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CONTRIBUTION OF HIGHER ORDER TERMS TO THE NONLINEAR SHALLOW WATER WAVES

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ABSTRACT. In this work, by utilizing the scaled multiple-space expansion method, we studied the propagation of weakly nonlinear waves in shallow water and obtained the governing evolution equations of various order terms in the perturbation expansion. Seeking a progressive wave solution to these evolution equations we obtained the speed correction terms so as to remove some possible secularities. The result obtained here is exactly the same with that of obtained by the modified reductive perturbation method [12]. We also proposed a method for the evolution equation governing the n th order term in the perturbation expansion. By defining a single time parameter we showed the connection of the modified reductive perturbation method to the scaled multiple-space expansion method.

Keywords: Shallow water waves, scaled multiple-space expansion, solitary waves.

AMS Subject Classification: 74J35

1. INTRODUCTION

The studies of nonlinear waves of various fields in physics and engineering, by use of the reductive perturbation method in the long-wave approximation, lead to the Korteweg-deVries equation as the evolution equation (Davidson [1], Antar and Demiray [2]). The examination of the higher order terms in the perturbation expansion by use of the reductive perturbation method gives some secularities(Ichikawa et al. [3], Aoyama and Ichikawa [4]). To remove such secularities Sugimoto and Kakutani [5] introduced additional slow variables both in space and time in reductive perturbation theory, Kodama and Taniuti [6] presented the renormalization procedure of the velocity of the KdV soliton. Nevertheless, the latter approach remains somewhat obscure, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. Another attempt to remove such secularities was made by Kraenkel and Manna [7] for long water waves by use of multiple-time scale expansion method but they could not obtain explicitly the speed correction terms. In order to remove this uncertainty, Malfielt and Wieers [8] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the long-wave limit, they expanded the field quantities into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the perturbation expansion. However, this approach is successful when one studies the progressive wave solution to

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the original nonlinear field equations and it does not give any idea about the form of the evolution equations governing various order terms in the perturbation expansion. The result obtained here is completely different from that of [4]. Another method, so called the " modified reductive perturbation method" is presented by Demiray [9-11] in which a scaling parameter, that assumes a perturbation expansion, is presented to balance the higher order nonlinearities with higher order dispersive effects so as to remove some possible secularities in the solution. The result obtained in this work is exactly the same with that of Malfliet and Wieers [8], but different from that of [4].

In this work, by utilizing the scaled multiple-space expansion method, we studied the propagation of weakly nonlinear waves in shallow water and obtained a set of Korteweg-de Vries equations as the evolution equations. By seeking a progressive wave solution to these evolution equations we obtained the speed correction terms so as to remove some possible secularities. The result obtained here is exactly the same with that of given by the modified reductive perturbation method [12].We also proposed a method for the evolution equation governing the n th order term in the perturbation expansion. By defining a single time parameter we showed the connection between the modified reductive perturbation method and the multiple-time scale expansion method.

2. MULTIPLE-TIME SCALE FORMALISM FOR WATER WAVES

We consider two dimensional incompressible inviscid fluid in a constant gravitational field g acting in the negative z^* - direction. The space coordinates are denoted by (x^*, z^*) and the corresponding velocity components by (u^*, w^*). The equations of motion describing such a fluid are:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial z^*} = 0, \text{ (incompressibility)}, \tag{1}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial \partial x^*} + w^* \frac{\partial u^*}{\partial z^*} + \frac{1}{\rho}\frac{\partial P^*}{\partial x^*} = 0, \tag{2}$$

$$\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial \partial x^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho}\frac{\partial P^*}{\partial z^*} + g = 0, \tag{3}$$

where t^* is the time parameter, ρ is the mass density and P^* is the fluid pressure function. Assuming that the flow is ir-rotational, the velocity components can be expressed in terms of scalar potential ϕ^* as

$$u^* = \frac{\partial \phi^*}{\partial x^*}, \quad w^* = \frac{\partial \phi^*}{\partial z^*}. \tag{4}$$

Then, the incompressibility condition becomes

$$\frac{\partial^2 \phi^*}{\partial x^{*2}} + \frac{\partial^2 \phi^*}{\partial z^{*2}} = 0, \tag{5}$$

and the Euler equation reads

$$\frac{P^* - P_0^*}{\rho} + \frac{\partial \phi^*}{\partial t^*} + \frac{1}{2}[(\frac{\partial \phi^*}{\partial x^*})^2 + (\frac{\partial \phi^*}{\partial z^*})^2] + gz^* = 0, \tag{6}$$

where P_0^* is the atmospheric pressure.

We consider the case of fluid of height h_0 , bounded above by a steady atmospheric pressure P_0^*. Let the upper surface be described by $z^* = \eta^*(x^*, t^*)$. The kinematical boundary condition on this surface can be expressed as:

$$\frac{\partial \phi^*}{\partial z^*} = \frac{\partial \eta^*}{\partial t^*} + \frac{\partial \phi^*}{\partial x^*}\frac{\partial \eta^*}{\partial x^*} = 0, \quad \text{on} \quad z^* = \eta^*. \tag{7}$$

From the equation (6), the dynamical boundary condition on this surface reads

$$\frac{\partial \phi^*}{\partial t^*} + \frac{1}{2}[(\frac{\partial \phi^*}{\partial x^*})^2 + (\frac{\partial \phi^*}{\partial z^*})^2] + g\eta^* = 0, \quad \text{on} \quad z^* = \eta^*. \tag{8}$$

Finally, the lower boundary is supposed to be rigid horizontal plane. Therefore, at $z^* = -h_0$, the normal component of the velocity must vanish, i.e.,

$$\frac{\partial \phi^*}{\partial z^*} = 0, \quad \text{at} \quad z^* = -h_0. \tag{9}$$

At this stage it is convenient to introduce the following non-dimensional quantities:

$$x^* = h_0 x, \quad z^* = h_0 z, \quad t^* = \frac{h_0}{c_0} t, \quad \phi^* = c_0 h_0 \hat{\phi}, \quad \eta^* = h_0 \hat{\eta},$$

$$P^* = \rho c_0^2 p, \quad P_0^* = \rho c_0^2 p_0, \quad c_0 = (g h_0)^{1/2}. \tag{10}$$

Introducing (10) into equations (5)-(9) the following non-dimensional equations are obtained:

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0, \tag{11}$$

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{\eta}}{\partial t} + \frac{\partial \hat{\phi}}{\partial x}\frac{\partial \hat{\eta}}{\partial x} = 0, \quad \text{at} \quad z = \hat{\eta}, \tag{12}$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2}[(\frac{\partial \hat{\phi}}{\partial x})^2 + (\frac{\partial \hat{\phi}}{\partial z})^2] + \hat{\eta} = 0, \quad \text{at } z = \hat{\eta}, \tag{13}$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0, \quad \text{at} \quad z = -1. \tag{14}$$

Now, we shall consider the long-wave in shallow water approximation to the above equations by applying the scaled multiple- space expansion formalism. For that purpose we shall propose the following slow variables:

$$\xi = \epsilon^{1/2}(t - x), \quad \tau_n = e^{n+3/2} x, \quad (n = 0, 1, 2, 3, ...) \tag{15}$$

where ϵ is a small parameter characterizing the smallness of certain physical entities. For our future purposes, we introduce the following new variables:

$$\hat{\phi} = \epsilon^{1/2}\phi, \quad \hat{\eta} = \epsilon\eta. \tag{16}$$

Introducing (15) and (16) into (11)-(14) we have

$$\frac{\partial^2 \phi}{\partial z^2} + \epsilon \frac{\partial^2 \phi}{\partial \xi^2} - 2\epsilon^2 \frac{\partial^2 \phi}{\partial \xi \partial \tau_0} + \epsilon^3 (\frac{\partial^2 \phi}{\partial \tau_0^2} - 2\frac{\partial^2 \phi}{\partial \xi \partial \tau_1}) + O(\epsilon^4) = 0, \tag{17}$$

$$\frac{\partial \phi}{\partial z} = \epsilon \frac{\partial \eta}{\partial \xi} + \epsilon^2 (-\frac{\partial \phi}{\partial \xi} + \sum_{n=0}^{\infty} \epsilon^{n+1} \frac{\partial \phi}{\partial \tau_n})(-\frac{\partial \eta}{\partial \xi} + \sum_{n=0}^{\infty} \epsilon^{n+1} \frac{\partial \eta}{\partial \tau_n}) = 0, \quad \text{at} \quad z = \epsilon\eta, \tag{18}$$

$$\frac{\partial \phi}{\partial \xi} + \frac{\epsilon}{2}(-\frac{\partial \phi}{\partial \xi} + \sum_{n=0}^{\infty} \epsilon^{n+1} \frac{\partial \phi}{\partial \tau_n})^2 + \frac{1}{2}(\frac{\partial \phi}{\partial z})^2 + \eta = 0, \quad \text{at} \quad z = \epsilon\eta, \tag{19}$$

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -1. \tag{20}$$

Now, we expand the functions ϕ and η into a suitable power series in the parameter ϵ as:

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + ...,$$

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + ... \quad . \tag{21}$$

Introducing the expansion (21) into the equations (17)-(20) and setting the coefficients of alike powers of ϵ equal to zero, the following sets of differential equations are obtained:

$O(1)$ equations:

$$\frac{\partial \phi_0}{\partial z^2} = 0. \tag{22}$$

and the boundary conditions

$$\frac{\partial \phi_0}{\partial z} = 0 \quad \text{at} \quad z = -1, \quad \frac{\partial \phi_0}{\partial z}|_{z=0} = 0, \quad [\frac{\partial \phi_0}{\partial \xi} + \frac{1}{2}(\frac{\partial \phi_0}{\partial z})^2]|_{z=0} + \eta_0 = 0. \tag{23}$$

$O(\epsilon)$ equations:

$$\frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_0}{\partial \xi^2} = 0, \tag{24}$$

and the boundary conditions

$$\frac{\partial \phi_1}{\partial z} = 0, \quad \text{at} \quad z = -1, \quad \frac{\partial \phi_1}{\partial z}|_{z=0} - \frac{\partial \eta_0}{\partial \xi} = 0,$$

$$[\frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}(\frac{\partial \phi_0}{\partial \xi})^2 + \frac{\partial \phi_0}{\partial z}\frac{\partial \phi_1}{\partial z}]|_{z=0} + \eta_1 = 0. \tag{25}$$

$O(\epsilon^2)$ equations:

$$\frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^2 \phi_1}{\partial \xi^2} - 2\frac{\partial^2 \phi_0}{\partial \xi \partial \tau_0} = 0, \tag{26}$$

and the boundary conditions

$$\frac{\partial \phi_2}{\partial z} = 0, \quad \text{at} \quad z = -1, \quad [\frac{\partial \phi_2}{\partial z} + \eta_0 \frac{\partial^2 \phi_1}{\partial z^2}]|_{z=0} - \frac{\partial \eta_1}{\partial \xi} - \frac{\partial \phi_0}{\partial z}|_{z=0}\frac{\partial \eta_0}{\partial \xi} = 0,$$

$$[\frac{\partial \phi_2}{\partial \xi} + \eta_0 \frac{\partial^2 \phi_1}{\partial z \partial \xi} + \frac{1}{2}(\frac{\partial \phi_1}{\partial z})^2 + \frac{\partial \phi_0}{\partial z}\frac{\partial \phi_2}{\partial z} + \frac{\partial \phi_0}{\partial \xi}(\frac{\partial \phi_1}{\partial \xi} - \frac{\partial \phi_0}{\partial \tau_0}]|_{z=0} + \eta_2 = 0. \tag{27}$$

$O(\epsilon^3)$ equations:

$$\frac{\partial^2 \phi_3}{\partial z^2} + \frac{\partial^2 \phi_2}{\partial \xi^2} - 2\frac{\partial^2 \phi_1}{\partial \xi \tau_0} + \frac{\partial^2 \phi_0}{\partial \tau_0^2} - 2\frac{\partial^2 \phi_0}{\partial \xi \partial \tau_1} = 0, \tag{28}$$

and the boundary conditions:

$$\frac{\partial \phi_3}{\partial z} = 0, \quad \text{at} \quad z = -1,$$

$$[\frac{\partial \phi_3}{\partial z} + \eta_0 \frac{\partial^2 \phi_2}{\partial z^2} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2}]|_{z=0} - \frac{\partial \eta_2}{\partial \xi} - \frac{\partial \phi_0}{\partial \xi}(\frac{\partial \eta_1}{\partial \xi} - \frac{\partial \eta_0}{\partial \tau_0}) - \frac{\partial \eta_0}{\partial \xi}(\frac{\partial \phi_1}{\partial \xi} - \frac{\partial \phi_0}{\partial \tau_0})|_{z=0} = 0,$$

$$[\frac{\partial \phi_3}{\partial \xi} + \eta_0 \frac{\partial^2 \phi_2}{\partial \xi \partial z} + \eta_1 \frac{\partial^2 \phi_1}{\partial \xi \partial z} + \frac{1}{2}\eta_0^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial \xi}]|_{z=0} + [\frac{\partial \phi_1}{\partial z}(\frac{\partial \phi_2}{\partial z} + \eta_0 \frac{\partial^2 \phi_1}{\partial z^2}]|_{z=0}$$

$$+\frac{1}{2}(\frac{\partial \phi_1}{\partial \xi} - \frac{\partial \phi_0}{\partial \tau_0})^2|_{z=0} + \frac{\partial \phi_0}{\partial \xi}(\eta_0 \frac{\partial^2 \phi_1}{\partial \xi \partial z} + \frac{\partial \phi_2}{\partial \xi} - \frac{\partial \phi_1}{\partial \tau_0} - \frac{\partial \phi_0}{\partial \tau_1})|_{z=0} + \eta_3 = 0. \tag{29}$$

2.1. **Solution of the field equations.** From the solution of the sets (22) and (23) one obtains

$$\phi_0 = \varphi_0(\xi, \tau_n), \quad \eta_0 = -\frac{\partial \varphi_0}{\partial \xi} \quad (n = 0, 1, 2, 3, ...), \tag{30}$$

where $\varphi_0(\xi, \tau_n)$ is an unknown function of its argument whose evolution equation will be obtained later.

Introducing (30) into (24) and (25), the solution of $O(\epsilon)$ equations gives the following result

$$\phi_1 = -\frac{1}{2}\frac{\partial^2 \varphi_0}{\partial \xi^2}(z^2 + 2z) + \varphi_1(\xi, \tau_n), \quad \eta_1 = -\frac{\partial \varphi_1}{\partial \xi} - \frac{1}{2}(\frac{\partial \varphi_0}{\partial \xi})^2, \tag{31}$$

where $\varphi_1(\xi, \tau_n)$ $(n = 0, 1, 2, 3, ..)$ is another unknown function whose governing evolution equation will be obtained from the higher order perturbation expansion.

Introducing (30) and (31) into equations (26) and (27), the solution of $O(\epsilon^2)$ equations may be obtained as

$$\phi_2 = \frac{1}{24}\frac{\partial^4 \varphi_0}{\partial \xi^4}(z^4 + 4z^3) + (\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau_0} - \frac{1}{2}\frac{\partial^2 \varphi_1}{\partial \xi^2]})z^2$$

$$+(-\frac{1}{3}\frac{\partial^4 \varphi_0}{\partial \xi^4} - \frac{\partial^2 \varphi_1}{\partial \xi^2} + 2\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau_0})z + \varphi_2(\xi, \tau_n),$$

$$\eta_2 = -\frac{\partial \varphi_2}{\partial \xi} - \frac{\partial \varphi_0}{\partial \xi}\frac{\partial^3 \varphi_0}{\partial \xi^3} - \frac{1}{2}(\frac{\partial \varphi_0}{\partial \xi})^2 - \frac{\partial \varphi_0}{\partial \xi}\frac{\partial \varphi_1}{\partial \xi} + \frac{\varphi_0}{\partial \xi}\frac{\partial \varphi_0}{\partial \tau_0}, \tag{32}$$

where $\varphi_2(\xi, \tau_n)$ is another unknown function whose evolution equation will be obtained from the higher order perturbation expansion. The use of the second boundary condition in (31) yields

$$\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau_0} + \frac{3}{2}\frac{\partial \varphi_0}{\partial \xi}\frac{\partial^2 \varphi_0}{\partial \xi^2} - \frac{1}{6}\frac{\partial^4 \varphi_0}{\partial \xi^4} = 0. \tag{33}$$

Noting the relation $\partial \varphi_0/\partial \xi = -\eta_0$, the equation (33) reduces to the following Korteweg-de Vries (KdV) equation

$$\frac{\partial \eta_0}{\partial \tau_0} - \frac{3}{2}\eta_0\frac{\partial \eta_0}{\partial \xi} - \frac{1}{6}\frac{\partial^3 \eta_0}{\partial \xi^3} = 0. \tag{34}$$

The equation (34) gives the evolution of η_0 with τ_0 and ξ. The remaining spatial variables remain as some parameters. The evolution of η_0 with τ_n $(n = 1, 2, 3, ...)$ should be obtained from the higher order perturbation expansion.

To obtain the solution for $O(\epsilon^3)$ equations we introduce (30), (31) and (32) into (28) and (29), which results in

$$\frac{\partial^2 \phi_3}{\partial z^2} + \frac{1}{24}\frac{\partial^6 \varphi_0}{\partial \xi^6}(z^4 + 4z^3) + (2\frac{\partial^4 \varphi_0}{\partial \xi^3 \partial \tau_0} - \frac{1}{2}\frac{\partial^4 \varphi_1}{\partial \xi^4})z^2$$

$$+(-\frac{1}{3}\frac{\partial^6 \varphi_0}{\partial \xi^6} - \frac{\partial^4 \varphi_1}{\partial \xi^4} + 4\frac{\partial^4 \varphi_0}{\partial \xi^3 \partial \tau_0})z + \frac{\partial^2 \varphi_2}{\partial \xi^2} - 2\frac{\partial^2 \varphi_1}{\partial \xi \partial \tau_0} + \frac{\partial^2 \varphi_0}{\partial \tau_0^2} - 2\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau_1} = 0, \tag{35}$$

and the boundary conditions

$$\frac{\partial \varphi_3}{\partial z} = 0 \quad \text{at} \quad z = -1, \quad \frac{\partial \phi_3}{\partial z}|_{z=0} - 4\frac{\partial \varphi_0}{\partial \xi}\frac{\partial^2 \varphi_0}{\partial \xi \partial \tau_0} - 2\frac{\partial^2 \varphi_0}{\partial \xi^2}\frac{\partial \varphi_0}{\partial \tau_0}$$

$$+3\frac{\partial}{\partial \xi}(\frac{\partial \varphi_0}{\partial \xi}\frac{\partial \varphi_1}{\partial \xi}) + \frac{3}{2}(\frac{\partial \varphi_0}{\partial \xi})^2\frac{\partial^2 \varphi_0}{\partial \xi^2} + \frac{\partial \varphi_0}{\partial \xi}\frac{\partial^4 \varphi_0}{\partial \xi^4} + 2\frac{\partial^2 \varphi_0}{\partial \xi^2}\frac{\partial^3 \varphi_0}{\partial \xi^3} + \frac{\partial^2 \varphi_2}{\partial \xi^2} = 0. \tag{36}$$

The solution of (35) after the use of the first boundary condition in (36) gives

$$\phi_3 = -\frac{1}{720}\frac{\partial^6\varphi_0}{\partial\xi^6}(z^6+6z^5) - \frac{1}{12}(2\frac{\partial^4\varphi_0}{\partial\xi^3\partial\tau_0} - \frac{1}{2}\frac{\partial^4\varphi_1}{\partial\xi^4})z^4$$
$$+(\frac{1}{18}\frac{\partial^6\varphi_0}{\partial\xi^6} + \frac{1}{6}\frac{\partial^4\varphi_1}{\partial\xi^4} - \frac{2}{3}\frac{\partial^4\varphi_0}{\partial\xi^3\partial\tau_0})z^3 - \frac{1}{2}(\frac{\partial^2\varphi_2}{\partial\xi^2} - 2\frac{\partial^2\varphi_1}{\partial\xi\partial\tau_0} - 2\frac{\partial^2\varphi_0}{\partial\xi\partial\tau_1} + \frac{\partial^2\varphi_0}{\partial\tau_0^2})z^2$$
$$+(-\frac{2}{15}\frac{\partial^6\varphi_0}{\partial\xi^6} - \frac{1}{3}\frac{\partial^4\varphi_1}{\partial\xi^4} + \frac{4}{3}\frac{\partial^4\varphi_0}{\partial\xi^3\partial\tau_0} + 2\frac{\partial^2\varphi_1}{\partial\xi\partial\tau_0} + 2\frac{\partial^2\varphi_0}{\partial\xi\partial\tau_1} - \frac{\partial^2\varphi_0}{\partial\tau_0^2} - \frac{\partial^2\varphi_2}{\partial\xi^2})z + \varphi_3, \quad (37)$$

where $\varphi_3(\xi,\tau_n)$ is another unknown function whose evolution equation will be obtained from the higher order perturbation expansion. Employing the last boundary condition in (36) the following evolution equation is obtained

$$\frac{\partial^2\varphi_1}{\partial\xi\partial\tau_0} + \frac{3}{2}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_0}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}) - \frac{1}{6}\frac{\partial^4\varphi_1}{\partial\xi^4} = S(\varphi_0), \quad (38)$$

where the function $S(\varphi_0)$ is defined by

$$S(\varphi_0) = \frac{1}{15}\frac{\partial^6\varphi_0}{\partial\xi^6} - \frac{2}{3}\frac{\partial^4\varphi_0}{\partial\xi^3\partial\tau_0} - \frac{\partial^2\varphi_0}{\partial\xi^2}\frac{\partial^3\varphi_0}{\partial\xi^3} - \frac{1}{2}\frac{\partial\varphi_0}{\partial\xi}\frac{\partial^4\varphi_0}{\partial\xi^4}$$
$$+2\frac{\partial\varphi_0}{\partial\xi}\frac{\partial^2\varphi_0}{\partial\xi\partial\tau_0} + \frac{\partial^2\varphi_0}{\partial\xi^2}\frac{\partial\varphi_0}{\partial\tau_0} - \frac{3}{4}(\frac{\partial\varphi_0}{\partial\xi})^2\frac{\partial^2\varphi_0}{\partial\xi^2} + \frac{1}{2}\frac{\partial^2\varphi_0}{\partial\tau_0^2} - \frac{\partial^2\varphi_0}{\partial\xi\partial\tau_1}. \quad (39)$$

Setting $\partial\varphi_0/\partial\xi = -\eta_0$, $\partial\varphi_1/\partial\xi = -\eta_1 - \eta_0^2/2$, the equations (38) and (39) takes the following form

$$\frac{\partial\eta_1}{\partial\tau_0} - \frac{3}{2}\frac{\partial}{\partial\xi}(\eta_0\eta_1) - \frac{1}{6}\frac{\partial^3\eta_1}{\partial\xi^3} = T(\eta_0), \quad (40)$$

where $T(\eta_0)$ is defined by

$$T(\eta_0) = \frac{1}{15}\frac{\partial^5\eta_0}{\partial\xi^5} - \frac{7}{12}\frac{\partial^3\eta_0}{\partial\xi^2\partial\tau_0} + \frac{4}{3}\frac{\partial\eta_0}{\partial\xi}\frac{\partial^2\eta_0}{\partial\xi^2} + \frac{2}{3}\eta_0\frac{\partial^3\eta_0}{\partial\xi^3} + \frac{3}{4}\eta_0^2\frac{\partial\eta_0}{\partial\xi} - \frac{9}{4}\eta_0\frac{\partial\eta_0}{\tau_0} - \frac{\partial\eta_0}{\partial\tau_1}. \quad (41)$$

The equation (40) is the degenerate (linearized) KdV equation with the non-homogeneous term $T(\eta_0)$. The left side of (40) gives the evolution of η_1 with respect to τ_0 and ξ, whereas the right side is the evolution of η_0 with respect to τ_1, τ_0 and ξ . These evolutions are related to each other through the equation (40). For this order, the slow variables τ_n $(n = 2, 3, ...)$ remain as some parameters.

2.2. **Solitary waves.** In this sub-section we shall study the localized travelling wave solution to the evolution equations (34) and (40). For that purpose we introduce

$$\eta_i = \eta_i(\zeta,\tau_n), \quad \zeta = \alpha(\tau_n)[\xi + g(\tau_0,\tau_n)], \quad (i = 0, 1), \quad (n = 1, 2, ...), \quad (42)$$

where the parameters $\alpha(\tau_n)$ and $g(\tau_0,\tau_n)$ $(n = 1, 2, 3, ...)$ are to be determined from the solution. As far as the equation (34) is concerned, only the variables are ξ and τ_0 ; and τ_n $(n = 1, 2, 3, ..)$ appears to be some parameters.

Introducing (42) for $i = 0$ into the evolution equation (34) we have

$$\frac{\partial g}{\partial\tau_0}\eta_0' - \frac{3}{2}\eta_0\eta_0' - \frac{\alpha^2}{6}\eta_0''' = 0, \quad (43)$$

where the prime denotes the differentiation of the corresponding quantity with respect to ζ . Integrating (43) with respect to ζ and utilizing the localization condition, i.e., η_0 and its various order derivatives vanish as $\zeta \to \pm\infty$, we obtain

$$\frac{\partial c}{\partial\tau_0}\eta_0 - \frac{3}{4}\eta_0^2 - \frac{\alpha^2}{6}\eta_0'' = 0. \quad (44)$$

The equation (44) admits the solitary wave solution of the form

$$\eta_0 = a\ \text{sech}^2\zeta, \tag{45}$$

where a is the amplitude of the solitary wave, and, in general, it is a function of the parameters τ_n. Inserting (45) into (44) and setting the coefficients of various powers of sechζ equal to zero we obtain

$$\alpha = (\frac{3a}{4})^{1/2}, \quad g = \frac{a}{2}\tau_0 + \theta(\tau_n), \quad (n = 1, 2, 3, ...). \tag{46}$$

To obtain the solution for the evolution equation (40) we introduce (42) for $i = 1$ into (40), which results in

$$\frac{\alpha a}{2}\eta_1' - \frac{3\alpha}{2}(\eta_0\eta_1)' - \frac{\alpha^3}{6}\eta_1''' = -\frac{1}{a}\frac{\partial a}{\partial \tau_1}\eta_0 - \frac{1}{\alpha}\frac{\partial \alpha}{\partial \tau_1}\zeta\eta_0' - \alpha\frac{\partial g}{\partial \tau_1}\eta_0' + \frac{\alpha^5}{15}\eta_0^{(5)}$$
$$-\frac{7a\alpha^3}{24}\eta_0''' + \frac{4\alpha^3}{3}\eta_0'\eta_0'' + \frac{2\alpha^3}{3}\eta_0\eta_0''' + \frac{3\alpha}{4}\eta_0^2\eta_0' - \frac{9a\alpha}{8}\eta_0\eta_0'. \tag{47}$$

Integrating (47) with respect to ζ and utilizing the localization condition we have

$$\frac{\alpha a}{2}\eta_1 - \frac{3\alpha}{2}(\eta_0\eta_1) - \frac{\alpha^3}{6}\eta_1'' = -\frac{1}{\alpha}\frac{\partial \alpha}{\partial \tau_1}\zeta\eta_0 - (\frac{1}{a}\frac{\partial a}{\partial \tau_1} - \frac{1}{\alpha}\frac{\partial \alpha}{\partial \tau_1}\int \eta_0 d\zeta$$
$$-\alpha\frac{\partial g}{\partial \tau_1}\eta_0 + \frac{\alpha^5}{15}\eta_0^{(4)} - \frac{7a\alpha^3}{24}\eta_0'' + \frac{\alpha^3}{3}(\eta_0')^2 + \frac{2\alpha^3}{3}\eta_0\eta_0'' + \frac{\alpha}{4}(\eta_0)^3 - \frac{9a\alpha}{16}(\eta_0)^2. \tag{48}$$

Since we are concerned with the localized waves, in order to remove the secularity in η_1, the coefficient of $\zeta\eta_1$ must vanish; which yields

$$\frac{\partial \alpha}{\partial \tau_1} = \frac{\partial a}{\partial \tau_1} = 0. \tag{49}$$

This equation states that the coefficients α and a are independent of τ_1. In a similar way one can prove that α and a are independent of τ_n $(n = 1, 2, 3, ...)$. The remaining part of the equation (48) takes the following form

$$\eta_1'' + (\frac{12}{a}\eta_0 - 4)\eta_1 = \frac{8}{a}\frac{\partial g}{\partial \tau_1}\eta_0 - \frac{3a}{10}\eta_0^{(4)} + \frac{7a}{4}\eta_0'' - 2(\eta_0')^2 - 4\eta_0\eta_0'' - \frac{2}{a}\eta_0^3 + \frac{9}{2}\eta_0^2. \tag{50}$$

Noting the relations

$$\eta_0'' = 4\eta_0 - \frac{6}{a}\eta_0^2, \quad (\eta_0')^2 = 4\eta_0^2 - \frac{4}{a}\eta_0^3, \quad \eta_0^{(4)} = 16\eta_0 - \frac{120}{a}\eta_0^2 + \frac{120}{a^2}\eta_0^3, \tag{51}$$

the equation (50) becomes

$$\eta_1'' + (\frac{12}{a}\eta_0 - 4)\eta_1 = (\frac{8}{a}\frac{\partial g}{\partial \tau_1} + \frac{11a}{5})\eta_0 + 6\eta_0^2 - \frac{6}{a}\eta_0^3. \tag{52}$$

The homogeneous differential equation obtained from (52) admits η_0' as one of the fundamental solutions. Therefore, the term proportional to η_0 on the right hand side of (52) causes the secularity in η_1. In order to remove this secularity the coefficient of η_0 must vanish, which yields

$$\frac{8}{a}\frac{\partial g}{\partial \tau_1} + \frac{11a}{5} = 0, \quad \text{or} \quad g = \frac{a}{2}\tau_0 - \frac{11a^2}{40}\tau_1 + \theta_1(\tau_n), \quad (n = 2, 3, 4, ...). \tag{53}$$

This result is exactly the same with that of [12] in which the modified reductive perturbation method was employed. The remaining part of equation (52) becomes

$$\eta_1'' + (\frac{12}{a}\eta_0 - 4)\eta_1 = 6\eta_0^2 - \frac{6}{a}\eta_0^3. \tag{54}$$

The particular solution of equation (54) gives

$$\eta_1 = -\frac{a}{2}\eta_0 + \frac{3}{4}\eta_0^2. \tag{55}$$

Since, in the present work, we are concerned with the contribution of zeroth order term to the higher order terms in the perturbation expansion, the non-homogeneous term $S_n(\eta_0, \eta_1, ...\eta_{n-1})$ will depend only on η_0 , i. e., $S_n(\eta_0)$. Thus, in studying the progressive wave solution, the linearized KdV equation for the n th order term will take the following form

$$\eta_n'' + (\frac{12}{a}\eta_0 - 4)\eta_n = (\frac{8}{a}\frac{\partial g}{\partial \tau_n} + a_n)\eta_0 + \sum_{k=1}^{n+1} d_k \eta_0^{k+1}, \tag{56}$$

where the coefficient a_n, d_k $(k = 1, 2, ..., n+1)$ are to be calculated from the higher order perturbation expansion. Again, the first term on the right hand side causes to the secularity in η_n ; thus, this coefficient must vanish

$$\frac{8}{a}\frac{\partial g}{\partial \tau_n} + a_n = 0. \tag{57}$$

From this equation one can determine the dependence of c on τ_n .

Since the solution obtained here is exactly the same with that of [12], then, one may raise the question whether a connection exists between the modified reductive perturbation method and the scaled multiple-space expansion method. The answer to this question is "yes" provided that the following substitution is made

$$\tau = \epsilon^{3/2} cx = c_0\tau_0 + c_1\tau_1 + c_2\tau_2 + 3_3\tau_3 + c_4\tau_4 + ..., \tag{58}$$

where c is the scale parameter defined in [12] and assumes a perturbation expansion of the form

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \epsilon^4 c_4 + ..., \tag{59}$$

and the coefficients c_i are related to g through

$$c_i = \frac{\partial g}{\partial \tau_i}, \quad (i = 0, 1, 2, 3, ...). \tag{60}$$

3. Conclusion

The study of the effects of the first order term to higher order terms in the reductive perturbation method leads to some secularities [3, 4]. To remove such secularities various methods, like the renormalization method by Kodama and Tanuiti [6] , multiple scale expansion method by Kraenkel and Manna [7] and the modified reductive perturbation method by Demiray [9]. The methods presented in [6] and [7] are quite complicated as compared to one given in [9].

In the present work, we studied an application of the scaled multiple-space expansion method for shallow water theory which was studied before by Demiray [12] through the use of the modified reductive perturbation method. The result reported here is exactly the same with that of [11]. We also proposed a method for the evolution equation governing the n th term in the perturbation expansion. The connection between this approach and the modified reductive perturbation method is also indicated.

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