TWMS J. App. Eng. Math. V.6, N.1, 2016, pp. 135-142

# ANALYTICAL AND NUMERICAL ASPECTS OF THE DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATION

C. BAYINDIR[1], §

Abstract. In this paper various analytical and numerical aspects of the dissipative nonlinear Schrödinger equation (d-NLS equation) are discussed. Decaying solitary wave type solutions derived by Demiray is reviewed and a new approximate solitary wave type solution of the d-NLS equation is introduced in order to make comparisons. Also a split-step Fourier scheme is proposed for numerical solution of the d-NLS equation and the analytical solutions are compared with the numerical results.

Keywords: Dissipative nonlinear Schrödinger equation, solitary waves, spectral methods, split-step Fourier method.

AMS Subject Classification: 35L05, 65T50, 65Z05

## 1. Introduction

In nonlinear wave theory it is known that the nonlinear Schrödinger (NLS) equation is the simplest equation that describes self-modulating monochromatic waves in dispersive medium [10]. In addition to its many uses in acoustics, optics, plasma physics and quantum mechanics it can also be used to model the evolution of the weakly nonlinear water wave packets on the surface of a deep water [19]. If not only a dispersive but also a dissipative medium is considered, then the first-order amplitude modulation can be described by the dissipative nonlinear Schrödinger equation (d-NLS equation). Therefore the d-NLS equation can be used to model the dissipative self-modulating monochromatic waves with dispersion.

In this paper we discuss various analytical and numerical aspects of the dissipative nonlinear Schrödinger equation. Decaying solitary wave solutions of sech type derived by Demiray [10] is reviewed and a new approximate dissipative solution of d-NLS equation is introduced to assess this solution. Additionally a split-step Fourier scheme is proposed for numerical solution of the d-NSE. The analytical solutions are compared with the numerical results. It is shown that both the approximate and Demiray solutions agree well with the numerical results for the solitary wave envelope however some phase mismatch can be observed in the real and imaginary parts. Agreement is much better in the zero dissipation limit as expected.

[1] Işık University, Civil Engineering Department, 34980, Şile, İstanbul.
e-mail: cihan.bayindir@isikun.edu.tr;

§ Manuscript received: July 14, 2015; Accepted: February 15, 2016.
TWMS Journal of Applied and Engineering Mathematics, Vol.6, No.1; 

## 2. Methodology

The d-NLS equation can be given as

$$i\eta_t + \mu_1\eta_{xx} + \mu_2 |\eta|^2 \eta + i\mu_3\eta = 0 \tag{1}$$

where $\mu_1, \mu_2, \mu_3$ are constants which represent dispersion, nonlinearity and dissipative effects, respectively. When the dissipative effects are weak, i.e, $\mu_3 \approx 0$, the d-NLS equation reduces to NLS equation of the form

$$i\eta_t + \mu_1\eta_{xx} + \mu_2 |\eta|^2 \eta = 0 \tag{2}$$

It is known that NLS equation has exact localized traveling wave solutions [10, 17] in the form of

$$\eta(x,t) = V(\zeta)\exp[i(Kx - \Omega t)], \quad \zeta = x - \nu_0 t, \quad \nu_0 = const., \tag{3}$$

where

$$V(\zeta) = a\text{sech}\left[\left(\frac{\mu_2}{2\mu_1}\right)a\zeta\right], \quad \Omega = \mu_1 K^2 - \mu_2 a^2/2, \tag{4}$$

for $\mu_1\mu_2 > 0$ and

$$V(\zeta) = a\tanh\left[\left(-\frac{\mu_2}{2\mu_1}\right)a\zeta\right], \quad \Omega = \mu_1 K^2 - \mu_2 a^2, \tag{5}$$

for $\mu_1\mu_2 < 0$. Here $a$ is the amplitude of the solitary wave and $v_0 = 2\mu_1 K$ is a constant which represents the solitary wave celerity.

**Dissipative sech-type solitons.**

*Review of the Demiray sech-type soliton.* In this section we give a brief review of the analytical solutions derived by Demiray [10]. Motivated with the solutions of NLS given in (3)-(5), we seek a solution to the d-NLS equation given in (1) in the form of

$$\eta(x,t) = a(t)V(\zeta)\exp\{i[Kx - \Omega(t)]\}, \quad \zeta = \alpha(t)[x - 2\mu_1 Kt], \tag{6}$$

where $K$ is constant and $a, \alpha, V$ are some real valued functions [10]. This leads to the ordinary differential equations of

$$\left[\frac{a'(t)}{a(t)} + \mu_3\right]V + \frac{\alpha'(t)}{\alpha(t)}\zeta V' = 0, \tag{7}$$

and

$$[\Omega'(t) - \mu_1 K^2]V + \mu_1\alpha(t)^2 V'' + \mu_2 a^2(t)V^3 = 0, \tag{8}$$

which arises from real and imaginary parts, respectively. Multiplying (7) by $V$ and integrating the product with respect to $\zeta$ from $-\infty$ to $\infty$ we obtain [10]

$$\left[\frac{a'(t)}{a(t)} - \frac{1}{2}\frac{\alpha'(t)}{\alpha(t)} + \mu_3\right]\left\langle V^2\right\rangle = 0, \qquad \left\langle V^2\right\rangle = \int_{-\infty}^{\infty} V^2 d\zeta. \tag{9}$$

Seeking a bounded and nonzero solution with the condition $\left\langle V^2\right\rangle < \infty$, we obtain[10]

$$\frac{a'(t)}{a(t)} - \frac{1}{2}\frac{\alpha'(t)}{\alpha(t)} + \mu_3 = 0. \tag{10}$$

Multiplying both sides of (8) by $V'$ and integrating the product over $\zeta$ results in

$$[\Omega'(t) - \mu_1 K^2]V^2 + \mu_1\alpha(t)^2(V')^2 + \frac{\mu_2 a(t)}{2}V^4 = 0, \tag{11}$$

Introducing a new variable as

$$\psi(\zeta) = V(\zeta)^2 \tag{12}$$

and using this in (11) results in

$$[\Omega'(t) - \mu_1 K^2]\psi^2 + \frac{\mu_1 \alpha(t)^2}{4}(\psi')^2 + \frac{\mu_2 a(t)^2}{2}\psi^3 = 0, \tag{13}$$

At this point one can introduce a new independent variable as[10]

$$y = \tanh\zeta, \qquad -1 \leq y \leq 1. \tag{14}$$

Seeking a solution of the form[10]

$$\psi = 1 - y^2 \tag{15}$$

to the (13) and realizing that $d/d\zeta = (1-y^2)d/dy$ and setting the coefficients of different powers of y equal to zero we arrive at[10]

$$\alpha(t)^2 = \frac{\mu_2}{2\mu_1}a(t)^2, \qquad \Omega'(t) = \frac{-\mu_2}{2}a(t)^2 + \mu_1 K^2. \tag{16}$$

In order $\alpha$ to be real the $\mu_1\mu_2 > 0$ condition must be satisfied. Also using (16) it can be shown that $\alpha'(t)\alpha(t) = a'(t)/a(t)$. Using this expression back in (10) we obtain[10]

$$\frac{a'(t)}{a(t)} + 2\mu_3 = 0. \tag{17}$$

The solution of this equation can be easily written as

$$a(t) = a_0 e^{-2\mu_3 t}, \tag{18}$$

where $a_0$ is a constant which represents the amplitude of the waveform. Using (18) back in (16) we obtain [10]

$$\alpha(t) = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2} a_0 e^{-2\mu_3 t}, \qquad \Omega(t) = \mu_1 K^2 t + \frac{\mu_2 a_0^2}{8\mu_3}[1 - e^{-4\mu_3 t}]. \tag{19}$$

Therefore the sech type Demiray soliton solution of the d-NLS equation becomes

$$\eta_D(x,t) = a_0 e^{-2\mu_3 t}\text{sech}[\zeta]\exp\{i[\Omega(t) - Kx]\}, \qquad \zeta = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2} a(t)(x - 2\mu_1 K t). \tag{20}$$

It is important to note that this solution is a solution in the averaged sense, not in the classical sense [10]. This equation shows that not only amplitude of the waveform decays but also the lobe width increases with time.

*A new approximate dissipative soliton.* In this section we propose a new approximate solution of the d-NLS equation to make an assessment of the Demiray solution. Seeking a solution of the form

$$\eta_{app}(x,t) \approx V(\zeta)\exp[i(Kx - \Omega t) - \phi(t)], \quad \zeta = x - \nu_0 t, \quad \nu_0 = const., \tag{21}$$

to the d-NLS equation given in (1), where $\phi$ is a real function, we arrive at the approximate solution as

$$\eta_{app}(x,t) \approx V(\zeta)\exp[i(Kx - \Omega t) - \mu_3 t], \quad \zeta = x - \nu_0 t, \quad \nu_0 = const., \tag{22}$$

It is obvious that, smaller the dissipation ($\mu_3$), better the approximation. Again $V(\zeta)$ is given by

$$V(\zeta) = a\text{sech}\left[\left(\frac{\mu_2}{2\mu_1}\right)a\zeta\right], \quad \Omega = \mu_1 K^2 - \mu_2 a^2/2, \tag{23}$$

for $\mu_1\mu_2 > 0$ and

$$V(\zeta) = a\tanh\left[\left(-\frac{\mu_2}{2\mu_1}\right)a\zeta\right], \quad \Omega = \mu_1 K^2 - \mu_2 a^2, \tag{24}$$

for $\mu_1\mu_2 < 0$.

**A split-step Fourier scheme for dissipative nonlinear Schrdinger equation.** One of the most popular choices for numerical simulations of the differential equations is spectral methods. Spectral methods are common tools in numerical mathematics and they are used in modeling various phenomena of including but not limited to acoustics, hydrodynamics [2, 5, 13, 14], optics, heat conduction [9] etc. In spectral simulations, the spatial derivatives are evaluated using basis functions of the orthogonal transforms. In periodic domain the most popular choice is the Fourier basis functions therefore fast and inverse fast Fourier transforms are employed very frequently.
Numerical time integration is generally performed by schemes such as $4^{th}$ order Runge-Kutta or Adams-Bashforth [1, 4, 6, 7]. Although its historical development is much later, one of the most popular techniques is the split-step method [11, 12, 17]. The fundamental idea in the split-step method is to approximate the exact solution of the governing equation as the seperate solutions of the linear and nonlinear equations in a given sequential order [3], in which the solution of the nonlinear part is used as an initial condition for the linear part or vice versa. This operation results in a splitting error due to the non-commutativity of $\mathcal{L}$ and $\mathcal{N}$ [16]. One option is to use the Baker-Campbell-Hausdorf formula [16] to reduce the splitting error however in this study we suffice with first order splitting. A possible way of writing the nonlinear part of the equation d-NLS equation given in (1) is

$$i\eta_t = \mathcal{N}\eta = -\left(\mu_2 |\eta|^2 + i\mu_3\right)\eta \tag{25}$$

where $\mathcal{N} = -\left(\mu_2 |\eta|^2 + i\mu_3\right)$ is the nonlinear operator. The solution of this equation can be written as

$$\widetilde{\eta}(x, t_0 + \Delta t) \approx e^{\left(i\mu_2|\eta|^2-\mu_3\right)\Delta t}\eta(x, t_0) \tag{26}$$

where $\Delta t$ is the time step. The linear part of the d-NLS equation given in (1) can be recognized as

$$i\eta_t = \mathcal{L}\eta = -\mu_1\eta_{xx} \tag{27}$$

where $\mathcal{L} = -\mu_1\partial^2()/\partial x^2$ is the linear operator. The solution of this equation can be found using Fourier series

$$\eta(x, t_0 + \Delta t) \approx F^{-1}[e^{-i\mu_1 k^2\Delta t}F[\widetilde{\eta}(x, t_0 + \Delta t)]] \tag{28}$$

where $k, F, F^{-1}$ are the wavenumber vector, Fourier and inverse Fourier transform operations, respectively. Combining (26) and (28) we arrive at

$$\eta(x, t_0 + \Delta t) \approx F^{-1}[e^{-i\mu_1 k^2\Delta t}F[e^{\left(i\mu_2|\eta|^2-\mu_3\right)\Delta t}\eta(x, t_0)]] \tag{29}$$

This equation can be used to obtain the numerical solution of the d-NLS equation given in (1) starting from the initial conditions. In the numerical simulations we use $\Delta t = 0.01$ which does not cause any stability problems and has only minor truncation errors.

## 3. Results and Discussion

We focus on sech type solutions in this study but similar results can be easily obtained for other type of solutions such as kinks. The parameters in this study are selected as $a = 1, K = -2, \mu_1 = -1, \mu_2 = -2$. For these parameters the exact solution of the NLS equation given in (2) becomes

$$\eta(x, t) = a\exp\left[-i\left(2x - [4 - a^2]t\right)\right]\operatorname{sech}\left[a(x - 4t)\right], \tag{30}$$

Similarly the dissipative Demiray soliton solution of the d-NLS equation given in (20) becomes

$$\eta_D(x,t) = ae^{-2\mu_3 t} \exp\left[-i\left(2x - [4-a^2]t\right)\right] \operatorname{sech}\left[ae^{-2\mu_3 t}(x-4t)\right], \tag{31}$$

and the approximate dissipative soliton solution we propose in (22) becomes

$$\eta_{app}(x,t) \approx a \exp\left[-i\left(2x - [4-a^2]t\right) - \mu_3 t\right] \operatorname{sech}\left[a(x-4t)\right], \tag{32}$$

Starting from the initial conditions which can be described by setting $t = 0$ in (31) and (32), the numerical solution is obtained by using (29).

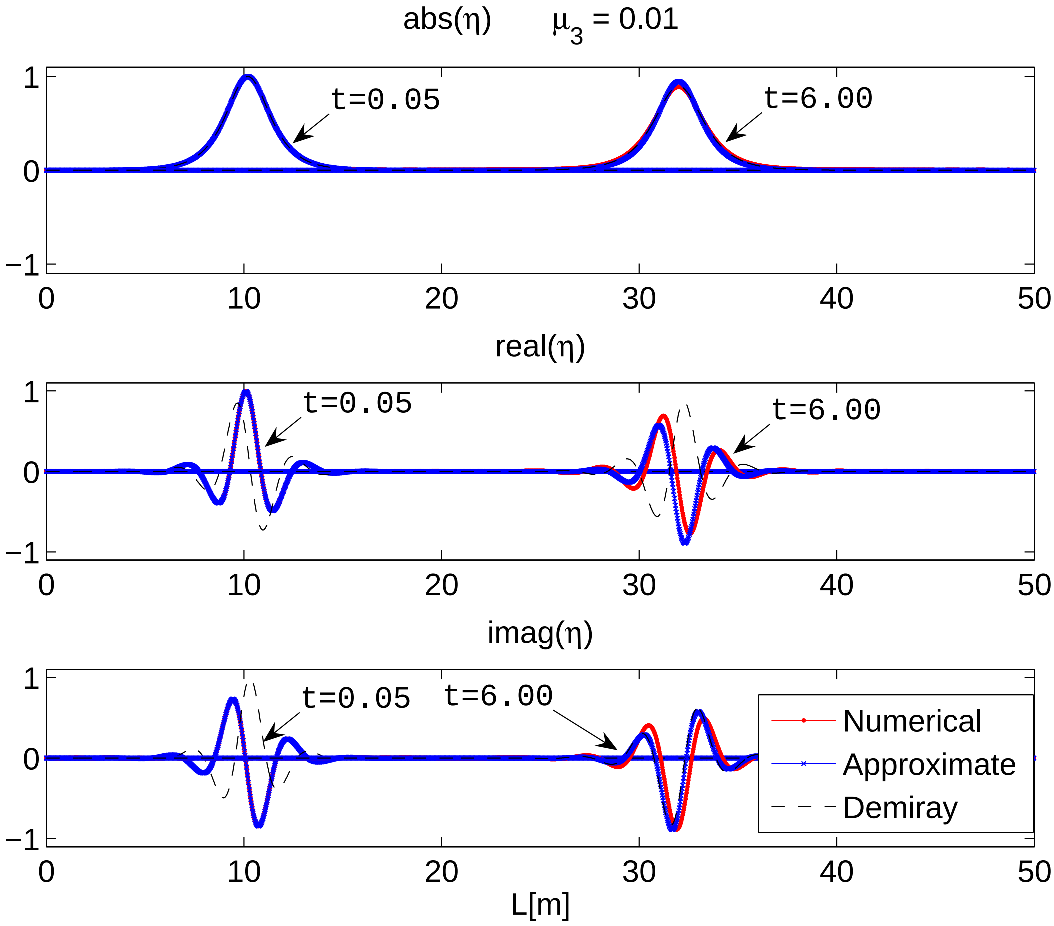


FIGURE 1. Comparison of the numerical (-.), approximate (-x) and Demiray (- -) solutions for $\mu_3 = 0.01$, a) absolute value b) real part c) imaginary part.

In the Figure 1 above we present the inter comparison of the numerical, approximate and Demiray solutions of the d-NLS equation. For this simulation weak dissipation is considered, so that the value of $\mu_3 = 0.01$ is used in the calculations. It can realized from the figure that, for the envelope (absolute value), all three solutions are in good agreement throughout the time stepping. Checking the real parts we can see that at $t = 0.05$, the approximate solution is in a good agreement with the numerical solution whereas the Demiray solution is subjected to a phase shift. As the time progresses and becomes $t = 6.00$, the approximate solution is also slightly shifted compared to the numerical solution, but there is a bigger shift in the Demiray solution compared to the numerical solution. A similar behavior can also be seen in the imaginary part of the solutions, however for $t = 6.00$ we can see that Demiray solution is in a good agreement with the numerical solutions. Therefore it can be understood that the phase shift between the Demiray and the numerical solutions behaves in an oscillatory manner. This occurs since Demiray solutions are solutions in the averaged sense, not in the classical sense. Furthermore one can realize that similar to the real part, in the imaginary part of the solution a small phase shift develops between the approximate and the numerical solutions as time progresses. This is due to the fact that the celerity of approximate solitary wave solution proposed does not include the amplitude damping.

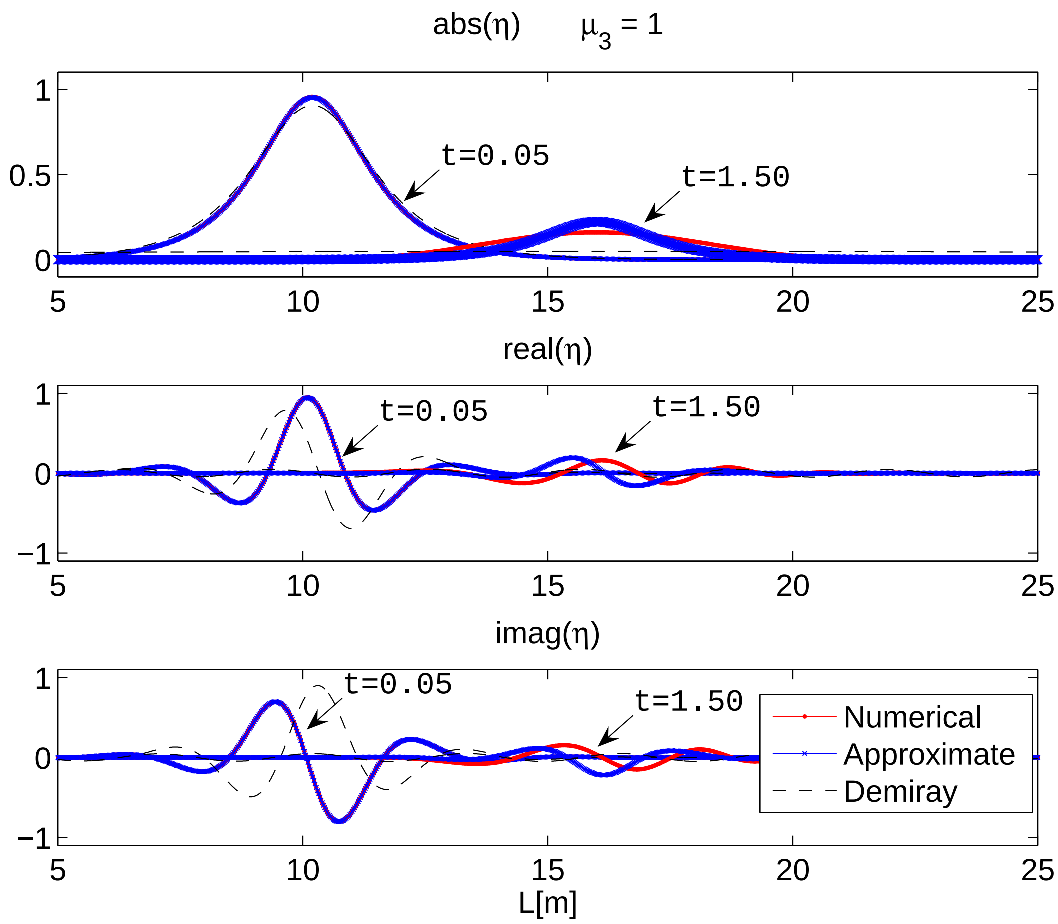


Figure 2. Comparison of the numerical (-.), approximate (-x) and Demiray (- -) solutions for $\mu_3 = 1.00$, a) absolute value b) real part c) imaginary part.

In the Figure 2 above we present the inter comparison of the numerical, approximate and Demiray solutions of the d-NLS equation for strong dissipation, so that the value of $\mu_3 = 1$ is used in the calculations. It can be realized from the figure that, for the envelope (absolute value), all three solutions are in good agreement initially at $t = 0.05$. As time progresses the amplitude decays but the lobe width of the solitary wave profile increases in all three models. However the decay in the wave amplitude and the increase in the lobe width is bigger for the Demiray soliton. The agreement between the numerical solution and the approximate solution is better for this case. Checking the real parts we can again see that at $t = 0.05$, the approximate solution is in a good agreement with the numerical solution whereas the Demiray solution is subjected to a phase shift. As the time progresses and becomes $t = 1.50$, the approximate solution is shifted compared to the numerical solution, but there is still a bigger shift in the Demiray solution compared to the numerical solutions. A similar behavior can also be seen in the imaginary part of the solutions, however for $t = 1.50$ we can see that Demiray solution is in a good agreement with the numerical solutions. Therefore again we see that the phase shift between the Demiray and the numerical solutions behaves in an oscillatory manner. This occurs since Demiray solutions are solutions in the averaged sense, not in the classical sense as discussed before. Furthermore one can realize that a small phase shift develops between the approximate and the numerical solutions as time progresses since the approximate solitary wave solution proposed does not include the amplitude dispersion effects. Also due to a stronger dissipation the decay in the solitary wave height becomes more dominant compared to Figure 1 as expected. Results obtained above confirms that both the Demiray and the proposed approximate solutions can be used as representative solutions of the dissipative nonlinear Schrödinger equation, especially for wave envelope calculations.

## 4. Conclusion and Future Work

In this paper various analytical and numerical aspects of the dissipative nonlinear Schrödinger equation are considered. Decaying solitary wave solutions of sech type derived by Demiray is reviewed. Also a new approximate dissipative solution of d-NLS equation is introduced in order to make comparisons. Additionally a split-step Fourier scheme is proposed for numerical solution of the d-NLS equation and implemented for simulations. The analytical solutions are compared with the numerical solutions and it is shown that both the dissipative Demiray solution and the proposed approximate dissipative solution agrees well with the numerical results especially for the envelope of the wavefield. It is also shown that some phase mismatch between the analytical and the numerical solutions can be observed in the real and imaginary parts. This occurs since Demiray solutions are valid in average sense, not in the pointwise sense and proposed form for the approximate solution does not include amplitude dispersion. However agreement is very good in the zero dissipation limit, as expected.

## Acknowledgments

The author thanks for the support of the Işık University.

## References

[1] Bayındır,C., (2009), Implementation of a Computational Model for Random Directional Seas and Underwater Acoustics, MS Thesis, University of Delaware.

[2] Bayındır,C., (2015), Compressive split-step Fourier method, TWMS J. Appl. and Engr. Math., vol. 5, no.2, pp. 298-306.

[3] Bayındır,C., (2016), Early detection of rogue waves by the wavelet transforms, Physics Letters A , vol. 380, pp. 156-161.

[4] Bayındır,C., (2016), Compressive spectral method for the simulation of the water waves, TWMS J. Appl. and Engr. Math., vol. 6, no.1, to appear.

[5] Bayındır,C., (2015), Hesaplamalı akışkanlar mekaniği çalışmaları için sıkıştırılabilir Fourier tayfı yöntemi, XIX. Türk Mekanik Kongresi, Trabzon, (In Turkish)

[6] Bayındır,C., (2015), Okyanus dalgalarının sıkıştırılabilir Fourier tayfı yöntemiyle hızlı modellenmesi, XIX. Türk Mekanik Kongresi, Trabzon. (In Turkish)

[7] Bayındır,C., (2015), Sönümlü değiştirilmiş Korteweg de-Vries (KdV) denkleminin analitik ve hesaplamalı çözüm karşılaştırması, Türk Mekanik Kongresi, Trabzon. (In Turkish)
[8] Bogomolov,Y.L. and Yunakovsky,A.D. , (2006), Split-step Fourier method for nonlinear Schrodinger equation, Proceedings of the International Conference Day on Diffraction, pp. 34-42.
[9] Canuto,C., Hussaini,M.Y., Quarteroni,A. and Zang,T.A. , (2006), Spectral Methods: Fundamentals in Single Domains, Springer-Verlag, Berlin.
[10] Demiray,H., (2003), An analytical solution to the dissipative nonlinear Schrodinger equation, Applied Mathematics and Computation, 145, pp. 179-184.
[11] Demiray,H. and Bayındır,C., (2015), A note on the cylindrical solitary waves in an electron-acoustic plasma with vortex electron distribution, Physics of Plasmas, 22, 092105.
[12] Hardin,R.H. and Tappert,F.D., (1973), Applications of the split-step Fourier method to the numerical solution of nonlinear and variable coefficient wave equation, SIAM Review Chronicles, 15, pp. 423-423.
[13] Karjadi,E.A., Badiey,M. and Kirby,J.T., (2010), Impact of surface gravity waves on high-frequency acoustic propagation in shallow water, The Journal of the Acoustical Society of America, 127, pp. 1787-1787.
[14] Karjadi,E.A., Badiey,M., Kirby,J.T. and Bayindir,C., (2012), The effects of surface gravity waves on high-frequency acoustic propagation in shallow water, IEEE Journal of Oceanic Engineering, 37, pp. 112-121.
[15] Lamb,H., (1945), Hydrodynamics, Dover Publications, New York.
[16] Sanz-Serna,J.M. and Calvo,M.P., (1994), Numerical Hamiltonian Problems, Chapman and Hall, London.
[17] Taha,T.R. and Ablowitz,M.J., (1984), Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical Nonlinear Schrodinger Equation, Journal of Computational Physics, 22, pp. 203-230.
[18] Trefethen,L.N., (2000), Spectral Methods in MATLAB, SIAM, Philadelphia.
[19] Zakharov,V.E., (1968), Stability of periodic waves of finite amplitude on the surface of a deep fluid, Soviet Physics JETP, 2, pp. 190-194.

**C. Bayındır** for the photography and short autobiography, see TWMS J. App. Eng. Math., V.5, N.2.