = \lfloor \frac{k}{2} \rfloor.$$

$$f(u_1v_j) = \begin{cases} \lfloor \frac{|E|}{3} \rfloor; & 1{\le}j{\le}m \text{ and } j \text{ odd} \\ \lfloor \frac{k}{2} \rfloor + 1; & 1{\le}j{\le}m \text{ and } j \text{ even.} \end{cases}$$

$$f(u_2v_j) = \begin{cases} \lfloor \frac{k}{2} \rfloor - 1; & 1{\le}j{\le}m \text{ and } j \text{ odd} \\ 1; & 1{\le}j{\le}m \text{ and } j \text{ even.} \end{cases}$$

Under this assignment the weight of edges of $P_2 + \overline{K_m}$ are:
For $1{\le}j{\le}m$ and j odd, $wt(u_1v_j) = 2m + 3 - \lfloor \frac{j}{2} \rfloor$.
When m is even, weights=$\{2m + 3, ..., \frac{3m}{2} + 4\}$.
When m is odd, weights=$\{2m + 3, ..., \frac{3m}{2} + \frac{7}{2}\}$.
For $1{\le}j{\le}m$ and j even, $wt(u_1v_j) = m + 3 + \frac{j}{2}$.
When m is even, weights=$\{\frac{3m}{2} + 3, ..., m + 4\}$.
When m is odd, weights=$\{\frac{3m}{2} + \frac{5}{2}, ..., m + 4\}$.
$wt(u_1u_2) = m + 3$.
For $1{\le}j{\le}m$ and j odd, $wt(u_2v_j) = m + 2 - \lfloor \frac{j}{2} \rfloor$.
When m is even, weights=$\{m + 2, ..., \frac{m}{2} + 3\}$.
When m is odd, weights=$\{m + 2, ..., \frac{m}{2} + \frac{5}{2}\}$.
For $1{\le}j{\le}m$ and j even, $wt(u_2v_j) = 2 + \frac{j}{2}$.
When m is even, weights=$\{\frac{m}{2} + 2, ..., 3\}$.
When m is odd, weights=$\{\frac{m}{2} + \frac{3}{2}, ..., 3\}$.
The weights of the $2m + 1$ edges of $P_2 + \overline{K_m}$, $m{\ge}3$ under the labeling f constitute the set $\{3, 4, ..., 2m + 3\}$ and f is a mapping from $V(P_2 + \overline{K_m}){\cup}E(P_2 + \overline{K_m})$ into $\{1, 2, ..., k\}$. Thus f is a total edge irregular k- labeling and $tes(P_2 + \overline{K_m}){\le}k$. Combining this with the lower bound, we conclude that $tes(P_2 + \overline{K_m}) = \lceil \frac{2m+3}{3} \rceil$.

(2) Let $V(P_3 + \overline{K_m}) = \{u_i : 1{\le}i{\le}3\} \cup \{v_j : 1{\le}j{\le}m\}$ and
$E(P_2 + \overline{K_m}) = \{u_iu_{i+1} : 1{\le}i{\le}2\} \cup \{u_iv_j : 1{\le}i{\le}3 \text{ and } 1{\le}j{\le}m\}$. We have

$tes(P_3+\overline{K_m})\geq\lceil\frac{3m+4}{3}\rceil$. Take $k=\lceil\frac{3m+4}{3}\rceil$.
Thus $tes(P_3+\overline{K_m})\geq k$. Now we define a k - labeling $f:V(P_3+\overline{K_m})\cup E(P_3+\overline{K_m})\rightarrow\{1,2,...k\}$ as :

$$f(u_i)=\begin{cases}k; & i=1\\ 4-i; & i=2,3.\end{cases}$$

$$f(v_j)=\begin{cases}k-\lfloor\frac{j}{2}\rfloor; & 1\leq j\leq m \text{ and } j \text{ odd}\\ \frac{j}{2}; & 1\leq j\leq m \text{ and } j \text{ even.}\end{cases}$$

$$f(u_iu_{i+1})=\begin{cases}m; & i=1\\ 2;\lfloor\frac{m}{2}\rfloor & i=2.\end{cases}$$

$$f(u_iv_j)=\begin{cases}m; & i=1;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ k; & i=1;\ 1\leq j\leq m \text{ and } j \text{ even}\\ m-\lceil\frac{j}{2}\rceil; & i=2,3;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \frac{j}{2}; & i=2,3;\ 1\leq j\leq m \text{ and } j \text{ even.}\end{cases}$$

Under this assignment the weight of edges of $P_3+\overline{K_m}$ are:
For $1\leq j\leq m$ and j odd, $wt(u_1v_j)=3m+\frac{9}{2}-\frac{j}{2}$.
When m is even, weights$=\{3m+4,...,\frac{5m}{2}+5\}$.
When m is odd, weights$=\{3m+4,...,\frac{5m}{2}+\frac{9}{2}\}$.
For $1\leq j\leq m$ and j even, $wt(u_1v_j)=2m+4+\frac{j}{2}$.
When m is even, weights$=\{\frac{5m}{2}+4,...,2m+5\}$.
When m is odd, weights$=\{\frac{5m}{2}+\frac{7}{2},...,2m+5\}$.
$wt(u_1u_2)=2m+4$.
For $i=2,3$; $1\leq j\leq m$ and j odd , $wt(u_iv_j)=6+2m-i-j$.
When m is even, weights$=\{2m+3,...,m+4\}$.
When m is odd, weights$=\{2m+3,...,m+3\}$.
$wt(u_2u_3)=3+2\lfloor\frac{m}{2}\rfloor$.
When m is even, weight$=m+3$.
When m is odd, weight$=m+2$.
For $i=2,3$; $1\leq j\leq m$ and j even, $wt(u_iv_j)=4-i+j$.
When m is even, weights$=\{m+2,...,3\}$.
When m is odd, weights$=\{m+1,...,3\}$.

The weights of the edges of $P_3+\overline{K_m}$, $m\geq 3$ under the labeling f constitute the set $\{3,4,...,3m+4\}$ and are distinct. Thus f is a total edge irregular k- labeling and $tes(P_3+\overline{K_m})\leq k$. Combining this with the lower bound, we conclude that $tes(P_3+\overline{K_m})=\lceil\frac{3m+4}{3}\rceil$. □

Finally, the total edge irregularity strength of $P_n+\overline{K_m}$ for $n\geq 4$ and $m\geq 4$:

Theorem 2.5. *For $n,m\geq 4$, $tes(P_n+\overline{K_m})=\lceil\frac{n+nm+1}{3}\rceil$.*

Proof. Let $V(P_n+\overline{K_m})=\{u_i:1\leq i\leq n\}\cup\{v_j:1\leq j\leq m\}$ and
$E(P_n+\overline{K_m})=\{u_iu_{i+1}:1\leq i\leq n-1\}\cup\{u_iv_j:1\leq i\leq n \text{ and } 1\leq j\leq m\}$.
We have $tes(P_n+\overline{K_m})\geq\lceil\frac{n+nm+1}{3}\rceil$. Take $k=\lceil\frac{n+nm+1}{3}\rceil$.
Thus $tes(P_n+\overline{K_m})\geq k$. Now we define a k - labeling $f:V(P_n+\overline{K_m})\cup E(P_n+\overline{K_m})\rightarrow$

$\{1, 2, ...k\}$ as :

$$f(u_i) = \begin{cases} k - i + 1; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ n - i + 1; & \lfloor \frac{n}{2} \rfloor < i \leq n. \end{cases}$$

$$f(v_j) = \begin{cases} k - \lfloor \frac{j}{2} \rfloor \lfloor \frac{n}{2} \rfloor; & 1 \leq j \leq m \text{ and } j \text{ odd} \\ 1 + \lfloor \frac{j-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor; & 1 \leq j \leq m \text{ and } j \text{ even}. \end{cases}$$

Case(1): When n is even and $|E| \equiv 0 \pmod 3$ or $|E| \equiv 1 \pmod 3$

$$f(u_i u_{i+1}) = \begin{cases} \lceil \frac{|E|}{3} \rceil - \frac{n}{2} \lceil \frac{m}{2} \rceil + i; & 1 \leq i < \frac{n}{2} \\ \lceil \frac{k}{2} \rceil; & i = \frac{n}{2} \\ \frac{n}{2} \lfloor \frac{m}{2} \rfloor + i - n + 1; & \frac{n}{2} < i \leq n - 1. \end{cases}$$

$$f(u_i v_j) = \begin{cases} \lceil \frac{|E|}{3} \rceil; & 1 \leq i \leq \frac{n}{2};\ 1 \leq j \leq m \text{ and } j \text{ odd} \\ \lceil \frac{k}{2} \rceil + \frac{n}{2}; & 1 \leq i \leq \frac{n}{2};\ 1 \leq j \leq m \text{ and } j \text{ even} \\ \lceil \frac{k}{2} \rceil - \frac{n}{2}; & \frac{n}{2} < i \leq n;\ 1 \leq j \leq m \text{ and } j \text{ odd} \\ 1; & \frac{n}{2} < i \leq n;\ 1 \leq j \leq m \text{ and } j \text{ even}. \end{cases}$$

Case(2): When n is even and $|E| \equiv 2 \pmod 3$

$$f(u_i u_{i+1}) = \begin{cases} \lfloor \frac{|E|}{3} \rfloor - \frac{n}{2} \lceil \frac{m}{2} \rceil + i; & 1 \leq i < \frac{n}{2} \\ \lfloor \frac{k}{2} \rfloor; & i = \frac{n}{2} \\ \frac{n}{2} \lfloor \frac{m}{2} \rfloor + i - n + 1; & \frac{n}{2} < i \leq n - 1. \end{cases}$$

$$f(u_i v_j) = \begin{cases} \lfloor \frac{|E|}{3} \rfloor; & 1 \leq i \leq \frac{n}{2};\ 1 \leq j \leq m \text{ and } j \text{ odd} \\ \lfloor \frac{k}{2} \rfloor + \frac{n}{2}; & 1 \leq i \leq \frac{n}{2};\ 1 \leq j \leq m \text{ and } j \text{ even} \\ \lfloor \frac{k}{2} \rfloor - \frac{n}{2}; & \frac{n}{2} < i \leq n;\ 1 \leq j \leq m \text{ and } j \text{ odd} \\ 1; & \frac{n}{2} < i \leq n;\ 1 \leq j \leq m \text{ and } j \text{ even}. \end{cases}$$

Under this assignment the weight of edges of $P_n + \overline{K_m}$ when n is even are:
For $1 \leq i \leq \frac{n}{2},\ 1 \leq j \leq m$ and j odd, $wt(u_i v_j) = \frac{5n}{4} + nm - i - \frac{nj}{4} + 2$.
When m is even, weights=$\{n + nm + 1, ..., n + \frac{3nm}{4} + 2\}$.
When m is odd, weights=$\{n + nm + 1, ..., \frac{3n}{4} + \frac{3nm}{4} + 2\}$.
For $1 \leq i < \frac{n}{2}$, $wt(u_i u_{i+1}) = n + nm - \frac{n}{2} \lceil \frac{m}{2} \rceil + 2 - i$.
When m is even, weights=$\{n + \frac{3nm}{4} + 1, ..., \frac{n}{2} + \frac{3nm}{4} + 3\}$.
When m is odd, weights=$\{\frac{3n}{4} + \frac{3nm}{4} + 1, ..., \frac{n}{4} + \frac{3nm}{4} + 3\}$.
For $1 \leq i \leq \frac{n}{2};\ 1 \leq j \leq m$ and j even, $wt(u_i v_j) = \frac{n}{2} + \frac{nm}{2} + \frac{nj}{4} + 3 - i$.
When m is even, weights=$\{\frac{n}{2} + \frac{3nm}{4} + 2, ..., \frac{n}{2} + \frac{nm}{2} + 3\}$.
When m is odd, weights=$\{\frac{n}{4} + \frac{3nm}{4} + 2, ..., \frac{n}{2} + \frac{nm}{2} + 3\}$.
For $i = \frac{n}{2}$, $wt(u_i u_{i+1}) = \frac{n}{2} + \frac{nm}{2} + 2$.
For $\frac{n}{2} < i \leq n;\ 1 \leq j \leq m$ and j odd, $wt(u_i v_j) = \frac{5n}{4} + \frac{nm}{2} - i - \frac{nj}{4} + 2$.
When m is even, weights=$\{\frac{n}{2} + \frac{nm}{2} + 1, ..., \frac{n}{2} + \frac{nm}{4} + 2\}$.
When m is odd, weights=$\{\frac{n}{2} + \frac{nm}{2} + 1, ..., \frac{n}{4} + \frac{nm}{4} + 2\}$.
For $\frac{n}{2} < i \leq n - 1$, $wt(u_i u_{i+1}) = n + \frac{n}{2} \lfloor \frac{m}{2} \rfloor + 2 - i$.
When m is even, weights=$\{\frac{n}{2} + \frac{nm}{4} + 1, ..., \frac{nm}{4} + 3\}$.
When m is odd, weights=$\{\frac{n}{4} + \frac{nm}{4} + 1, ..., \frac{nm}{4} - \frac{n}{4} + 3\}$.
For $\frac{n}{2} < i \leq n;\ 1 \leq j \leq m$ and j even, $wt(u_i v_j) = \frac{n}{2} + 3 + \frac{nj}{4} - i$.
When m is even, weights=$\{\frac{nm}{4} + 2, ..., 3\}$.

When m is odd, weights=$\{\frac{nm}{4}-\frac{n}{4}+2,...,3\}$.

Case(3): When n is odd and $|E|\equiv 0 \pmod 3$

$$f(u_iu_{i+1})=\begin{cases}\frac{|E|}{3}-\lfloor\frac{n}{2}\rfloor\lceil\frac{m}{2}\rceil+i; & 1\leq i<\lfloor\frac{n}{2}\rfloor\\ \lfloor\frac{|E|}{6}\rfloor+\lceil\frac{m}{2}\rceil; & i=\lfloor\frac{n}{2}\rfloor\\ \lceil\frac{n}{2}\rceil\lfloor\frac{m}{2}\rfloor+i-n+1; & \lfloor\frac{n}{2}\rfloor<i\leq n-1.\end{cases}$$

$$f(u_iv_j)=\begin{cases}\frac{|E|}{3}; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \lfloor\frac{k}{2}\rfloor+\lceil\frac{m+n}{2}\rceil; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ even}\\ \lfloor\frac{k}{2}\rfloor+\lceil\frac{m-n}{2}\rceil-\lfloor\frac{j}{2}\rfloor; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \frac{j}{2}; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ even.}\end{cases}$$

Case(4): When n is odd and $|E|\equiv 1 \pmod 3$

$$f(u_iu_{i+1})=\begin{cases}\lceil\frac{|E|}{3}\rceil-\lfloor\frac{n}{2}\rfloor\lceil\frac{m}{2}\rceil+i; & 1\leq i<\lfloor\frac{n}{2}\rfloor\\ \lceil\frac{|E|}{6}\rceil+\lfloor\frac{m}{2}\rfloor; & i=\lfloor\frac{n}{2}\rfloor\\ \lceil\frac{n}{2}\rceil\lfloor\frac{m}{2}\rfloor+i-n+1; & \lfloor\frac{n}{2}\rfloor<i\leq n-1.\end{cases}$$

$$f(u_iv_j)=\begin{cases}\lceil\frac{|E|}{3}\rceil; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \lceil\frac{k}{2}\rceil+\lceil\frac{m+n}{2}\rceil; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ even}\\ \lceil\frac{k}{2}\rceil+\lceil\frac{m-n}{2}\rceil-\lfloor\frac{j}{2}\rfloor; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \frac{j}{2}; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ even.}\end{cases}$$

Case(5): When n is odd and $|E|\equiv 2 \pmod 3$

$$f(u_iu_{i+1})=\begin{cases}\lfloor\frac{|E|}{3}\rfloor-\lfloor\frac{n}{2}\rfloor\lceil\frac{m}{2}\rceil+i; & 1\leq i<\lfloor\frac{n}{2}\rfloor\\ \lfloor\frac{|E|}{6}\rfloor+\lceil\frac{m}{2}\rceil; & i=\lfloor\frac{n}{2}\rfloor\\ \lceil\frac{n}{2}\rceil\lfloor\frac{m}{2}\rfloor+i-n+1; & \lfloor\frac{n}{2}\rfloor<i\leq n-1.\end{cases}$$

$$f(u_iv_j)=\begin{cases}\lfloor\frac{|E|}{3}\rfloor; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \lfloor\frac{k}{2}\rfloor+\lfloor\frac{m+n}{2}\rfloor; & 1\leq i\leq\lfloor\frac{n}{2}\rfloor;\ 1\leq j\leq m \text{ and } j \text{ even}\\ \lfloor\frac{k}{2}\rfloor+\lfloor\frac{m-n}{2}\rfloor-\lfloor\frac{j}{2}\rfloor; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ odd}\\ \frac{j}{2}; & \lfloor\frac{n}{2}\rfloor<i\leq n;\ 1\leq j\leq m \text{ and } j \text{ even.}\end{cases}$$

Under this assignment the weight of edges of $P_n+\overline{K_m}$ when n is odd are:
For $1\leq i\leq\lfloor\frac{n}{2}\rfloor$, $1\leq j\leq m$ and j odd, $wt(u_iv_j)=\frac{5n}{4}+nm-i-\frac{(n-1)j}{4}+\frac{7}{4}$.
When m is even, weights=$\{n+nm+1,...,n+\frac{3nm}{4}+\frac{m}{4}+2\}$.
When m is odd, weights=$\{n+nm+1,...,\frac{3n}{4}+\frac{3nm}{4}+\frac{m}{4}+\frac{9}{4}\}$.
For $1\leq i<\lfloor\frac{n}{2}\rfloor$, $wt(u_iu_{i+1})=n+nm-(\frac{n-1}{2})\lceil\frac{m}{2}\rceil+2-i$.
When m is even, weights=$\{n+\frac{3nm}{4}+\frac{m}{4}+1,...,\frac{n}{2}+\frac{3nm}{4}+\frac{m}{4}+\frac{7}{2}\}$.
When m is odd, weights=$\{\frac{3n}{4}+\frac{3nm}{4}+\frac{m}{4}+\frac{5}{4},...,\frac{n}{4}+\frac{3nm}{4}+\frac{m}{4}+\frac{15}{4}\}$.
For $1\leq i\leq\lfloor\frac{n}{2}\rfloor$; $1\leq j\leq m$ and j even, $wt(u_iv_j)=\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{(n-1)j}{4}+\frac{7}{2}-i$.
When m is even, weights=$\{\frac{n}{2}+\frac{3nm}{4}+\frac{m}{4}+\frac{5}{2},...,\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{7}{2}\}$.
When m is odd, weights=$\{\frac{n}{4}+\frac{3nm}{4}+\frac{m}{4}+\frac{11}{4},...,\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{7}{2}\}$.
For $i=\lfloor\frac{n}{2}\rfloor$, $wt(u_iu_{i+1})=\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{5}{2}$.
For $\lfloor\frac{n}{2}\rfloor<i\leq n$; $1\leq j\leq m$ and j odd, $wt(u_iv_j)=\frac{5n}{4}+\frac{nm}{2}+\frac{m}{2}-i-\frac{(n+1)j}{4}+\frac{9}{4}$.
When m is even, weights=$\{\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{3}{2},...,\frac{n}{2}+\frac{nm}{4}+\frac{m}{4}+\frac{5}{2}\}$.

When m is odd, weights=$\{\frac{n}{2}+\frac{nm}{2}+\frac{m}{2}+\frac{3}{2},...,\frac{n}{4}+\frac{nm}{4}+\frac{m}{4}+\frac{9}{4}\}$.
For $\lfloor\frac{n}{2}\rfloor<i\leq n-1$, $wt(u_iu_{i+1})=n+\lceil\frac{n}{2}\rceil\lfloor\frac{m}{2}\rfloor+2-i$.
When m is even, weights=$\{\frac{n}{2}+\frac{nm}{4}+\frac{m}{4}+\frac{3}{2},...,\frac{nm}{4}+\frac{m}{4}+3\}$.
When m is odd, weights=$\{\frac{n}{4}+\frac{nm}{4}+\frac{m}{4}+\frac{5}{4},...,\frac{nm}{4}-\frac{n}{4}+\frac{m}{4}+\frac{11}{4}\}$.
For $\lfloor\frac{n}{2}\rfloor<i\leq n$; $1\leq j\leq m$ and j even, $wt(u_iv_j)=\frac{n}{2}+\frac{5}{2}+\frac{(n+1)j}{4}-i$.
When m is even, weights=$\{\frac{nm+m}{4}+2,...,3\}$.
When m is odd, weights=$\{\frac{nm+m}{4}-\frac{n}{4}+\frac{7}{4},...,3\}$.
From the above cases, the weights of the $n+nm-1$ edges of $P_n+\overline{K_m}$, for $n\geq 4$ and $m\geq 4$ under the labeling f constitute the set $\{3,4,...,n+nm+1\}$ and are distinct. Thus f is a total edge irregular k- labeling and $tes(P_n+\overline{K_m})\leq k$. Combining this with the lower bound, we conclude that $tes(P_n+\overline{K_m})=\lceil\frac{n+nm+1}{3}\rceil$. □

3. Conclusion

The exact value of the total edge irregularity strength is known only for few classes of graphs. The lower and upper bound of the total edge irregularity strength of a graph was found earlier. In this paper, we obtained the exact value of total edge irregularity strength of join of path and complement of complete graph. We found that the lower bound itself is its total edge irregularity strength. Similar results can be obtained for other classes of graphs also. Future study can be done to find the exact value of total edge irregularity strength of join of path and complete graph.

Acknowledgement. The first author thanks the University Grants Commission of India for providing financial support for carrying out research through their Junior Research Fellowship (JRF) scheme.

References

[1] Aigner, M. and Triesch, E., (1990), Irregular assignments of trees and forests, SIAM J. Discrete Math, 3, pp. 439-449.
[2] Bača, M., Jendrol', S., Miller, M. and Ryan, J., (2007), On irregular total labeling, Discrete Math, 307, pp. 1378-1388.
[3] Chartrand, G. , Jacobson, M.S., Lehel, J., Oellermann, O. R., Ruiz, S. and Saba, F., (1988), Irregular networks, Congr. Numer, 64, pp. 355-374.
[4] Douglas B. West, (2015), Introduction to graph theory, Pearson, 2nd Edition.
[5] Gallian, J. A., (2015), A dynamic survey of graph labeling , The Electronic Journal of Combinatorics, 18, pp. 247-252.
[6] Indriati, D., Widodo, Wijayanti, I. E. and Sugeng, K. A., (2013), On the total edge irregularity Strength of generalized helm, AKCE International Journal Graphs and Combinatorics, 10(2), pp. 147-155.
[7] Ivančo, J., Jendrol',S., (2006), The total edge irregularity strength of trees, Discuss. Math. Graph Theory, 26, pp. 449-456.
[8] Jendrol', S., Miskůf, J. and Soták, R., (2010), Total edge irregularity strength of complete graphs and complete bipartite graphs, Discrete Mathemathics, 310(3), pp. 400–407.
[9] Jinnah, M. I. and Santhosh Kumar, K. R., (2012) Irregularity strength of triangular snake and double triangular snake, Advances and Applications in Discrete Mathematics, 9(2), pp. 83-92.
[10] Kalkowski, M., Karonski, M. and Pfender, F., (2011), A new upper bound for the irregularity strength of graphs, SIAM J. Discrete Math, 25(3), pp. 1319-132
[11] Muthu Guru Packiam, K., Manimaran, T. and Thuraiswamy, A., (2016), On total edge irregularity strength of graph, Ars Combin, 129, pp. 173-183.
[12] Pryzbylo, J., (2008), Irregularity strength of regular Graphs, Electron. J. of Combin., 15, R82.

A. S. Saranya is a PhD scholar in Mathematics at the Department of Mathematics, University College, Thiruvananthapuram. She completed her graduation and post graduation in mathematics at the Government Women's College, Thiruvananthapuram. She was qualified UGC JRF on 2016. She was awarded Mphil degree in 2017 from University College, Thiruvananthapuram. She is doing her PhD under the guidance of Dr. K. R. Santhosh Kumar. Her research area is graph theory. Her special focus is on graph labeling.

K. R. Santhosh Kumar is an Assosciate Professor of Mathematics, at the University College, Thiruvananthapuram. He completed his graduation and post graduation at the University of Kerala. In 2013, he was awarded with PhD from Department of Mathematics, University of Kerala under the guidance of Dr M. I. Jinnah. His major research area is Graph Theory. He is interested mainly in Graph labeling, Algebraic Graph Theory and Domination in graphs.