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HIGHER ORDER HERMITE-FEJÉR INTERPOLATION ON THE UNIT CIRCLE

S. BAHADUR[1], VARUN[1*], §

Abstract. The aim of this paper is to study the approximation of functions using a higher-order Hermite-Fejér interpolation process on the unit circle. The system of nodes is composed of vertically projected zeros of Jacobi polynomials onto the unit circle with boundary points at ± 1. Values of the polynomial and its first four derivatives are fixed by the interpolation conditions at the nodes. Convergence of the process is obtained for analytic functions on a suitable domain, and the rate of convergence is estimated.

Keywords: Unit circle, Non-uniform nodes, Jacobi Polynomial, Rate of Convergence, Lagrange Interpolation, Hermite-Fejér interpolation.

AMS Subject Classification: 41A10, 97N50, 41A05, 30E10.

1. Introduction

Approximation of continous functions can be done using different methods by constructing algebraic or trigonometric polynomials. Hermite interpolation attracted the attention of many researchers in the last century.
Hermite interpolation [14]: It is the process of finding a polynomial which coincides with the continous function at certain pre-assigned points, called the nodes of interpolation, and its successsive derivatives coinciding with arbitrarily chosen numbers.

An important step was taken when Fejér [10] in 1916 proved a theorem where the values of the derivatives in the Hermite scheme were equal to zero.

Fejér's theorem : If $f \in C[-1,1]$, then $H_n(f,x)$ converges to $f(x)$ uniformly on [-1,1] as n tends to infinty. Interpolation polynomials $H_n(f,x)$ is defined by

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})(1 - x_{kn}x)\left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2,$$

[1] University of Lucknow, Faculty of Science, Department of Mathematics and Astronomy, Lucknow, Uttar Pradesh, India.
e-mail: swarnimabahadur@ymail.com; ORCID: https://orcid.org/0000-0001-8734-943X.
e-mail: varun.kanaujia.1992@gmail.com; ORCID: https://orcid.org/0000-0001-6816-1167.
[*] Corresponding author.
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where x_{kn} are the zeros of the Chebyshev polynomial of the first kind.
$H_n(f,x)$ satisfies the below given conditions where $k = 1, 2, ..., n$.

$$H_n(f, x_{kn}) = f(x_{kn}) \quad and \quad H'_n(f, x_{kn}) = 0.$$

Mills [13] in his paper highlights Hermite and Hemite Fejér interpolation as important techniques in the approximation theory. Knoop and Locher [12] modified Hermite Fejer interpolation at the zeros of Jacobi polynomials by introducing more boundary conditions and obtaining pointwise convergence for arbitrary $\alpha, \beta > -1$. Fejér's theorem has been extended to more general nodal systems. For example, in 2001, Daruis and González-Vera [9] extended Fejér's result to the unit circle by considering the nodal system constituted by the complex n^{th} roots of unity. They proved that the sequence of Hermite-Fejér interpolation polynomials uniformly converge for continous functions on the unit circle.

Berriochoa, Cachafeiro and García-Amor [5] extended the Fejér's second theorem to the unit circle. Then Berriochoa, Cachafeiro, Díaz, and Martínez Brey [6] obtained the supremum norm of the error of interpolation for analytic functions and computed the order of convergence of Hermite-Fejér interpolation on the unit circle considering the same set of nodes as of [9].

Apart from the uniform nodal system (where nodes are equally spaced on the unit circle), Hermite-Fejér interpolation on the unit circle have been also studied on some non-uniformly distributed nodes on the unit circle (see [1], [2], [3], [4] and [8]).
Higher order Hermite-Fejér interpolation: It is a process of finding a polynomial which coincides with a continous function at the nodes of the interpolation and the derivatives upto r^{th} order ($r > 1$) are null at the nodal points.

A considerable number of papers on higher order Hermite-Fejér interpolation processes on real nodes have been published (see [15] and [18]). This motivated us to consider a higher order Hermite-Fejér interpolation problem on non-uniform set of complex nodes on the unit circle. Let us denote nodal system containing the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ by Gauss Jacobi point system. Let us also define two sets $\mathbb{T} = \{z : |z| = 1\}$ and $\mathbb{D} = \{z : |z| < 1\}$.

In the present paper, we consider a Hermite-Fejér interpolation problem on the nodal system constituted of ± 1 and the projections of the Gauss Jacobi point system vertically onto the unit circle by the transformation $x = \dfrac{1+z^2}{2z}$. The aim of this paper is to extend the Hermite-Fejér interpolation on the unit circle problem on all the above said projected nodes upto the fourth derivative and prove the following convergence theorem:

Theorem 1.1. *Let $f(z)$ is a function continous on $\mathbb{T} \cup \mathbb{D}$ and analytic on $\mathbb{D}$. For $\beta \leq \alpha \leq \frac{1}{2}$, the sequence of interpolatory polynomial $\{Q_n(z)\}$ satisfies the below relation*

$$| Q_n(z) - f(z) | = \boldsymbol{O}\big(\omega(f, n^{-1}) \log n\big), \tag{1}$$

where $\omega(f, n^{-1})$ represents the modulus of continuity of the function $f(z)$, α and β are parameters of Jacobi Polynomial $P_n^{(\alpha,\beta)}(x)$ and $\boldsymbol{O}$ notation refers to as $n \to \infty$.

The paper has been organised in following manner. Preliminaries are given in section 2. Section 3 covers the interpolation problem and explicit representation of the interpolatory polynomial. Section 4 is devoted to finding estimates and the proof of theorem 1.1 has been assigned section 5.

2. Preliminaries

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0.$$

Using the Szegő transformation $x=\dfrac{1+z^2}{2z}$,

$$\begin{aligned}&(z^2-1)^4P_n^{(\alpha,\beta)''}(x)+4z(z^2-1)\left[\{(\alpha+\beta+2)z^2+1\}(z^2-1)-2z^3(\beta-\alpha)\right]P_n^{(\alpha,\beta)'}(x)\\&-16z^6n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0.\end{aligned} \tag{2}$$

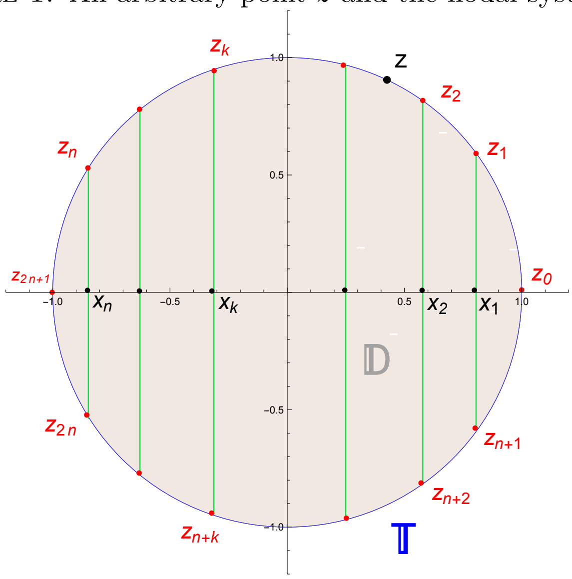
Let Z_n be set of nodes

$$\begin{aligned}Z_n=\{z_0=1,z_{2n+1}=-1,z_k=x_k+iy_k=\cos\theta_k+i\,\sin\theta_k\,;\,z_{n+k}=\overline{z_k};\\ k=1,2,3,...,n\,;\,x_k,y_k\in R\},\end{aligned}$$

which are obtained by projecting vertically the Gauss Jacobi point system on the unit circle together with ± 1.

The polynomial defined on Z_n are given by (3),

Figure 1. An arbitrary point z and the nodal system Z_n



$$R(z)=(z^2-1)W(z), \tag{3}$$

where

$$W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n, \tag{4}$$

$$K_n=2^{2n}n!\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}.$$

The fundamental polynomials of Lagrange interpolation on the zeros of $R(z)$ are given by

$$L_k(z) = \frac{R(z)}{(z-z_k)R'(z_k)}, \qquad k = 0,1,...,2n+1. \tag{5}$$

We can write $z = x + iy$, where $x, y \in R$. If $z \in \mathbb{T}$, then

$$\mid z^2 - 1 \mid = 2\sqrt{1-x^2} \tag{6}$$

and

$$\mid z - z_k \mid = \sqrt{2}\sqrt{1 - xx_k - \sqrt{1-x^2}\sqrt{1-x_k^2}}. \tag{7}$$

In order to evaluate the estimates of the fundamental polynomials formed in section 3, we will be using below results.
All the estimates from (8)-(13) are obtained under the restriction $\beta \le \alpha$.
For $-1 \le x \le 1$, we have

$$(1-x^2)^{1/2} \mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}), \tag{8}$$

$$\mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha}), \tag{9}$$

$$\mid P_n^{(\alpha,\beta)'}(x) \mid = O(n^{\alpha+2}), \tag{10}$$

$$\mid P_n^{(\alpha,\beta)''}(x) \mid = O(n^{\alpha+4}). \tag{11}$$

Considering set of nodes Z_n, where $x_k = \cos\theta_k\,, k = 1,2,...,n$ are the zeros of $P_n^{(\alpha,\beta)}(x)$, then

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{12}$$

$$\mid P_n^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}. \tag{13}$$

For more details, refer pg.164-166 of [17].

Let $f(z)$ be continous on $\mathbb{T} \cup \mathbb{D}$ and analytic on $\mathbb{D}$. Then, there exists a polynomial $F_n(z)$ of degree less than $(2n+2)(r+1)$ satisfying Jackson's inequality.[11]

$$\mid f(z) - F_n(z) \mid \le C\,\omega(f, n^{-1}), \tag{14}$$

where $\omega(f, n^{-1})$ represents the modulus of continuity of the function $f(z)$ and C is independent of n and z.

3. The problem and explicit representation of interpolatory polynomial

Here, we are interested in determining the convergence of interpolatory polynomial $Q_n(z)$ of degree less than $(2n+2)(r+1)$ on the distinct set of nodes $\{z_k\}_{k=0}^{2n+1}$ with Hermite conditions at all points satisfying

$$\begin{cases} Q_n(z_k) = \alpha_k, & k = 0,1,...,2n+1 \\ Q_n^{(r)}(z_k) = 0, & k = 0,1,...,2n+1,\ r = 1,2,3,4\,, \end{cases} \tag{15}$$

where α_k's are arbitrary complex constants.

Theorem 3.1. *We shall write $Q_n(z)$ satisfying (15)*

$$Q_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_{0k}(z), \tag{16}$$

where $A_{0k}(z)$ is a polynomial of degree less than $(2n+2)(r+1)$ satisfying the conditions given in (17).

For $j, k = 0, 1, ..., 2n+1$,

$$\begin{cases} A_{0k}(z_j) = \delta_{kj}, \\ A_{0k}^{(r)}(z_j) = 0 \qquad ; r = 1, 2, 3, 4 \ , \end{cases} \tag{17}$$

where

$$A_{0k}(z) = [L_k(z)]^5 + \sum_{p=1}^{4} c_{pk} A_{pk}(z), \tag{18}$$

$$A_{pk}(z) = [R(z)]^p (L_k(z))^{5-p}, \tag{19}$$

$$c_{1k} = -\frac{5\, L_k'(z_k)}{R'(z_k)}, \tag{20}$$

$$c_{2k} = -\frac{5}{2!\, [R'(z_k)]^2} \left[L_k''(z_k) + 10\, [L_k'(z_k)]^2\right], \tag{21}$$

$$c_{3k} = -\frac{5}{3!\, [R'(z_k)]^3} \left[-18\, L_k''(z_k)\, L_k'(z_k) \ + \ L_k'''(z_k) \ - \ 198 [L_k'(z_k)]^3\right], \tag{22}$$

$$\begin{aligned} c_{4k} = -\frac{5}{4!\, [R'(z_k)]^4} \Bigg[& L_k''''(z_k) - 24 L_k'''(z_k)\, L_k'(z_k) \\ & + [L_k''(z_k)]^2 - 156 [L_k'(z_k)]^2 L_k''(z_k) + 2544 [L_k'(z_k)]^4 \Bigg]. \end{aligned} \tag{23}$$

Proof. Let $A_{0k}(z)$ be written as

$$A_{0k}(z) = [L_k(z)]^5 + \sum_{p=1}^{4} c_{pk} [R(z)]^p (L_k(z))^{5-p}. \tag{24}$$

At $z = z_j$, where $j = 0, 1, ..., 2n+1$,

$$A_{0k}(z_j) = [L_k(z_j)]^5 + \sum_{p=1}^{4} c_{pk} [R(z_j)]^p (L_k(z_j))^{5-p}.$$

Using (3), we have $R(z_j) = 0$ and from (5), we have

$$A_{0k}(z_j) = \delta_{kj}. \tag{25}$$

Clearly, the first set of condition in (17) is satisfied.
In order to determine the c_{pk}'s, we use the second set of conditions of (17).
On differentiating $A_{0k}(z)$ in (24) one time with respect to z, we get

$$A_{0k}'(z) = 5 L_k'(z) [L_k(z)]^4 + c_{1k} [R(z) (L_k(z))^4]' + \Big[\sum_{p=2}^{4} c_{pk} [R(z)]^p (L_k(z))^{5-p} \Big]'. \tag{26}$$

Clearly, at $z = z_j$ $(j \neq k)$, we have $A'_{0k}(z) = 0$.
At $z = z_k$, $A'_{0k}(z)$ must be equal to zero. We have

$$5\, L'_k(z_k) + c_{1k}\, R'(z_k) = 0,$$

which provides (20). In a similar manner, differentiating (24) two, three and four times with respect to z gives (21), (22) and (23) respectively by using conditions given in (17). □

4. Estimation of the fundamental polynomials

In order to find the estimates, we intend to represent the constants c_{pk} in general form as given under (p=1,2,3,4)

$$c_{pk} = \frac{5}{p![R'(z_k)]^p} \sum_{s=0}^{[\frac{p}{2}]} \sum_{r=s}^{p-s} e_{psr} [L_k^{(s)}(z_k)]^r L_k^{(p-sr)}(z_k), \tag{27}$$

where e_{psr} are the constants independent of n and z and $\left[\dfrac{p}{2}\right]$ denotes greatest integer function Also, $L_k^{(s)}(z_k)$ denotes s^{th} derivative of $L_k(z)$ at $z = z_k$.

Lemma 4.1. *Let $L_k(z)$ be given by (5), then for $z \in \mathbb{T} \cup \mathbb{D}$, we have*

$$| L_k(z) | = \boldsymbol{O}\left(\frac{1}{k^{-\alpha+\frac{3}{2}}}\right) \tag{28}$$

Proof. For $k = 1, 2, ..., 2n$

$$L_k(z) = \frac{R(z)}{(z - z_k)R'(z_k)}. \tag{29}$$

Taking modulus on the both sides,

$$\begin{aligned} | L_k(z) | &= \frac{| R(z) |}{| z - z_k || R'(z_k) |}, \\ &= \frac{| (z^2-1)W(z) |}{| z - z_k || \{2zW(z) + (z^2-1)W'(z))\}_{z=z_k} |}. \end{aligned}$$

Since $z'_k s$ are the zeros of $W(z)$, using (4), we get

$$\begin{aligned} | L_k(z) | &= \frac{\left|(z^2-1)K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n\right|}{| z - z_k | \left|(z_k^2-1)\left\{K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n\right\}'_{z=z_k}\right|} \\ &= \frac{\left|(z^2-1)P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n\right|}{| z - z_k | \left|(z_k^2-1)\left\{nz_k^{n-1}P_n^{(\alpha,\beta)}(x_k) + z_k^n P_n^{(\alpha,\beta)'}(x_k)\left(\frac{z_k^2-1}{2z_k^2}\right)\right\}\right|} \\ &= \frac{2|z^2-1||P_n^{(\alpha,\beta)}(x)||z|^n}{| z - z_k | \, |z_k|^{n-2}|z_k^2-1|^2|P_n^{(\alpha,\beta)'}(x_k)|}. \end{aligned}$$

Using (6) and (7), we get

$$\begin{aligned}
\mid L_k(z) \mid &= \frac{2.2\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)||z|^n}{4(1-x_k^2)\sqrt{2}\sqrt{1-xx_k-\sqrt{1-x^2}\sqrt{1-x_k^2}}|P_n^{(\alpha,\beta)'}(x_k)|} \\
&= \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)||z|^n\sqrt{1-xx_k+\sqrt{1-x^2}\sqrt{1-x_k^2}}}{\sqrt{2}(1-x_k^2)\sqrt{(1-xx_k)^2-(1-x^2)(1-x_k^2)}|P_n^{(\alpha,\beta)'}(x_k)|} \\
&= \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)||z|^n\sqrt{1-xx_k+\sqrt{1-x^2-x_k^2+x^2x_k^2}}}{\sqrt{2}(1-x_k^2)\ \mid x-x_k\mid |P_n^{(\alpha,\beta)'}(x_k)|} \\
&= \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)||z|^n\sqrt{1-xx_k+\sqrt{(1-xx_k)^2-(x-x_k)^2}}}{\sqrt{2}(1-x_k^2)\ \mid x-x_k\mid |P_n^{(\alpha,\beta)'}(x_k)|} \\
&\le \frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)|\sqrt{1-xx_k}}{(1-x_k^2)\ \mid x-x_k\mid |P_n^{(\alpha,\beta)'}(x_k)|}.
\end{aligned}$$

For $\mid x-x_k \mid\ge \frac{1}{2}\mid 1-x_k^2\mid$, we get

$$\mid L_k(z)\mid \ \le\ C\frac{\sqrt{1-x^2}|P_n^{(\alpha,\beta)}(x)|}{(1-x_k^2)^{3/2}\,|P_n^{(\alpha,\beta)'}(x_k)|},$$

where C is constant independent of n and z. Using (8), (12) and (13), we have

$$\mid L_k(z)\mid \ =\ \boldsymbol{O}\left(\frac{1}{k^{-\alpha+\frac{3}{2}}}\right). \tag{30}$$

Similarly, for $\mid x-x_k\mid\le\frac{1}{2}\mid 1-x_k^2\mid$, we get the same result as (30).
For $k=0$ and $k=2n+1$, we have

$$\mid L_0(z)\mid=\mid L_{2n+1}(z)\mid=\boldsymbol{O}(1). \tag{31}$$

From (30) and (31), we have Lemma 4.1. □

Lemma 4.2. *Let c_{pk} be given by (27), then*

$$\mid c_{pk}\mid=\boldsymbol{O}\left(\frac{1}{K_n^p\,n^{p(\alpha-1)}\,k^{p/2-p\alpha}}\right). \tag{32}$$

Proof. From (3) and (4), we have

$$R'(z_k)=\left(\frac{K_n}{2}\right)z_k^{n-2}(z_k^2-1)^2P_n^{(\alpha,\beta)'}(x_k).$$

Taking modulus on both the sides, we get

$$\mid R'(z_k)\mid=\left(\frac{K_n}{2}\right)\mid z_k\mid^{n-2}\mid(z_k^2-1)\mid^2\mid P_n^{(\alpha,\beta)'}(x_k)\mid.$$

From (6), (12) and (13), we have

$$| R'(z_k) |= \boldsymbol{O}(K_n\, k^{-\alpha+\frac{1}{2}} n^{\alpha}). \tag{33}$$

Similarly, from (2), (5) and (6), we have

$$| L_k^{(s)}(z_k) |= \boldsymbol{O}(n^s). \tag{34}$$

Using (33) and (34) in (27), we have Lemma 4.2.

□

Lemma 4.3. *Let $A_{0k}(z)$ be given by (18) and c_{pk} given by (27), then for $z \in \mathbb{T} \cup \mathbb{D}$,*

$$\sum_{k=0}^{2n+1} | A_{0k}(z) | = \boldsymbol{O}(\log n), \tag{35}$$

where $-1 < \alpha \leq \dfrac{1}{2}$.

Proof. From (3) and (4), we have

$$R(z) = (z^2-1)K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right) z^n. \tag{36}$$

Taking modulus on both the sides and using (6) and (8), we have

$$| R(z) |= \boldsymbol{O}(K_n n^{\alpha-1}). \tag{37}$$

For $| x_k - x | \geq \dfrac{1}{2} | 1 - x_k^2 |$ and from (18) and (19), we have

$$\sum_{k=0}^{2n+1} | A_{0k}(z) | \;=\; \sum_{k=0}^{2n+1} | L_k(z) |^5 + \sum_{k=0}^{2n+1}\sum_{p=1}^{4} | c_{pk} || R(z) |^p | L_k(z) |^{5-p} .$$

Using (37), Lemma 4.1 and Lemma 4.2, we get

$$\sum_{k=0}^{2n+1} | A_{0k}(z) | \;=\; \boldsymbol{O}\left(\sum_{k=0}^{2n+1} \frac{1}{k^{-5\alpha+\frac{15}{2}}} + \sum_{k=0}^{2n+1}\sum_{p=1}^{4} \frac{1}{k^{-5\alpha-p+\frac{15}{2}}}\right),$$

$$\sum_{k=0}^{2n+1} | A_{0k}(z) |= \boldsymbol{O}\left(\sum_{k=0}^{2n+1} \frac{1}{k}\right) = \boldsymbol{O}(\log n), \qquad \left\{ -1 < \alpha \leq \frac{13}{10} - \frac{p}{5} \right\}. \tag{38}$$

Similarly, for $| x_k - x |\leq \frac{1}{2} | 1 - x_k^2 |$, we get the same result.
Since, range of α with $p = 4$ lies in the intersection of all the cases. Hence, the lemma follows. □

5. Proof of theorem 1.1

Let $f(z)$ be a function that is continous on $\mathbb{T} \cup \mathbb{D}$ and analytic on $\mathbb{D}$. Since $Q_n(z)$ is the uniquely determined polynomial of degree less than $(2n+2)(r+1)$ and the polynomial $F_n(z)$ satisfying equation (14) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) A_k(z). \tag{39}$$

Then

$$| Q_n(z) - f(z) | \leq | Q_n(z) - F_n(z) | + | F_n(z) - f(z) | . \tag{40}$$

Using (16) and (39), we have

$$| Q_n(z) - f(z) | \quad \leq \quad \underbrace{\sum_{k=0}^{2n+1} | f(z_k) - F_n(z_k) || A_k(z) |}_{N_1} + \underbrace{| F_n(z) - f(z) |}_{N_2}.$$

We have

$$| Q_n(z) - f(z) | \leq N_1 + N_2. \tag{41}$$

From (14) and (35), we have

$$N_1 = \boldsymbol{O}\big(\omega(f, n^{-1}) \log n\big). \tag{42}$$

From (14), we have

$$N_2 = \boldsymbol{O}\big(\omega(f, n^{-1})\big). \tag{43}$$

Using (42) and (43) in (41), we get

$$| Q_n(z) - f(z) | = \quad \boldsymbol{O}\big(\omega(f, n^{-1}) \log n\big).$$

Hence, Theorem 1.1 follows.

6. Conclusions

This research paper poses a completely new problem where Hermite-Fejér interpolation on the unit circle is extended upto the fourth derivative on the nodal system constituted of ± 1 and the projections of the Gauss Jacobi point system vertically onto the unit circle. On comparing our main convergence result (1) with the theorem 14.6 of [17] ,we can conclude that for $\alpha = \frac{1}{2}$, we get a good approximation of a function which is continous on $\mathbb{T} \cup \mathbb{D}$ and analytic on $\mathbb{D}$. The reason behind this is to make use of first modulus of continuity instead of the second modulus of continuity used in theorem 14.6 of [17]. Since the present problem invloves extension upto fourth derivative, a subtle open problem is to generalize the problem upto m^{th} derivative, where m can be even or odd. This will provide a much broader aspect of convergence and comparisions to the present problem.

Author contributions:
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Dr. Swarnima Bahadur is currently working as an Assistant Professor in Department of Mathematics and Astronomy, University of Lucknow, Lucknow. Her area of specialization is in Approximation Theory and Mathematical Analysis. She is continously contributing in the field of analysing the convergence behaviour of different types of interpolatory polynomials on various set of nodes.

Varun is currently pursuing his Ph.D. under the guidance of Dr. Swarnima Bahadur in the Department of Mathematics and Astronomy, University of Lucknow. He works on different types of interpolation techniques to analyse and approximate analytic and continous function in a unit disk via interpolatory polynomial. He is keen to find the applications of interpolation on the unit circle and numerically justify the works done in this direction.