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AN ESTIMATE TO FUNCTIONS WITH SECOND DERIVATIVES IN HÖLDER CLASS BY MODULI OF CONTINUITY AND SOLUTION OF CHANDRASEKHAR'S WHITE DWARF EQUATION BY CHEBYSHEV WAVELET

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ABSTRACT. In this paper, modulus of continuity, second kind Chebyshev wavelet and Hölder class are studied. The moduli of continuity and approximations of functions whose second derivative belonging to Hölder class have been determined by second kind Chebyshev wavelet. The operational matrix of integration for second kind Chebyshev wavelet has been framed. Using this, numerical solutions of non-linear singular differential equations have been obtained. The modulus of continuity, approximations, solution of Lane-Emden equation of index p=1 as well as the comparison with the exact solution and applicability of second kind Chebyshev wavelet method in finding numerical solution of Chandrasekhar's white dwarf equation are significant achievements of this paper.

Keywords: Second kind Chebyshev wavelet, moduli of continuity, wavelet approximations, Chebyshev wavelet operational matrix of integration.

AMS Subject Classification: 41A50, 42C40, 65T60, 65L05, 65L99.

1. Introduction

The approximation of functions using wavelet series has been widely used in recent years. The approximation theory was initiated by Natanson([13]) and Zygmund([19]) using polynomials and summability theory. Since then, many approaches like spline, finite element, etc. have been employed, but the approximation of functions using wavelets has become a more interesting and effective tool. Wavelets have been frequently employed in approximation theory by researchers due to qualities such as compact support, orthonormality, and simple applicability.

There are various types of wavelets available in the literature such as Haar wavelet([6], [16]), Meyer wavelet([12]), Legendre wavelet([15]), Laguerre wavelet([7]), Chebyshev wavelet ([4].[18]).These are used to find approximation of functions. One of the objectives of this research paper is to obtain the moduli of continuity $W(f,\delta)$ of a function f such that its second derivative f'' belong to $H^{\alpha}[0,1)$ class by second kind Chebyshev wavelet

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expansion. The modulus of continuity of f, denoted by $W(f,\delta)$, is defined by $W(f,\delta) = \sup_{0<h\leq\delta} ||f(t+h) - f(t)||_2$, for every t belonging to a finite interval, has the property $\lim_{\delta\to 0^+} W(f,\delta) = 0$(Chui[2]). Due to this property, the better estimations of the rate of approximation of functions in different classes are obtained.

Wavelets have a wide range of applications due to their various properties like orthonormality, compact support which is applied for the inclusion of initial and boundary conditions, symmetry and well localization. These can be used in quantum field theory, numerical solutions to physical problems, digital processing and many more. Nowadays, wavelets are being used in solution of physical and engineering problems formulated as differential and integral equations. Unlike the traditional bases, wavelet basis provides a sparse representation of signals due to orthogonality and can also identify the singularities in an efficient way. These qualities of wavelets are highly appreciated for solving various linear and non-linear differential equations having singularities, since due to singularities, solving these equations by other methods is a challenging task.

In order to solve non-linear initial value problems involving singularity at some point in [0,1], a second kind Chebyshev operational integration matrix has been developed in this paper. Using this matrix of integration, the highest order derivative appearing in the differential equations and the initial conditions can be expressed as the second kind Chebyshev wavelet matrices. This approach has been applied to a non-linear differential equation called Lane-Emden equation of index p=1 given as:

$$y''(t) + \frac{2}{t}y'(t) + y(t) = 0,\ t \geq 0$$

with initial conditions $y(0) = 1, y'(0) = 0$ ([17]).

The same technique has also been employed to obtain the numerical solution of Chandrasekhar's white dwarf equation

$$y''(t) + \frac{2}{t}y'(t) + (y^2 - c)^{\frac{3}{2}} = 0,\ t \geq 0$$

with initial conditions $y(0) = 1, y'(0) = 0$ ([1]) which has a prominent role in the theory of stellar structure and astrophysics.

This paper is arranged as: Section 2. contains some definitions and preliminaries used in this paper. Section 3. gives the moduli of continuity of functions whose second-order derivatives belong to Hölder class $H^\alpha[0,1)$. Section 4. derives the second kind Chebyshev wavelet matrix of integration, methodology of solving non-linear singular ordinary differential equation and the numerical solution of Lane-Emden and Chandrasekhar's white dwarf equation. Section 5. contains the concluding remarks and lastly the references have been written used in framing this paper.

2. Definitions and Preliminaries

2.1. **Chebyshev polynomials of second kind.** Chebyshev polynomial $U_m(t)$ of second kind is a polynomial of degree m in t, defined by

$$U_m(t) = \frac{\sin(m+1)\theta}{\sin\theta}$$

when $t = \cos\theta$ (Mason et al.[11]). These polynomials are defined on the interval [-1,1] and are orthogonal with respect to the weight function $w(t) = \sqrt{1-t^2}$ as

$$\int_{-1}^{1} U_m(t)U_n(t)w(t)dt = \begin{cases} \frac{\pi}{2}, & m = n \\ 0, & \text{otherwise,} \end{cases}$$

for $m, n = 0, 1, 2, 3...$.
Chebyshev polynomials of second kind are: $U_0(t) = 1$, $U_1(t) = 2t$, $U_2(t) = 4t^2$, $U_3(t) = 8t^3 - 4t...$.
The recurrence formula for these polynomials are $U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$,
$m = 1, 2, 3, ...$ ([11]).

2.2. **Chebyshev wavelets of second kind.** Chebyshev wavelets $\psi_{n,m}(t)=\psi(k, n, m, t)$ have four arguments, where $n = 1, 2, 3, ..., 2^{k-1}, k \in \mathbb{Z}^+, m$ being the degree of Chebyshev polynomials and t the normalized time. These are defined on the interval [0,1) by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise,} \end{cases}$$

where $m = 0, 1, 2, ..., M$ and $n = 1, 2, ..., 2^{k-1}$ (Sahu et al.[18]).
Here, the coefficient $\sqrt{\frac{2}{\pi}}$ is for orthonormality, $(2n - 1)2^{-k}$ is the translation parameter and 2^{-k} is the dilation parameter. Also, the weight function $w(t) = \sqrt{1 - t^2}$ has to be dilated and translated as $w_n(t) = w(2^k t - 2n + 1)$.
$\{\psi_{n,m}\}$ is an orthonormal basis of $L^2_w[0, 1)$ w.r.t. weight function $w(t)$ i.e. $\langle\psi_{n,m}, \psi_{n',m'}\rangle_{w_n} = \delta_{n,n'}\delta_{m,m'}$, Kronecker's delta.

2.3. **Second kind Chebyshev expansion and approximation.** A function $f \in L^2_w[0, 1)$ can be expanded in terms of second kind Chebyshev wavelet as

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \quad \text{where } c_{n,m} = \int_0^1 f(t)\psi_{n,m}(t)w_n(t)dt \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) + \sum_{n=1}^{2^{k-1}}\sum_{M+1}^{\infty} c_{n,m}\psi_{n,m}(t) + \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) \\ &+ \sum_{n=2^{k-1}+1}^{\infty}\sum_{M+1}^{\infty} c_{n,m}\psi_{n,m}(t) \\ &= S_{2^{k-1},M} + \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(t), \quad \text{where } S_{2^{k-1},M} = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) \end{aligned}$$

(For $n \in [2^{k-1} + 1, \infty), 1 \le t < \infty$, so $\psi_{n,m} = 0$ for $n = 2^{k-1} + 1, ...\infty$)
Now, if $f(t)$ is truncated by $S_{2^{k-1},M}$, then $f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) = C^T\psi(t)$,
where C and $\psi(t)$ are $2^{k-1}(M + 1)$ vectors of the form
$C = [c_{1,0}\ c_{1,1}\ ...\ c_{1,M}\ c_{2,0}\ c_{2,1}\ ...\ c_{2,M}\ ...\ c_{2^{k-1},0}\ ...\ c_{2^{k-1},M}]^T$ and
$\psi(t) = [\psi_{1,0}\ \psi_{1,1}\ ...\psi_{1,M}\ \psi_{2,0}\ \psi_{2,1}\ ...\ \psi_{2,M}\ ...\ \psi_{2^{k-1},0}\ ...\ \psi_{2^{k-1},M}]^T$.

2.4. **Modulus of Continuity.** The modulus of continuity(Chui[2]) of a function $f \in L^2_w[0, 1)$ is defined as

$$\begin{aligned} W(f, \delta) &= \sup_{0<h\le\delta} ||f(t + h) - f(t)||_2, \quad \forall t \in [0, 1) \\ &= \sup_{0<h\le\delta} \left(\int_0^1 |f(t + h) - f(t)|^2 w(t)dt\right)^{1/2}. \end{aligned}$$

It is remarkable to note that $W(f,\delta)$ is a non-decreasing function of δ and $W(f,\delta) \to 0$ as $\delta \to 0^+$ for $f \in L^2_w[0,1)$.

2.5. **Hölder class** $H^\alpha[0,1)$**.** A function f is said to be in Hölder class(Das et al.[3]) $H^\alpha[0,1)$ of order $\alpha \in (0,1]$ if f is continuous on [0,1) and satisfies the inequality,

$$f(x+t) - f(x) = O(|t|^\alpha), \qquad \forall x+t, x \in [0,1).$$

3. Moduli of Continuity of functions having second derivative in Hölder class

Lemma 3.1. *Let $f(t) \in L^2_w[0,1)$ and has the second kind Chebyshev wavelet expansion*

$$f(t) \;=\; \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \tag{1}$$

then the series converges uniformly to $f(t)$ in $L^2_w[0,1)$.

Proof. Consider

$$f(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t) \;=\; \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \langle f, \psi_{n,m}\rangle_{w_n}\psi_{n,m}(t)$$

$$\begin{aligned}\therefore \|\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}(t)\|^2 &= \|\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \langle f, \psi_{n,m}\rangle_{w_n}\psi_{n,m}(t)\|^2 \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} |\langle f, \psi_{n,m}\rangle_{w_n}|^2 \,\|\psi_{n,m}(t)\|^2 \\ &\le \|f\|^2 \quad (\because \|\psi_{n,m}(t)\|^2 = 1).\end{aligned}$$

Hence, the series (2) converges to $f(t)$. □

In this paper, the following convergence theorem has been proved:

Theorem 3.1. *If a function $f \in L^2_w[0,1)$ such that its second derivative f'' belongs to $H^\alpha[0,1)$ i.e.$f''(x+t) - f''(x) = O(|t|^\alpha), 0 < \alpha \le 1$ and its Chebyshev wavelet expansion is $f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)$, where $c_{n,m} = \langle f, \psi_{n,m}\rangle_{w_n}$ then the modulus of continuity $W(f - S_{2^{k-1},M}(f), \frac{1}{2^k})$ of $(f - S_{2^{k-1},M}(f))$ satisfies:*
(i) for $f(t) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(t)$,

$$\begin{aligned}W(f - S_{2^{k-1},0}(f), \frac{1}{2^k}) &= \sup_{0<h\le\frac{1}{2^k}} ||(f - S_{2^{k-1},0}(f))(.+h) - (f - S_{2^{k-1},0}(f))(.)||_2 \\ &= O\left(\frac{1}{2^{k\alpha}}\right), k \ge 1,\end{aligned}$$

(ii) for $f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{1} c_{n,m}\psi_{n,m}(t)$,

$$\begin{aligned}W(f - S_{2^{k-1},1}(f), \frac{1}{2^k}) &= \sup_{0<h\le\frac{1}{2^k}} ||(f - S_{2^{k-1},1}(f))(.+h) - (f - S_{2^{k-1},1}(f))(.)||_2 \\ &= O\left(\frac{1}{2^{k(\alpha+2)}}\right), k \ge 1, \quad \textit{and}\end{aligned}$$

(iii) for $f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(t)$,

$$W(f-S_{2^{k-1},M}(f),\frac{1}{2^k})=\sup_{0<h\leq\frac{1}{2^k}}||(f-S_{2^{k-1},M}(f))(.+h)-(f-S_{2^{k-1},M}(f))(.)||_2$$
$$=O\left(\frac{1}{2^{k(\alpha+2)}.M^{\frac{3}{2}}}\right),k\geq 1,M\geq 2.$$

Proof. (i)For $m=0$,

The error between $f(t)$ and its Chebyshev wavelet expansion in the interval $\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)$ is given by

$$e_n(f) = c_{n,0}\psi_{n,0}-f\chi_{[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}})},\quad \frac{n-1}{2^{k-1}}\leq t<\frac{n}{2^{k-1}}. \tag{2}$$

$$\begin{aligned}
c_{n,0}=\langle f,\psi_{n,0}\rangle_{w_n} &= \int_0^1 f(t)\psi_{n,0}(t)w_n(t)dt\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_0(2^kt-2n+1)w(2^kt-2n+1)dt\\
&= \int_{-1}^1 f\left(\frac{2n-1+v}{2^k}\right)2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_0(v)w(v)\frac{dv}{2^k},\quad 2^kt-2n+1=v\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^1\left[f\left(\frac{2n-1}{2^k}\right)+\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)+\left(\frac{v}{2^k}\right)^2\frac{1}{2!}\right.\\
&\quad \left. f''\left(\frac{2n-1}{2^k}+\frac{\theta v}{2^k}\right)\right]w(v)dv=I_1+I_2+I_3.\\
I_1 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^1 f\left(\frac{2n-1}{2^k}\right)\sqrt{1-v^2}dv=\frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\pi}{2}}f\left(\frac{2n-1}{2^k}\right)\\
I_2 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}f'\left(\frac{2n-1}{2^k}\right)\int_{-1}^1\frac{v}{2^k}\sqrt{1-v^2}dv=0
\end{aligned}$$

$$\begin{aligned}
I_3 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\frac{1}{2!}\frac{1}{(2^k)^2}\int_{-1}^1 v^2f''\left(\frac{2n-1}{2^k}+\frac{\theta v}{2^k}\right)\sqrt{1-v^2}dv\quad ,0<\theta<1\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\frac{1}{2!}\frac{1}{(2^k)^2}f''\left(\frac{2n-1}{2^k}+\frac{\theta v_1}{2^k}\right)2.\int_0^1 v^2\sqrt{1-v^2}dv\quad ,v_1\in(-1,1)\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\pi}{2}}.\frac{1}{4}.\frac{1}{2!}\frac{1}{(2^{2k})}f''\left(\frac{2n-1}{2^k}+\frac{\theta v_1}{2^k}\right).
\end{aligned}$$

Therefore,

$$c_{n,0}=\frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\pi}{2}}\left[f\left(\frac{2n-1}{2^k}\right)+\frac{1}{4.2!}.\frac{1}{2^{2k}}f''\left(\frac{2n-1}{2^k}+\frac{\theta v_1}{2^k}\right)\right]. \tag{3}$$

Substituting the values of $c_{n,0}$ in eq.(2),

$$\begin{aligned}
|e_n(f)| &= \left|f\left(\frac{2n-1}{2^k}\right)+\frac{1}{4.2!}.\frac{1}{2^{2k}}f''\left(\frac{2n-1+\theta v_1}{2^k}\right)-f\left(\frac{2n-1}{2^k}\right)\right.\\
&- \left.\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)-\frac{v^2}{2!}\left(\frac{1}{2^{2k}}\right)f''\left(\frac{2n-1+\theta v}{2^k}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\frac{v^2}{2!.2^{2k}.4}f''\left(\frac{2n-1+\theta v_1}{2^k}\right)-\frac{v^2}{2!.2^{2k}}f''\left(\frac{2n-1+\theta v}{2^k}\right)\right.\\
&- \left.\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)\right|\\
&\leq \left(\frac{v^2}{2!.2^{2k}}\right)\left|\frac{\theta(v_1-v)}{2^k}\right|^{\alpha}+\left(\frac{v^2}{2^{k\alpha}}\right) \qquad (\because f''\in H^{\alpha}[0,1), 0<\alpha\leq 1)\\
&\leq \frac{1}{(2!).2^{k(\alpha+2)}}+\left(\frac{1}{2^{k\alpha}}\right) \qquad (\because v\in[-1,1) \;\; and \;\; 0<\theta<1)\\
&\leq \left(\frac{2}{2^{k\alpha}}\right).
\end{aligned}
$$

$$
\begin{aligned}
\|e_n\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|e_n(f)|^2|w_n(t)|dt\\
&\leq \int_{-1}^{1}\frac{4}{2^{2k\alpha}}\frac{\sqrt{1-v^2}}{2^k}dv=\frac{2\pi}{2^{k(2\alpha+1)}}.\\
\|f-S_{2^{k-1},0}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}}||e_n||_2^2\leq\sum_{n=1}^{2^{k-1}}\frac{2\pi}{2^{k(2\alpha+1)}}=\frac{\pi}{2^{2k\alpha}}\\
\|f-S_{2^{k-1},0}(f)\|_2 &\leq \frac{\sqrt{\pi}}{2^{k\alpha}}.
\end{aligned}
$$

$$
\begin{aligned}
W\left(f-S_{2^{k-1},0}(f),\frac{1}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}}||(f-S_{2^{k-1},0}(f))(t+h)-(f-S_{2^{k-1},0}(f))(t)||_2\\
&\leq \sup_{0<h\leq\frac{1}{2^k}}\left[||(f-S_{2^{k-1},0}(f))(t+h)||_2+||(f-S_{2^{k-1},0}(f))(t)||_2\right]\\
&\leq 2||f-S_{2^{k-1},0}(f)||_2\\
&\leq 2\frac{\sqrt{\pi}}{2^{k\alpha}}=O\left(\frac{1}{2^{k\alpha}}\right), \; k\geq 1.
\end{aligned}
$$

(ii)For $m=0,1,$

The error between $f(t)$ and its Chebyshev wavelet expansion in the interval $\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)$ is given by

$$
e_n(f)=c_{n,0}\psi_{n,0}+c_{n,1}\psi_{n,1}-f\chi_{[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}})}, \quad \frac{n-1}{2^{k-1}}\leq t<\frac{n}{2^{k-1}} \tag{4}
$$

$$
\begin{aligned}
c_{n,0} &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\pi}{2}}\left[f\left(\frac{2n-1}{2^k}\right)+\frac{1}{4.2!}.\frac{1}{2^{2k}}f''\left(\frac{2n-1}{2^k}+\frac{\theta v_1}{2^k}\right)\right].\\
c_{n,1} &= \langle f,\psi_{n,1}\rangle_{w_n}\\
&= \int_0^1 f(t)\psi_{n,1}(t)w_n(t)dt\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}f(t)2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_1(2^kt-2n+1)w(2^kt-2n+1)dt\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^{1}f\left(\frac{2n-1}{2^k}+\frac{v}{2^k}\right)2v\sqrt{1-v^2}dv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^{1}\left[f\left(\frac{2n-1}{2^k}\right)+\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)+\left(\frac{v}{2^k}\right)^2\frac{1}{2!}f''\left(\frac{2n-1}{2^k}+\frac{\theta v}{2^k}\right)\right] \\
&\quad .v\sqrt{1-v^2}dv, \quad 0<\theta<1 \\
&= J_1+J_2+J_3. \\
J_1 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\int_{-1}^{1} f\left(\frac{2n-1}{2^k}\right)v\sqrt{1-v^2}dv=0 \\
J_2 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}f'\left(\frac{2n-1}{2^k}\right)\frac{1}{2^k}\int_{-1}^{1}v^2\sqrt{1-v^2}dv \\
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}f'\left(\frac{2n-1}{2^k}\right)\frac{1}{2^k}.\frac{\pi}{8} \\
J_3 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\frac{1}{2!}\frac{1}{(2^k)^2}\int_{-1}^{1}v^3 f''\left(\frac{2n-1}{2^k}+\frac{\theta v}{2^k}\right)\sqrt{1-v^2}dv \quad ,0<\theta<1 \\
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}}\frac{1}{2!}\frac{1}{(2^k)^2}f''\left(\frac{2n-1}{2^k}+\frac{\theta v_2}{2^k}\right)\int_{-1}^{1}v^3\sqrt{1-v^2}dv \quad ,v_2\in(-1,1) \\
&= 0.
\end{aligned}
$$

Therefore,

$$
c_{n,1}=\frac{1}{2}.\sqrt{\frac{\pi}{2}}.\frac{1}{2^{\frac{3k}{2}}}f'\left(\frac{2n-1}{2^k}\right).
$$

Substituting the values of $c_{n,0}$ *and* $c_{n,1}$ in eq.(4),

$$
\begin{aligned}
|e_n(f)| &= \left|f\left(\frac{2n-1}{2^k}\right)+\frac{1}{4.2!}.\frac{1}{2^{2k}}f''\left(\frac{2n-1+\theta v_1}{2^k}\right)+\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)\right. \\
&- \left. f\left(\frac{2n-1}{2^k}\right)-\frac{v}{2^k}f'\left(\frac{2n-1}{2^k}\right)\frac{v^2}{2!}\left(\frac{1}{2^k}\right)f''\left(\frac{2n-1+\theta v}{2^k}\right)\right| \\
&\le \left|\frac{v^2}{2!.2^{2k}.4}f''\left(\frac{2n-1+\theta v_1}{2^k}\right)-\frac{v^2}{2!.2^{2k}}f''\left(\frac{2n-1+\theta v}{2^k}\right)\right| \\
&\le \left(\frac{v^2}{2!.2^{2k}}\right)\left|f''\left(\frac{2n-1+\theta v_1}{2^k}\right)-f''\left(\frac{2n-1+\theta v}{2^k}\right)\right| \\
&\le \left(\frac{v^2}{2!.2^{2k}}\right)\left|\frac{\theta(v_1-v)}{2^k}\right|^{\alpha} \qquad \left(\because f''\in H^{\alpha}[0,1)\right) \\
&\le \frac{1}{(2!).2^{k(\alpha+2)}}. \qquad (\because v\in[-1,1) \;\; and \;\; 0<\theta<1) \\
\|e_n\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|e_n(f)|^2|w_n(t)|dt \\
&\le \int_{-1}^{1}\frac{1}{(2!)^2.2^{2k(\alpha+2)}}\frac{\sqrt{1-v^2}}{2^k}dv=\frac{\pi}{2(2!)^2 2^{k(2\alpha+5)}}. \\
\|f-S_{2^{k-1},1}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}}||e_n||_2^2
\end{aligned}
$$

$$\leq \sum_{n=1}^{2^{k-1}} \frac{\pi}{2(2!)^2 2^{k(2\alpha+5)}} = \frac{\pi}{16.2^{2k(\alpha+2)}}$$

$$\therefore \|f - S_{2^{k-1},1}(f)\|_2 \leq \frac{\sqrt{\pi}}{4.2^{k(\alpha+2)}}.$$

Hence,

$$W\left(f - S_{2^{k-1},1}(f), \frac{1}{2^k}\right) = \sup_{0<h\leq\frac{1}{2^k}} ||(f - S_{2^{k-1},1}(f))(t+h) - (f - S_{2^{k-1},1}(f))(t)||_2$$

$$= O\left(\frac{1}{2^{k(\alpha+2)}}\right), \ k \geq 1.$$

(ii)For $m \geq 2$,

$$\|f - S_{2^{k-1},M}\|_2^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} |c_{n,m}|^2.$$

$$c_{n,m} = \int_0^1 f(t)\psi_{n,m}(t)w_n(t)dt$$

$$= \sqrt{\frac{2}{\pi}}.2^{\frac{k}{2}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)U_m(2^k t - 2n + 1)w(2^k t - 2n + 1)dt$$

$$= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}} \int_0^\pi f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) U_m(\cos\theta)w(\cos\theta)\sin\theta\frac{d\theta}{2^k},$$

$$= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2}{\pi}} \int_0^\pi f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) U_m(\cos\theta)w(\cos\theta)\sin\theta d\theta$$

$$= \frac{1}{2^{\frac{(k-1)}{2}}}\frac{1}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin(m+1)\theta\sin\theta d\theta$$

$$= \frac{1}{2^{\frac{(k+1)}{2}}}\frac{1}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) [\cos m\theta - \cos(m+2)\theta]d\theta$$

$$= \frac{1}{2^{\frac{(k+1)}{2}}}\frac{1}{\sqrt{\pi}} \left[f\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\left(\frac{\sin m\theta}{m} - \frac{\sin(m+2)\theta}{m+2}\right)_0^\pi\right] \frac{1}{2^{\frac{(k+1)}{2}}}$$
$$\frac{1}{\sqrt{\pi}}\left[\int_0^\pi f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\left(\frac{-\sin\theta}{2^k}\right)\left(\frac{\sin m\theta}{m} - \frac{\sin(m+2)\theta}{m+2}\right)d\theta\right],$$

$$= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}} \int_0^\pi f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\gamma_m(\theta)d\theta, \ \textit{again integrating by parts}$$

where $\gamma_m(\theta) = \frac{\sin\theta}{m}\left(\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1}\right) - \frac{\sin\theta}{m+2}\left(\frac{\sin(m+1)\theta}{m+1} - \frac{\sin(m+3)\theta}{m+3}\right).$

$$c_{n,m} = \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}\left(\int_0^\pi \left[f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right) - f''\left(\frac{2n-1}{2^k}\right)\right]\gamma_m(\theta)d\theta\right.$$
$$\left. + \ f''\left(\frac{2n-1}{2^k}\right)\int_0^\pi \gamma_m(\theta)d\theta\right)$$
$$= I_1 + I_2.$$

$$I_2 = \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}f''\left(\frac{2n-1}{2^k}\right)\int_0^\pi \gamma_m(\theta)d\theta = 0.$$

$$\begin{aligned}
c_{n,m} &= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}\int_0^{\pi}\left[f''\left(\frac{\cos\theta+2n-1}{2^k}\right)-f''\left(\frac{2n-1}{2^k}\right)\right]\gamma_m(\theta)d\theta \\
|c_{n,m}| &= \left|\frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}\int_0^{\pi}\left[f''\left(\frac{\cos\theta+2n-1}{2^k}\right)-f''\left(\frac{2n-1}{2^k}\right)\right]\gamma_m(\theta)d\theta\right| \\
&\leq \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}\int_0^{\pi}\left|\frac{\cos\theta}{2^k}\right|^{\alpha}|\gamma_m(\theta)|d\theta, \qquad (\because f''\in H^{\alpha}[0,1)) \\
&\leq \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{1}{\sqrt{\pi}}.\frac{1}{2^{k\alpha}}\int_0^{\pi}|\gamma_m(\theta)|d\theta. \\
\int_0^{\pi}|\gamma_m(\theta)|d\theta &= \int_0^{\pi}\left|\frac{\sin\theta}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)-\frac{\sin\theta}{m+2}\right. \\
&\quad \left.\left(\frac{\sin(m+1)\theta}{m+1}-\frac{\sin(m+3)\theta}{m+3}\right)\right|d\theta \\
&\leq \left[\frac{1}{m}\left(\frac{1}{m-1}+\frac{1}{m+1}\right)+\frac{1}{m+2}\left(\frac{1}{m+1}+\frac{1}{m+3}\right)\right]\int_0^{\pi}d\theta \\
&= \pi\left[\left(\frac{1}{m-1}-\frac{1}{m}\right)+\left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)\right. \\
&+ \left.\left(\frac{1}{m+2}-\frac{1}{m+3}\right)\right] \\
&= \frac{4\pi}{(m-1)(m+3)}. \\
|c_{n,m}| &\leq \frac{4\sqrt{\pi}}{2^{(5k+1)/2}(m+3)(m-1)}.\frac{1}{2^{k\alpha}} \\
\therefore \|f-S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\frac{16\pi}{2^{(5k+1)}(m+3)^2(m-1)^2}.\frac{1}{2^{2k\alpha}} \\
&= \frac{16\pi}{2^{(5k+1)}}\frac{1}{2^{2k\alpha}}\sum_{m=M+1}^{\infty}\frac{2^{k-1}}{(m+3)^2(m-1)^2} \\
&\leq \frac{16\pi}{2^{(4k+2)}}\frac{1}{2^{2k\alpha}}\left[\frac{4}{3M^3}\right]. \\
\|f-S_{2^{k-1},M}(f)\|_2 &\leq \frac{8\sqrt{\pi}}{2^{(2k+1)}}\frac{1}{2^{k\alpha}}\frac{4}{\sqrt{3}.M^{3/2}} \\
&= O\left(\frac{1}{2^{k(\alpha+2)}.M^{3/2}}\right), \ k\geq 1, \quad M\geq 2.
\end{aligned}$$

Hence,

$$\begin{aligned}
W\left(f-S_{2^{k-1},M}(f),\frac{1}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}}||(f-S_{2^{k-1},M}(f))(t+h)-(f-S_{2^{k-1},M}(f))(t)||_2 \\
&= O\left(\frac{1}{2^{k(\alpha+2)}.M^{3/2}}\right), \ k\geq 1, \quad M\geq 2.
\end{aligned}$$

□

4. Solution of differential equations using second kind Chebyshev wavelet

4.1. Second kind Chebyshev wavelet Operational matrix of Integration. In this section, second kind Chebyshev wavelet operational matrix of integration is obtained for k=2, M=3.

In this case, basis functions are as follows:

$$\left.\begin{aligned}\psi_{1,0}(t)&=2\sqrt{\tfrac{2}{\pi}}\\ \psi_{1,1}(t)&=2\sqrt{\tfrac{2}{\pi}}(8t-2)\\ \psi_{1,2}(t)&=2\sqrt{\tfrac{2}{\pi}}(64t^2-32t+3)\\ \psi_{1,3}(t)&=2\sqrt{\tfrac{2}{\pi}}(512t^3-384t^2+80t-4)\end{aligned}\right\}0\le t<\tfrac{1}{2}$$

$$\left.\begin{aligned}\psi_{2,0}(t)&=2\sqrt{\tfrac{2}{\pi}}\\ \psi_{2,1}(t)&=2\sqrt{\tfrac{2}{\pi}}(8t-6)\\ \psi_{2,2}(t)&=2\sqrt{\tfrac{2}{\pi}}(64t^2-96t+35)\\ \psi_{2,3}(t)&=2\sqrt{\tfrac{2}{\pi}}(512t^3-1152t^2+848t-204)\end{aligned}\right\}\tfrac{1}{2}\le t<1$$

Integrating above functions from 0 to t and expressing in terms of basis functions,

$$\int_0^t\psi_{1,0}(t')dt'=\begin{cases}2\sqrt{\dfrac{2}{\pi}}t, & 0\le t<\frac{1}{2}\\[2ex] \sqrt{\dfrac{2}{\pi}}, & \frac{1}{2}\le t<1\end{cases}$$

$$=\tfrac{1}{4}\psi_{1,0}(t)+\tfrac{1}{8}\psi_{1,1}(t)+\tfrac{1}{2}\psi_{2,0}(t)$$

$$\int_0^t\psi_{1,1}(t')dt'=\begin{cases}4\sqrt{\dfrac{2}{\pi}}(2t^2-t), & 0\le t<\frac{1}{2}\\[2ex] 0, & \frac{1}{2}\le t<1\end{cases}$$

$$=\tfrac{-3}{16}\psi_{1,0}(t)+\tfrac{1}{16}\psi_{1,2}(t).$$

Similarly, $\int_0^t\psi_{1,2}(t')dt'=\frac{1}{12}\psi_{1,0}(t)+\frac{-1}{24}\psi_{1,1}(t)+\frac{1}{24}\psi_{1,3}(t)+\frac{1}{6}\psi_{2,0}(t)$

$$\int_0^t\psi_{1,3}(t')dt'=\tfrac{-1}{16}\psi_{1,0}(t)+\tfrac{1}{32}\psi_{1,2}(t)$$

$$\int_0^t\psi_{2,0}(t')dt'=\tfrac{1}{4}\psi_{2,0}(t)+\tfrac{1}{8}\psi_{2,1}(t)$$

$$\int_0^t\psi_{2,1}(t')dt'=\tfrac{-3}{16}\psi_{2,0}(t)+\tfrac{1}{16}\psi_{2,2}(t)$$

$$\int_0^t\psi_{2,2}(t')dt'=\tfrac{1}{12}\psi_{2,0}(t)+\tfrac{-1}{24}\psi_{2,1}(t)+\tfrac{1}{24}\psi_{2,3}(t)$$

$$\int_0^t\psi_{2,3}(t')dt'=\tfrac{-1}{16}\psi_{2,0}(t)+\tfrac{-1}{32}\psi_{2,2}(t).$$

Thus, $\int_0^t\psi_{8\times1}(t')dt'=P_{8\times8}\psi_{8\times1}(t)$, where $\psi(t)=[\psi_{1,0}\ \ \psi_{1,1}\ \ \psi_{1,2}\ \ \psi_{1,3}\ \ \psi_{2,0}\ \ \psi_{2,1}\ \ \psi_{2,2}\ \ \psi_{2,3}\ ]^T$.

Hence, second kind Chebyshev wavelet operational matrix of integration is given by

$$P_{8\times 8} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{16} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & \frac{-1}{24} & 0 & \frac{1}{24} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{-1}{16} & 0 & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-3}{16} & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{-1}{24} & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & \frac{-1}{16} & 0 & \frac{-1}{32} & 0 \end{bmatrix} \tag{5}$$

4.2. **Algorithm for solving non-linear singular differential equations using second kind Chebyshev wavelet operational matrix of integration.** Consider the following non-linear singular differential equation

$$y''(t) + \frac{m}{t}y'(t) + f(t,y) = g(t), \qquad 0 \le t < 1,\ m \ge 1 \tag{6}$$

with initial conditions

$$y(0) = a,\ y'(0) = b. \tag{7}$$

For solving the differential eq.(9), multiply it by t,

$$ty''(t) + my'(t) + tf(t,y) = tg(t), \qquad 0 \le t < 1,\ m \ge 1 \tag{8}$$

and approximate $y''(t)$ as

$$y''(t) \approx \sum_{n=1}^{2}\sum_{m=0}^{3} c_{n,m}\psi_{n,m}(t) = C^T\psi(t). \tag{9}$$

Integrating eq.(12) w.r.t. t twice from 0 to t,

$$\begin{aligned} y'(t) &\approx C^T P\psi(t) + y'(0) \\ &= C^T P\psi(t) + A^T\psi(t), \end{aligned} \tag{10}$$

$$\begin{aligned} y(t) &\approx C^T P^2\psi(t) + ty'(0) + y(0) \\ &= C^T P^2\psi(t) + B^T\psi(t), \end{aligned} \tag{11}$$

where A and B are unknown which can be calculated from initial conditions (10). Here, P is the 8×8 second kind Chebyshev wavelet operational matrix of integration given by eq.(8). Also approximate $e(t) = t,\ f(t,y)$ and $g(t)$ as

$$e(t) = t \approx E^T\psi(t), \tag{12}$$

$$\begin{aligned} f(t,y) &\approx \sum_{j=0}^{n} \frac{1}{j!}\left((t-t_0)\frac{\partial}{\partial t} + (y-y_0)\frac{\partial}{\partial y}\right)^j f(t_0,y_0) \\ &= F^T\psi(t), \end{aligned} \tag{13}$$

$$\begin{aligned} g(t) &\approx \sum_{j=0}^{n} \frac{g^{(j)}(t_0)}{j!}(t-t_0)^j \\ &= G^T\psi(t). \end{aligned} \tag{14}$$

Substituting the values from eqs.(12)-(17) in eq.(11),

$$E_T\psi(t)C_T\psi(t) + m(C^T P\psi(t) + A^T\psi(t)) + E_T\psi(t)F^T\psi(t) = E_T\psi(t)G^T\psi(t). \tag{15}$$

Solving the system of algebraic equations (18), the eq.(9) can be solved.

4.3. **Solution of Lane-Emden equation of index p=1.** Consider the general Lane-Emden equation of index p,

$$y''(t) + \frac{2}{t}y'(t) + y^p(t) = 0, \qquad 0 \le t < 1, \tag{16}$$

with initial conditions $y(0) = 1,\ y'(0) = 0$. ([17])
This eq.(19) is reduced from eq.(9) when $f(t, y) = y^p(t), g(t) = 0$.
For p=1, the eq.(19) reduces to

$$y''(t) + \frac{2}{t}y'(t) + y(t) = 0, \qquad y(0) = 1,\ y'(0) = 0. \tag{17}$$

The exact solution of eq.(20) is $y(t) = \dfrac{\sin t}{t}$.

The numerical solution of eq.(20) using the method described in the section (4.2) is given in Table 1.

The Table 1 compares the numerical solution of Lane-Emden equation of index p=1 with the exact solution. The absolute error shows that the numerical values obtained using second kind Chebyshev wavelet are in good agreement with the exact values at those points. The graphs of exact and approximate solutions obtained by second kind Chebyshev wavelet are plotted in Figure 1. The graphs of exact and approximate solutions coincide almost everywhere.

t	**Cheby. sol.**	**Exact sol.**	**Abs. error**
0.0000001	0.99999008	0.99999998	9.9×10^{-6}
0.000001	0.99999008	0.99999998	9.9×10^{-6}
0.00001	0.99999008	0.99999998	9.9×10^{-6}
0.0001	0.99999011	0.99999998	9.87×10^{-6}
0.001	0.99999023	0.99999983	9.6×10^{-6}
0.01	0.99997634	0.99998333	6.99×10^{-6}
0.1	0.99833665	0.99833416	2.49×10^{-6}
0.2	0.98509001	0.99334665	8.25×10^{-3}
0.3	0.98506630	0.98506735	1.05×10^{-6}
0.4	0.97354834	0.97354855	2.1×10^{-7}
0.5	0.95884292	0.958851077	8.16×10^{-6}
0.6	0.94107292	0.94107078	2.14×10^{-6}
0.7	0.92031014	0.92031098	8.4×10^{-7}
0.8	0.89669436	0.89669511	7.5×10^{-7}
0.9	0.87036538	0.87036323	2.15×10^{-6}

Table 1. Chebyshev solution of Lane-Emden equation of index 1

4.4. **Solution of Chandrasekhar's white dwarf equation.** Consider the white dwarf equation ([1])
$y''(t) + \frac{2}{t}y'(t) + (y^2(t) - c)^{\frac{3}{2}} = 0, \forall t \in [0, 1), \qquad y(0) = 1,\ y'(0) = 0.$
The numerical solutions of eq.(21), for different values of c *i.e.* c=0, c=0.1, c=0.2, c=0.3, using the method described in the section (4.2) are given in Table 2. The numerical solutions of Chandrasekhar's white dwarf equation using second kind Chebyshev wavelet, for different values of c, have also been plotted as shown in Figure 2.

t	c=0	c=0.1	c=0.2	c=0.3
0.0000001	0.99997051	0.99997596	0.99998087	0.99998523
0.000001	0.99997051	0.99997596	0.99998087	0.99998523
0.00001	0.99997051	0.99997596	0.99998087	0.99998523
0.0001	0.99997060	0.99997603	0.99998093	0.99998528
0.001	0.99997128	0.99997658	0.99998136	0.99998560
0.01	0.99996256	0.99996883	0.99997459	0.99997982
0.1	0.99834326	0.99858506	0.99881389	0.99902875
0.2	0.99337005	0.99433773	0.99525348	0.99611331
0.3	0.98519882	0.98735409	0.98939482	0.99131203
0.4	0.97397751	0.97775430	0.98133311	0.98469805
0.5	0.95986121	0.96566332	0.97116653	0.97634603
0.6	0.94320905	0.95137700	0.95913323	0.96644229
0.7	0.92424055	0.93507135	0.94537067	0.95509060
0.8	0.90336430	0.91708308	0.93015004	0.94250321
0.9	0.88048890	0.89774890	0.91374252	0.92889238

TABLE 2. Chebyshev solution of Chandrasekhar's white dwarf equation

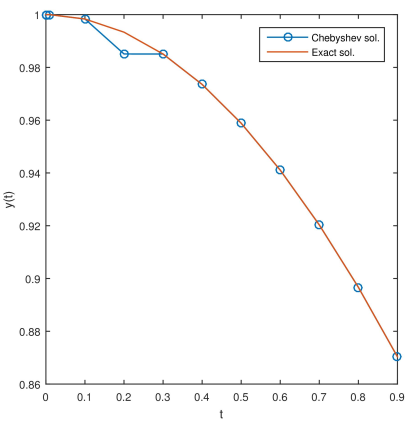

FIGURE 1. Solution of Lane-Emden eqn.

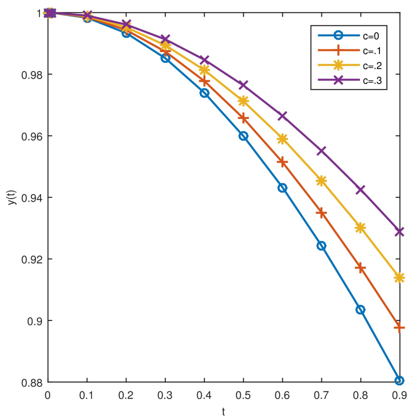

FIGURE 2. Sol. of Chandrasekhar's white dwarf eqn.

5. Conclusions

1. By theorem (3.1),

(i) $W(f - S_{2^{k-1},0}(f), \frac{1}{2^k}) = O\left(\frac{1}{2^{k\alpha}}\right)$

(ii) $W(f - S_{2^{k-1},1}(f), \frac{1}{2^k}) = O\left(\frac{1}{2^{k(\alpha+2)}}\right)$

(iii) $W(f - S_{2^{k-1},M}(f), \frac{1}{2^k}) = O\left(\frac{1}{2^{k(\alpha+2)}.M^{\frac{3}{2}}}\right)$, all vanishes as $k, M \to \infty$.

2. By cor.1,

(ii) $E_{2^{k-1},0} = O\left(\frac{1}{2^{k\alpha}}\right), k \geq 1,$

(ii) $E_{2^{k-1},1} = O\left(\frac{1}{2^{k(\alpha+2)}}\right), k \geq 1,$

(iii) $E_{2^{k-1},M} = O\left(\frac{1}{2^{k(\alpha+2)}(M)^{\frac{3}{2}}}\right), k \geq, M \geq 2.$

$$W(f - S_{2^{k-1},0}(f), \frac{1}{2^k}) \leq 2E_{2^{k-1},0}, \quad W(f - S_{2^{k-1},1}(f), \frac{1}{2^k}) \leq 2E_{2^{k-1},1}$$
$$\text{and} \quad W(f - S_{2^{k-1},M}(f), \frac{1}{2^k}) \leq 2E_{2^{k-1},M},$$

therefore, moduli of continuity, $W(f - S_{2^{k-1},0}(f), \frac{1}{2^k}), W(f - S_{2^{k-1},1}(f), \frac{1}{2^k})$, $W(f - S_{2^{k-1},M}(f), \frac{1}{2^k})$ are better and sharper than the approximations, $E_{2^{k-1},0}, E_{2^{k-1},1}$, $E_{2^{k-1},M}$ respectively.

3. The second kind Chebyshev wavelet operational matrix of integration has been constructed which helps in solving non-linear singular differential equations efficiently, since solving these equations with other methods is a tedious work.

4. The method of operational matrix of integration developed in this paper has been employed to Lane-Emden equation of index p=1, which gives high agreement with the exact solution, hence verifying the applicability of the method developed.

5. The method of operational matrix of integration has been employed for getting the numerical solution of Chandrasekhar's white dwarf equation using lesser order matrix than used in Kaur et al.([8]), demonstrating the efficiency of the method developed.

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