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COMPARATIVE STUDY ON HYERS-ULAM-RASSIAS STABILITY OF PEXIDER TYPE FUNCTIONAL EQUATION IN BANACH SPACES USING DIRECT METHOD AND FIXED POINT METHOD

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Abstract. In this paper, we have established a Hyers-Ulam-Rassias stability result of a pexider type functional equation. The work is in the framework of Banach spaces. The result is established using direct method as well as a fixed point theorem approach followed by some corollaries. Apart from its main objective of obtaining the stability result, the present paper is also a demonstration of the comparative study of the results in Banach spaces.

Keywords:Keywords: Hyers-Ulam -Rassias stability, Pexider type functional equation, Banach space, contraction mapping, fixed point.

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1. Introduction

The paper consists of results on Hyers-Ulam-Rassias stability of a certain functional equation in the framework of direct and fixed point method. There are several kinds of them and diverse mathematical ideas from different fields of mathematics have contributed to their foundations.

The Hyers-Ulam-Rassias stability is the most general type of stability which arises in diverse mathematical domains. In general, this type of stability problem investigates whether any mathematical object which behaves approximately like a class of mathematical entities has actually an approximation from that class. There are a considerable number of contributions in the recent literatures to the field we consider here [4, 17, 18, 19, 22].

The stability problem of functional equations had originated from a question of S. M. Ulam [20] in 1940 concerning the stability of group homomorphisms and he posed a question as follows:

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" Let G_1 be a group and G_2 be a metric group with the metric $d(\cdot, \cdot)$. For any given $\varepsilon > 0$ can we be able to find a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there would exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d\ (h(x), H(x)) < \varepsilon$ for all $x \in G_1$?."
In the following year, the above question was partially answered by Hyers [9] under the assumption that the groups are Banach spaces. He showed that for $\delta > 0$ and for function $f : E \longrightarrow E_1$ with E and E_1 Banach spaces, satisfying

$$\| f(x + y) - f(x) - f(y) \| \leq \delta$$

, for all $x, y \in E$ there exists a unique $g : E \longrightarrow E_1$ such that

$$g(x + y) = g(x) + g(y)$$

and

$$\| f(x) - g(x) \| \leq \delta$$

for all $x, y \in E$.

Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. In 1978, Themistocles M.Rassias [16] had succeeded in extending the result of Hyers's theorem and his exciting result attracted a number of Mathematicians who investigated the stability problems of several functional equations. Due to influence of S.M. Ulam and D.H. Hyers, in the work of Th.M. Rassias, regarding the study of stability problems of functional equations, the stability phenomenon proved by Th.M. Rassias is termed as the Hyers-Ulam-Rassias stability. Thereafter for the last thirty five years many results concerning the Hyers-Ulam-Rassias stability of various functional equations have been obtained and a number of definitions of stability have been introduced in various aspects [5, 7, 8, 10, 11, 12, 13, 14, 15, 21].

The functional equation

$$F(2x + y) - F(x + 2y) = 3F(x) - 3F(y) \tag{1}$$

is known as quadratic functional equation because quadratic mapping $f(x) = cx^2$ is a solution of the functional equation (1). Jun et.al. [10] found out some properties of solution of the functional equation (1) and established Hyers-Ulam-Rassias stability of this functional equation in that context.
The functional equation

$$F(x + y) - G(x) - H(y) = 0\ \forall x, y \in X, \tag{2}$$

is known as Pexider type functional equation [3] corresponding to Cauchy functional equation.Using the idea of Gavruta [8], the authors Jun et al. [11] investigated the Hyers-Ulam -Rassias stability of the functional equation (2).

Accordingly, the functional equation

$$p(2x + y) - p(x + 2y) = 3q(x) - 3r(y) \qquad \cdots \tag{3}$$

a generalization of the equation (1), can be considered as a Pexider type functional equation of the quadratic functional equation (1).
In this paper, we would compare the Hyers-Ulam-Rassias stability by applying direct method and the fixed point method of the functional equation (3).

Here, in section 2, we discussed some preliminaries. In section 3, we prove the generalized Hyers-Ulam-Rassias stability of the function equation (3) by using direct method. In section 4, we examined the same by using fixed point method. Finally some concluding remarks and comparative study of the results are specified in section 5.

2. Preliminaries

In this section, we recall a definition and a result concerning the metric spaces.

Definition 2.1. *Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies*
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then (X, d) is called a generalized metric space.

Theorem 2.1. *([2] and [1]) Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant L, $0 < L < 1$, that is,*

$$d(Jx, Jy) \leq L d(x, y),$$

for all $x, y \in X$.
Then for each $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \ \forall\, n \geq 0$$

or,

$$d(J^n x, J^{n+1} x) < \infty \ \ \forall\, n \geq n_o$$

for some non-negative integers n_0. Moreover, if the second alternative holds then
(1) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J;
(2) $y^\star$ is the unique fixed point of J in the set

$$Y = \{y \in X : d(J^{n_0} x, y) < \infty\};$$

(3) $d(y, y^\star) \leq (\frac{1}{1-L}) d(y, Jy)$ for all $y \in Y$.

3. The Generalized Hyers-Ulam-Rassias Stability of The Pexider Type Functional Equation (3): Direct Method

Let $(G, +)$ be an abelian group, $(X, \|.\|)$ be a real Banach space and let $C : G \to X$ be such that

$$C(2x+y) - C(x+2y) = C(x) - C(y) \, \forall x, y \in X \in G. \tag{4}$$

It is easy to see that $C(x) = A(x) + b$ satisfies (4), where A is an additive mapping.

Let $\phi : G \times G \to [0, \infty)$ be a mapping such that

$$\lim_{n\to\infty} \frac{\phi(2^n x\ ,\ 2^n y)}{2^n} = 0. \tag{5}$$

Then ϕ is called a control function and clearly,

$$\varepsilon(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\phi\left(2^k x, 0\right) + \frac{3}{2}\phi\left(2^k x, -2^k x\right) + \frac{1}{2}\phi\left(2^k x, 2^k x\right)}{2^k} < \infty$$

and

$$\varepsilon_1(x) = \sum_{k=1}^{\infty} \frac{\phi(2^{k-1} x, 0)}{4^k} < \infty.$$

Theorem 3.1. *Let $p\,,\,q\,,\,r\,:\,G\,\rightarrow\,X$ be odd mappings satisfying the inequality*

$$\|\,p\,(2x+y)-p\,(x+2y)-3q\,(x)+3r\,(y)\,\|\,\leq\,\,\phi\,(x\,,\,y)\;\;\forall x\,,\,y\,\in\,G\,. \tag{6}$$

Then there exits a unique mapping $C\,:\,G\,\rightarrow\,X$ such that

$$C\,(2x+y)\;-\;C\,(x+2y)\;=\;C\,(x)\;-\;C\,(y)\;, \tag{7}$$

$$\|\,p\,(x)-C\,(x)\,\|\,\leq\,\epsilon\,(\,x\,)\,, \tag{8}$$

$$\|3\,q\,(x)+3\,r\,(x)\;-2\,C\,(x)\,\|\,\leq\,2\,\epsilon\,(\,x\,)+\phi\,(x\,,\,-x)\qquad \textit{for all } x\,,\,y\in G. \tag{9}$$

Where $u\,(x,x)\;=\;\phi\,(x\,,\,0)+\dfrac{3}{2}\,\phi\,(x\,,\,-x)+\dfrac{1}{2}\,\phi\,(x\,,\,x)\;.$

and

$$\varepsilon\,(\,x\,)\;=\;\frac{1}{2}\sum_{k=0}^{\infty}\frac{\phi\left(2^k\,x\,,\,0\right)+\frac{3}{2}\phi\left(2^k\,x\,,\,-\,2^k\,x\right)+\frac{1}{2}\phi\left(2^k\,x\,,\,2^k\,x\right)}{2^k}\;<\;\infty$$

Proof. For $x\,=\,y$ inequality (6) implies

$$\|\,0-3q\,(x)+3r\,(x)\,\|\,\leq\,\,\phi\,(x\,,\,x)\quad \text{for all } x\,\in\,G. \tag{10}$$

Again for $y\,=\,-\,x$, we have from (6)

$$\|2\,p\,(x)-3q\,(x)-3r\,(x)\,\|\,\leq\,\,\phi\,(x\,,\,-x)\quad \text{for all } x\,\in\,G\,. \tag{11}$$

Also for $y\,=\,0$,

$$\|p\,(2\,x)-p\,(x)-3q\,(x)\,\|\,\leq\,\,\phi\,(x\,,\,0) \tag{12}$$

Then,

$$\|2\,p\,(x)-6r\,(x)\,\|\,=\|2\,p\,(x)-3q\,(x)-3r\,(x)\,+\,3q\,(x)-3r\,(x)\,\|$$

$$\leq\|2\,p\,(x)-3q\,(x)-3r\,(x)\,\|\,+\,\|3q\,(x)-3r\,(x)\,\|$$

$$\leq\,\phi\,(x\,,\,-\,x)+\phi\,(x\,,\,x)\,,\;\text{ by using (11) and (10)}$$

$$\text{or},\;\;\|p\,(\,x\,)\,-\,3r\,(x)\,\|\,\leq\,\frac{1}{2}\,\phi\,(x\,,\,-x)\,+\,\frac{1}{2}\,\phi\,(x\,,\,x)\quad \text{for all } x\,\in\,G\,. \tag{13}$$

Then we have

$$\|p\,(2\,x)-2p\,(x)\,\|\,=\,\|p\,(2\,x)-p\,(x)-3\,q\,(x)-2\,p\,(x)+3\,q\,(x)+3\,r\,(x)+p\,(x)-3\,r\,(x)\,\|$$

$$\leq\,\phi\,(x\,,\,0)+\phi\,(x\,,-\,x)+\frac{1}{2}\,\phi\,(x\,,\,-\,x)+\frac{1}{2}\,\phi\,(x\,,\,x)\;\;\text{by using (12), (11) and (13)}$$

$$=\,\phi\,(x\,,\,0)+\frac{3}{2}\,\phi\,(x\,,\,-\,x)+\frac{1}{2}\,\phi\,(x\,,\,x)\;=\;u\,(x,x)\;.$$

$$\textit{i.e,}\;\;\|\,p\,(2\,x)\,-\,2p\,(x)\,\|\;\leq\;u\,(x,x) \tag{14}$$

$$\textit{i.e,}\;\;\left\|\,\frac{p\,(2\,x)}{2}\,-\,p\,(x)\,\right\|\;\leq\;\frac{1}{2}\,u\,(x,x) \tag{15}$$

Now, by induction on positive integer n, we show that

$$\|\,p\,(2^{\,n}x)\,-\,2^{\,n}\,p\,(x)\,\|\;\leq\;\sum_{k=1}^{n}2^{\,k\,-\,1}\,u\left(2^{\,n\,-\,k}x,\,2^{\,n\,-\,k}x\right) \tag{16}$$

Replacing x by $2\,x$ in (14) we get

$$\left\|\,p\left(2^2\,x\right)\,-\,2\,p\,(2\,x)\,\right\|\;\leq\;u\,(2\,x,2\,x)\qquad \text{for all } x\,\in\,G.$$

Therefore

$$\left\| p\left(2^2 x\right) - 2^2 p(x) \right\| \leq \left\| p\left(2^2 x\right) - 2^2 p(x) - 2p(2x) + 2p(2x) \right\|$$
$$\leq u(2x, 2x) + 2u(x, x) \quad \text{for all } x \in G. \tag{17}$$

Thus from (14) and (17) we see that (16) is true for $n = 1$ and $n = 2$. Assume that (16) is true for all k satisfying $n = 1, 2, 3, ..., m$. Then

$$\| p(2^m x) - 2^m p(x) \| \leq \sum_{k=1}^{m} 2^{k-1} u\left(2^{m-k}x, \; 2^{m-k}x\right). \tag{18}$$

Replacing x by $2x$ in (18) we get

$$\left\| p\left(2^{m+1}x\right) - 2^m p(2x) \right\| \leq \sum_{k=1}^{m} 2^{k-1} u\left(2^{m+1-k} x, 2^{m+1-k} x\right). \tag{19}$$

Now

$$\left\| p\left(2^{m+1}x\right) - 2^{m+1} p(x) \right\| = \left\| p\left(2^{m+1}x\right) - 2^m p(2x) + 2^m p(2x) - 2^{m+1} p(x) \right\|$$
$$\leq \left\| p\left(2^{m+1}x\right) - 2^m p(2x) \right\| + 2^m \| p(2x) - 2p(x) \|$$

$$\leq \sum_{k=1}^{m} 2^{k-1} u\left(2^{m+1-k} x, 2^{m+1-k} x\right) + 2^m u(x, x) \quad [\text{by (14) and (19)}]$$
$$= \sum_{k=1}^{m+1} 2^{k-1} u\left(2^{m+1-k} x, \; 2^{m+1-k} x\right)$$

Thus (16) is true for $n = m + 1$ whenever it is true for $n = m$. Hence, by induction principle (16) is proved for all $n \in N$. Therefore,

$$\left\| \frac{p(2^n x)}{2^n} - p(x) \right\| \leq 2^{-n} \sum_{k=1}^{n} 2^{k-1} u\left(2^{n-k}x, \; 2^{n-k}x\right)$$

$$i.e., \left\| \frac{p(2^n x)}{2^n} - p(x) \right\| \leq \sum_{k=1}^{n} 2^{k-n-1} u\left(2^{n-k} x, \; 2^{n-k} x\right) \tag{20}$$

Therefore for $m < n$ we have

$$\left\| \frac{p(2^m x)}{2^m} - \frac{p(2^n x)}{2^n} \right\| = \left\| \frac{p(2^m x)}{2^m} - p(x) + p(x) - \frac{p(2^n x)}{2^n} \right\|$$
$$\leq \left\| \frac{p(2^m x)}{2^m} - p(x) \right\| + \left\| \frac{p(2^n x)}{2^n} - p(x) \right\|$$

$$\leq \sum_{k=1}^{m} 2^{k-m-1} u(2^{m-k} x, 2^{m-k} x) + \sum_{k=1}^{n} 2^{k-n-1} u(2^{n-k} x, 2^{n-k} x) \; [\text{by (20)}]$$

$$\leq \sum_{k=1}^{n} 2^{k-n} u(2^{n-k} x, 2^{n-k} x)$$
$$= \sum_{s=0}^{n-1} \frac{u(2^s x, 2^s x)}{2^s}$$

$$= \sum_{s=0}^{n-1} \frac{\phi(2^s x, 0) + \frac{3}{2}\phi(2^s x, -2^s x) + \frac{1}{2}\phi(2^s x, 2^s x)}{2^s} < \epsilon(x) < \infty$$

for all $x \in G$ and integers $0 \le m < n$.
Hence the sequence $\left\{\frac{p(2^n x)}{2^n}\right\}$ is a Cauchy sequence in X.
Since X is a Banach space, the sequence $\left\{\frac{p(2^n x)}{2^n}\right\}$ converges to some $C(x) \in X$ where $C : G \to X$ is defined by

$$C(x) := \lim_{n\to\infty} \frac{p(2^n x)}{2^n} \tag{21}$$

for all $x \in G$.
Now replacing x by $2^n x$ and y by $2^n y$ in (6) we have
$\| p(2^n(2x+y)) - p(2^n(x+2y)) - 3q(2^n x) + 3r(2^n y) \| \le \phi(2^n x, 2^n y)$
i.e.,, $\left\| \frac{p(2^n(2x+y))}{2^n} - \frac{p(2^n(x+2y))}{2^n} - \frac{3q(2^n x)}{2^n} + \frac{3r(2^n y)}{2^n} \right\| \le \frac{\phi(2^n x, 2^n y)}{2^n}$
Therefore,

$$\lim_{n\to\infty} \left\| \frac{p(2^n(2x+y))}{2^n} - \frac{p(2^n(x+2y))}{2^n} - \frac{3q(2^n x)}{2^n} + \frac{3r(2^n y)}{2^n} \right\| \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n}$$

$$\text{i.e., } 0 \le \lim_{n\to\infty} \left\| \frac{p(2^n(2x+y))}{2^n} - \frac{p(2^n(x+2y))}{2^n} - \frac{3q(2^n x)}{2^n} + \frac{3r(2^n y)}{2^n} \right\| \le 0 \tag{22}$$

$$as\ \frac{\phi(2^n x, 2^n y)}{2^n} \to 0 \text{ as } n \to \infty,\ \phi \text{ being finite.}$$

$$\text{i.e., } \left\| \lim_{n\to\infty} \frac{p(2^n(2x+y))}{2^n} - \lim_{n\to\infty} \frac{p(2^n(x+2y))}{2^n} - \lim_{n\to\infty} \frac{3q(2^n x)}{2^n} + \lim_{n\to\infty} \frac{3r(2^n y)}{2^n} \right\| = 0 \tag{23}$$

Again from (10) replacing x by $2^n x$ and y by $2^n y$ we get

$$\left\| \frac{3q(2^n x)}{2^n} - \frac{3r(2^n x)}{2^n} \right\| \le \frac{\phi(2^n x, 2^n x)}{2^n}$$

$$\Longrightarrow \lim_{n\to\infty} \left\| \frac{3q(2^n x)}{2^n} - \frac{3r(2^n x)}{2^n} \right\| \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n x)}{2^n} = 0$$

$$\Longrightarrow \lim_{n\to\infty} \frac{q(2^n x)}{2^n} = \lim_{n\to\infty} \frac{r(2^n x)}{2^n} \quad \forall x \in G. \tag{24}$$

Similarly from (13), we have

$$\lim_{n\to\infty} \frac{p(2^n x)}{2^n} = \lim_{n\to\infty} \frac{3\,r(2^n x)}{2^n} \quad \forall x \in G. \tag{25}$$

Then from (23) using (21), (24) and (25), we get
$\| C(2x+y) - C(x+2y) - C(x) + C(y) \| = 0$

$$\text{i. e., } C(2x+y) - C(x+2y) = C(x) - C(y) \tag{26}$$

Now taking limit as $n \to \infty$ in (20), we have

$$\| p(x) - C(x) \| \le \lim_{n\to\infty} \sum_{k=1}^{n} 2^{k-n-1} u\left(2^{n-k} x,\ 2^{n-k} x\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{u(2^k x,\ 2^k x)}{2.\,2^k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\phi\left(2^k x, 0\right) + \frac{3}{2}\phi\left(2^k x, -2^k x\right) + \frac{1}{2}\phi\left(2^k x, 2^k x\right)}{2^k}$$

$$= \varepsilon(x) \tag{27}$$

Again, we obtain

$$\| 3q(x) + 3r(x) - 2C(x) \| = \| 2C(x) - 2p(x) + 2p(x) - 3q(x) - 3r(x) \|$$

$$\leq 2 \| C(x) - p(x) \| + \| 2p(x) - 3q(x) - 3r(x) \|$$

$$\leq 2\varepsilon(x) + \phi(x, -x) \quad \text{by using (27) and (11)}$$

for all $x \in G$.

Uniqueness: Let D be another mapping satisfying (7) (8) and (9). Also by (21) we have

$$D(x) = \lim_{n\to\infty} \frac{p(2^n x)}{2^n}$$

Now putting $y = 0$ in (26) we get

$$C(2x) = 2C(x).$$

From (21) we have

$$C\left(2^k x\right) = \lim_{n\to\infty} \frac{p\left(2^{n+k} x\right)}{2^n} = 2^k \lim_{n\to\infty} \frac{p\left(2^{n+k} x\right)}{2^{n+k}} = 2^k C(x)$$

Then by induction, it follows that

$C(2^n x) = 2^n C(x)$ and also $D(2^n x) = 2^n D(x)$.

So, $\| C(x) - D(x) \| = \left\| \frac{C(2^n x)}{2^n} - \frac{D(2^n x)}{2^n} \right\|$

$$= \left\| \frac{C(2^n x)}{2^n} - \frac{p(2^n x)}{2^n} + \frac{p(2^n x)}{2^n} - \frac{D(2^n x)}{2^n} \right\|$$

$$\leq \frac{\| C(2^n x) - p(2^n x) \| + \| D(2^n x) - p(2^n x) \|}{2^n}$$

$$\leq \frac{\varepsilon(2^n x) + \varepsilon(2^n x)}{2^n} = \frac{\varepsilon(2^n x)}{2^{n-1}} \to 0 \text{ as } n \to \infty, \text{ since } \phi \text{ is finite}$$

for all $x \in G$.

Hence, we have $C(x) = D(x)$. □

Corollary 3.1. *Let $\delta > 0$ and $p, q, r : G \to X$ be odd mappings satisfying the inequality*

$$\| p(2x + y) - p(x + 2y) - 3q(x) + 3r(y) \| \leq \delta$$

for all $x, y \in G$. Then there exits a unique mapping $C : G \to X$ such that

$$C(2x + y) - C(x + 2y) = C(x) - C(y),$$

$$\| p(x) - C(x) \| \leq 3\delta,$$

$$\| 3q(x) + 3r(x) - 2C(x) \| \leq 7\delta, \text{ for all } x \in G.$$

Corollary 3.2. *Let E_1, E_2 be two Banach space and let $p, q, r : E_1 \to E_2$ be odd mapping. Also let $\theta \geq 0$ with $0 \leq s < 1$ such that*

$$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \theta(\|x\|^s + \|y\|^s)$$

for all $x, y \in E_1$. Then there exits a unique mapping $C : E_1 \to E_2$ such that

$$C(2x+y) - C(x+2y) = C(x) - C(y),$$

$$\| p(x) - C(x) \| \leq \frac{5.2^s \theta \|x\|^s}{2(2^s - 2)},$$

$$\| 3q(x) + 3r(x) - 2C(x) \| \leq \frac{7.2^s - 4}{2^s - 2} \theta \|x\|^s$$

for all $x \in G$.

Corollary 3.3. *Let $0 \leq s + t < 1$ where s and t are the non-negative real numbers also $\theta \geq 0$ and let $p, q, r : G \to X$ be odd mappings. such that*

$$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \theta \|x\|^s \|y\|^t$$

for all $x, y \in E_1$. Then there exits a unique mapping $C : G \to X$ such that

$$C(2x+y) - C(x+2y) = C(x) - C(y),$$

$$\| p(x) - C(x) \| \leq \frac{2^{s+t} \theta}{2^{s+t} - 2} \|x\|^{s+t},$$

$$\| 3q(x) + 3r(x) - 2C(x) \| \leq \frac{3.2^{s+t} - 2}{2^{s+t} - 2} \theta \|x\|^{s+t}$$

for all $x \in G$.

Theorem 3.2. *Let $p, q, r : G \to X$ be mapping with $q(x) = p(x) + r(0)$ satisfying the inequality*

$$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \phi(x, y)$$

for all $x, y \in G$. Then there exits a unique mapping $C : G \to X$ such that

$$C(2x+y) - C(x+2y) = C(x) - C(y),$$

$$\| p(x) - C(x) \| \leq \epsilon_1(x),$$

$$\| C(x) - r(x) \| \leq \epsilon_1(x) + \frac{\phi(x, x)}{3} + \|r(0)\| \quad \text{for all } x \in G.$$

Proof. Prove of this theorem is same as the Theorem 3.1 □

Corollary 3.4. *Let $\delta > 0$ and $p, q, r : G \to X$ be mappings with $q(x) = p(x) + r(0)$ satisfying the inequality*

$$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \delta$$

for all $x, y \in G$. Then there exits a unique mapping $C : G \to X$ such that

$$C(2x+y) - C(x+2y) = C(x) - C(y),$$

$$\| p(x) - C(x) \| \leq \frac{\delta}{3},$$

$$| C(x) - r(x) | \leq \frac{2\delta}{3} + \|r(0)\| \text{ for all } x \in G.$$

Corollary 3.5. *Let E_1 , E_2 be two Banach space and let p , q , r : $E_1 \to E_2$ be mappings with $q(x) = p(x) + r(0)$ Also let $\theta \geq 0$ with $0 \leq s < 2$ such that*

$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \theta(\|x\|^s + \|y\|^s)$

for all $x , y \in E_1$. Then there exits a unique mapping $C : E_1 \to E_2$ such that

$C(2x+y) - C(x+2y) = C(x) - C(y)$,

$\| p(x) - C(x) \| \leq \dfrac{1}{4} \times \dfrac{2^s \theta \|x\|^s}{2^s - 4}$,

$\| C(x) - r(x) \| \leq \dfrac{11 \times 2^s - 32}{12(2^s - 4)} \theta \|x\|^s$

for all $x \in G$.

4. The Generalized Hyers-Ulam-Rassias Stability of The Pexider Type Functional Equation (3): Fixed point Approach

Theorem 4.1. *Let p , q , r : $G \to X$ be odd mappings for which there exists a mapping $\phi : X^2 \to [0, \infty)$ such that*

$$\phi(2x, 2x) \leq 2\,\alpha\,\phi(x, x) \text{ for } 0 < \alpha < 1$$

$$\text{and } \| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \phi(x, y)$$

for all $x , y \in G$. Then there exits a unique mapping C' : $G \to X$ such that

$$C'(2x+y) - C'(x+2y) = C'(x) - C'(y) ,$$

$$\text{and } \| p(x) - C'(x) \| \leq \frac{1}{2(1-\alpha)} u(x, x), \forall x, y \in G \tag{28}$$

where $u(x,x) = \phi(x, 0) + \dfrac{3}{2}\phi(x, -x) + \dfrac{1}{2}\phi(x, x)$.

Proof. Consider the set $\rho := \{g : X \to Y, g(0) = 0\}$
and introduce a generalized metric d on ρ by

$$d(g, h) = \inf\{c \in R^+ : \|g(x) - h(x)\| \leq cu(x, x)\ \forall x \in X\}$$

where $g , h \in \rho$.
It is easy to prove that (ρ, d) is complete [1].

Also consider a mapping $J : \rho \to \rho$ such that

$$J\,g(x) := \frac{1}{2} g(2x)$$

for all $g \in \rho$ and $x \in X$. We now prove that J is a strictly contracting mapping of ρ with the Lipschitz constant α.
Let $g , h \in \rho$ and $\epsilon > 0$. Then there exists $c_1 \in R^+$ satisfying

$$\|g(x) - h(x)\| \leq c_1 u(x, x)\ \forall x \in X$$

such that $d(g, h) \leq c_1 < d(g, h) + \epsilon$.
Then

$$\begin{aligned} &\inf\{c \in R^+ : \|g(x) - h(x)\| \leq cu(x, x)\ \forall x \in X\} \leq c_1 < d(g, h) + \epsilon \\ \Longrightarrow\ &\inf\{c \in R^+ : \|g(2x) - h(2x)\| \leq cu(2x, 2x)\ \forall x \in X\} \leq c_1 < d(g, h) + \epsilon \\ \Longrightarrow\ &\inf\{c \in R^+ : \|\frac{g(x)}{2} - \frac{h(x)}{2}\| \leq \frac{c}{2} u(2x, 2x)\ \forall x \in X\} \leq c_1 < d(g, h) + \epsilon \end{aligned}$$

$$\implies \inf\{c \in R^+ : \|J\, g(x) - J\, h(x)\| \le c\,\alpha\, u(x,x)\ \forall x \in X\} \le c_1 < d(g,h) + \epsilon$$

$$\implies d\left\{\frac{1}{\alpha}(J\,g\,,\,J\,h)\right\} < d(g\,,\,h) + \epsilon$$

$$\implies d\{(J\,g\,,\,J\,h)\} < \alpha\,\{d(g\,,\,h) + \epsilon\}$$

Taking $\epsilon \to 0$ we get $d\{(J\,g\,,\,J\,h)\} \le \alpha\,\{d(g\,,\,h)\}$
Therefore J is a strictly contractive mapping with Lipschitz constant $\alpha < 1$.

Also similarly as before from ((14)) we have

$$\|\,p\,(2\,x) - 2p\,(x)\,\| \le u\,(x,x)$$

$$\implies \left\|\frac{p\left(2^{n+1}\,x\right)}{2^{n+1}} - \frac{p\,(2^n\,x)}{2^n}\right\| \le \frac{u\,(2^n\,x, 2^n\,x)}{2^{n+1}}$$

$$\le \frac{\alpha^n\,u(x,\,x)}{2},$$

$$\text{Hence}, d\,(\,J^{\,n+1}\,f\,,\,J^{\,n}\,f\,) \le \frac{\alpha^n}{2} < \infty$$

as Lipschitz constant $\alpha < 1$ for $n \ge n_0 = 1$.

Therefore by Theorem 2.1 there exists a mapping $C' : X \to Y$ satisfying the following:

1. C' is a fixed point of J, that is, $C'\,(\,x\,) = \frac{1}{2}\,C'\,(\,2\,x\,)$ for all $x \in X$. Since $p : X \to Y$ is an odd mapping, therefore $C' : X \to Y$ is also an odd mapping and

2. The mapping C' is a unique fixed point of J in the set
$\rho_1 = \{\,g \in \rho : d\,(\,J^{\,n_0}\,p\,,\,g\,) = d\,(\,J\,p\,,\,g\,) < \infty\,\}$
Therefore $d\,(\,J\,p\,,\,C'\,) < \infty$.
Also from (15), $d\,(\,J\,p\,,\,p\,) \le \frac{1}{2} < \infty$
Thus $p \in \rho_1$
Now, $d\,(\,p\,,\,C'\,) \le d\,(\,p\,,\,J\,p\,) + d\,(\,J\,p\,,\,C'\,) < \infty$.
Thus there exists $c \in (\,0\,,\,\infty\,)$ satisfying

$$\|p(x) - C'(x)\| \le cu(x,x)\ \forall x \in X$$

$$\implies \|p(2^n\,x) - C'(2^n\,x)\| \le cu(2^n\,x, 2^n\,x)$$

$$\|\frac{p(2^n\,x)}{2^n} - \frac{C'(2^n\,x)}{2^n}\| \le \frac{c}{2^n}u(2^n\,x, 2^n\,x) \le c\,\alpha^n\,u(x,\,x)$$

$$\implies \|\frac{p(2^n\,x)}{2^n} - C'(x)\| \le c\,\alpha^n\,u(x,\,x)$$

$$\text{since, } C'(x) = \frac{1}{2}C'(2x) = ... = \frac{1}{2^n}C'(2^n\,x)$$

Therefore $d\,(\,J^{\,n}\,f\,,\,C'\,) \le \alpha^n\,c \to 0$ as $n \to \infty$ and $\alpha < 1$.
This implies the equality

$$C'\,(\,x\,) := \lim_{n\to\infty} J^{\,n}\,p\,(\,x\,) = \lim_{n\to\infty} \frac{1}{2^n}\,p\,(\,x\,)$$

for all $x \in X$.

3. $d(p, C') \leq \frac{1}{1-L} d(p, Jp)$ with $p \in \rho_1$ which implies the inequality

$$d(p, C') \leq \frac{1}{1-\alpha} \times \frac{1}{2} = \frac{1}{2(1-\alpha)}$$

This implies that the inequality (28) holds.
The uniqueness of C' follows from the the fixed point Theorem 2.1. □

Theorem 4.2. *Let $p, q, r : G \to X$ be mappings with $q(x) = p(x) + r(0)$ such that $0 < \alpha < 1$ satisfying the inequality*

$$\phi(2x, 2x) \leq 4\,\alpha\,\phi(x, x)$$

$$\| p(2x+y) - p(x+2y) - 3q(x) + 3r(y) \| \leq \phi(x, y)$$

for all $x, y \in G$. Then there exits a unique mapping $C' : G \to X$ such that

$$C'(2x+y) - C'(x+2y) = C'(x) - C'(y),$$

$$\| p(x) - C'(x) \| \leq \frac{1}{4(1-\alpha)} u(x, x), \forall x, y \in G.$$

Proof. Prove of this theorem is same as in the Theorem 4.1. □

5. Comparative Study of The Results by the Two Proposed Methods and Conclusion

We have proposed two diversified ways of the study of the Hyers-Ulam-Rassias stability for the pexider type functional equation. The first one is the direct method in which we explore the Hyers-Ulam-Rassias stability with some suitable conditions and apply properties of Cauchy sequence. Here we aim to generalized the class of the possible control function ϕ and its effect in Theorem 3.1 and Theorem 3.5.In this method, we consider the sum of convergent series of the control function ϕ.
In the second method, that is, in the fixed point method we consider sum of some control functions. We consider an additional particular condition of ϕ that is $\phi(2x, 2y) \leq 2\,\alpha\phi(x, y)$ in the result of Theorem 4.1 and $\phi(2x, 2y) \leq 4\,\alpha\phi(x, y)$ in the result of Theorem 4.2. Also, in the second method further restriction $0 < \alpha < 1$ is imposed in the theorem. Further, if we analyze the results of both the cases, we see that if $\alpha = 0$, the result of the fixed point coincides with the direct method. If we consider the control function to be a fixed number, then also the result of the both cases will be same.
In future, we will investigate the Hyers-Ulam-Rassias stability for the pexider type functional equation in Pythagorean fuzzy normed spaces, probabilistic normed spaces, modular spaces and others using the proposed method.

References

[1] Batool, A., Nawaz, S., Ege, O., Sen, M. da la, (2020), Hyers–Ulam stability of functional inequalities: a fixed point approach, J. Inequalities and Applications.

[2] Cadariu, L., Radu, V., (2003), Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math., 4(1), Article ID 4.

[3] Chmielinski, J., Tabor, J., (1993), On approximate solutons of the Pexider equation, Aequationes Math. 46, pp. 143-163.

[4] Ciepli´nsk, K., (2021), Ulam stability of functional equations in 2-Banach spaces via the fixed point method, J. Fixed Point Theory Appl., 23:33.

[5] Czerwik, S., (1992), On the stability of the quadratic mappings in normed spaces, Abh. Math. Sem. Univ. Hamburg, 62, pp. 59-64.

[6] Chang, I. S., Kim, H. M., (2002), On the Hyers-Ulam stability of quadratic functional equations, J. Ineq. Pure App. Math., 3 Art. 33, pp. 1-12.

[7] Forti, G. L., (1995), Hyers-Ulam stability of functional equations in several variables, Aeq. Math. 50, pp. 143-190.

[8] Gavruta, P., (1982), A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Func. Anal., 46, pp. 126-130.

[9] Hyers, D. H., (1941), On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U. S. A., 27, pp. 222-224.

[10] Jun, K. W., Kim, H. M., Lee, D. O., (2002), On the stability of a quadratic functional equation, J. Chung. Math. Sci., 15(2), pp. 73-84.

[11] Jun, K. W., Shin, D. S., Kim, B. D., (1999), On Hyers-Ulam-Rassias stability of the Pexider equation, J. Math. Anal. Appl., 239, pp. 20-29.

[12] Jung, S. M., (1999), On the Hyers-Ulam-Rassias stability of a quadratic functional equations, J. Math. Anal. Appl., 232, pp. 384-393.

[13] Jung S. M., Popa, D., Rassias M.T., (2014), On the stability of the linear functional equation in a single variable on complete metric groups, Journal of Global Optimization, 59, pp. 165–171.

[14] Jung S. M., Rassias M.T., Mortici C., (2015), On a functional equation of trigonometric type, Applied Mathematics and Computation, 252, pp. 294-303.

[15] Lee, Y. H., Jung, S.M., Rassias M.T., (2018), Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation, Journal of Mathematical Inequalities, 12(1), pp. 43–61.

[16] Rassias, Th. M., (1978), On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72, pp. 297-300.

[17] Saha, P., Samanta, T. K., Mondal, P., Choudhury, B. S., Sen, M. D. L., (2020), Applying fixed point techniques to stability problems in intuitionistic fuzzy Banach spaces, Mathematics, 8(6), pp.974.

[18] Saha, P., Mondal, P., Choudhury, B. S., (2021), Stability property of functional equations in modular spaces: A fixed-point approach, Mathematical Notes, 2(109), pp. 262-269.

[19] Samanta, T. K., Mondal, P., Kayal, N. C., (2013), The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation in fuzzy Banach spaces, Ann. Fuzzy Math. Inform., 6, pp. 59-68.

[20] Ulam, S. M., (1960), Problems in Modern Mathematics, Cahp. VI, Wiley, New York.

[21] Wu, T. Z., Rassias, J. M., Xu, W. X., (2010), Generalized Ulam-Hyers Stability of a General Mixed AQCQ-functional Equation in Multi-Banach Spaces: a Fixed Point Approach, European J. Pure and Applied Mathematics, 6, pp. 1032-1047.

[22] Phochai, T., Saejung, S., (2019), Some notes on the Ulam stability of the general linear equation. Acta Math. Hungar., pp. 158, 40-52.

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