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MORE ON PICTURE FUZZY SETS AND THEIR PROPERTIES

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Abstract. In this paper, some basic properties of the set of a picture fuzzy set truth values $\mathbb{D}^*$ are studied. Also, using an adequate order $\preceq$ of $\mathbb{D}^*$, some picture fuzzy sets operations are introduced by meaning a punctual order (point by point). As well as the order of $\mathbb{D}^*$ is used to show some characteristic sets of a picture fuzzy set, such as support, kernel, α-cut, strong α-cut and picture fuzzy line of degree α of a picture fuzzy set, where $\alpha \in \mathbb{D}^*$, have been defined, some properties of them have been established and some decomposition theorems of picture fuzzy sets have been proved. Finally, some of Atanassov's modal operators are extended to the picture fuzzy case.

Keywords: Picture fuzzy set, picture fuzzy sets operations, α-cuts, picture fuzzy line of degree α, modal operators.

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1. Introduction

To deal with uncertainty, Zadeh (1965) [28] proposed the notion of fuzzy sets and related concepts. Since its creation, this notion has not ceased to develop. Two years after this invention, Goguen [20] introduced a fundamental generalization of fuzzy sets by replacing the unit interval with a complete lattice. Interval-valued fuzzy sets are a further development of fuzzy sets (IVFSs) introduced simultaneously by several researchers Zadeh, Grattan-Guiness, Jahn and Sambuc [29, 21, 17, 25] (1975).

Intuitionistic fuzzy sets (IFSs) were first proposed by Atanassov in 1983 to address the issue of non-membership. This concept has been found to be quite helpful in dealing with vagueness, and it was followed by a general intuitionistic fuzzy set, the "intuitionistic L-fuzzy set," introduced in 1984. As an extension of the fuzzy set, Gau and Buehrer [19] presented the theory of vague sets in 1993. (later proven to be intuitionistic fuzzy sets; see [7]).

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The same thing as Zadeh's fuzzy set has been introduced under the term "type-1 fuzzy set" (T1FS) in recent years. Some authors e.g., [8, 24], emphasize that T1FSs serve as the foundation for fuzzy set expansions.

It is evident that a significant portion of these extensions consists of minor alterations to previous fuzzy set extensions and already exist that have been "re-branded" with new names.

Picture fuzzy sets, a concept coined by Cuong Bui Cong and Vladik Kreinovich [10], is an interesting expansion of both fuzzy sets and intuitionistic fuzzy sets (IFS).

The main work of Cuong Bui Cong and Vladik Kreinovich is the incorporation of the concept of the positive, negative and neutral membership degree of an element. In other words, an element of a picture fuzzy set A is characterized by three memberships degrees, positive, negative and neutral membership degree $\mu_A(x), \eta_A(x), \nu_A(x)$, so that the total of the three degrees cannot be greater than one. This gives an unusual but wonderful idea of a mathematician and many valued-logic.

Not only does the resulting notion have a beautiful mathematical structure with connections to various fields of mathematics, but it also has a broad range of applications outside mathematics, for example in decision-making [2, 18, 27], Medical Diagnosis [15], investment risk [6] and other applications [1, 26].

The picture fuzzy set is one of the most reliable techniques to handle the uncertainties in the data throughout the decision-making process, in which an intuitionistic fuzzy set may fail to produce good results. It is an effective mathematical tool for dealing with uncertain real-life issues. It can function extremely effectively in ambiguous situations that call for responses of the yes, no, abstain, and rejection types.

Fetanat and Tayebi are doing research to try to combine a new decision support system (DSS) with a picture fuzzy set based combined compromise solution (PFS-CoCoSo) [16].

The rest of this paper is structured as follows: Preliminary concepts related to fuzzy sets, L-fuzzy sets, intuitionistic fuzzy sets theory and picture fuzzy sets are summarized in Section 2. In Section 3, we study some properties of $\mathbb{D}^*$ (the set of membership values of a picture fuzzy set). In Section 4, based on the study of $\mathbb{D}^*$, we introduce the concepts of picture fuzzy set union, picture fuzzy set intersection and we define the picture fuzzy complement by using the standard negator. In Section 5, we provide some characteristic sets of a picture fuzzy set such as support, kernel, α-cuts, strong α-cuts and picture fuzzy line of degree α of a picture fuzzy set for $\alpha \in \mathbb{D}^*$, and investigate their properties, we finish this section by giving some decomposition theorems for a picture fuzzy set. In Section 6, we extend some of Attanassov's modal operators to the picture fuzzy case. Finally, we present some concluding remarks in Section 7.

2. Preliminaries

In this paper, $\mathbf{U}$ stands for a referential set ($\mathbf{U} \neq \emptyset$).

2.1. Fuzzy sets and L-fuzzy sets.

Definition 2.1. [28] *A* fuzzy subset *E extends the traditional bi-valence by defining a generalized characteristic function $\mu_E : \mathbf{U} \longrightarrow [0,1]$, μ_E is said to be the membership function of the subset E on $\mathbf{U}$.*

Definition 2.2. [20] *For a complete lattice L, an* L-fuzzy set *μ is any application $\mu : \mathbf{U} \longrightarrow L$.*

2.2. **Intuitionistic fuzzy sets.**

Definition 2.3. [3, 4] *An intuitionistic fuzzy set E on $\mathbf{U}$ is defined as the expression $E = \{\langle a, \mu_E(a), \nu_E(a)\rangle | a \in \mathbf{U}\}$, where $\mu_E(a) + \nu_E(a) \leq 1$, for all $a \in \mathbf{U}$. $\mu_E : \mathbf{U} \longrightarrow [0,1]$ and $\nu_E : \mathbf{U} \longrightarrow [0,1]$ denote, respectively, the degree of membership and the degree of non-membership of the element a in the set E.*

Obviously, when $\nu_E(a) = 1 - \mu_E(a)$ for every a in $\mathbf{U}$, the intuitionistic fuzzy set E is a fuzzy set.

Dealing with intuitionistic fuzzy ideas, need to investigate the mathematical model that depicts this phenomenon, namely the set L^* defined in the following definition.

Definition 2.4. [9, 14] *Let $L^* = \left\{(a_1, a_2) \in [0,1]^2 | a_1 + a_2 \leq 1\right\}$ and $\leq_{L^*}$ be an order in L^* defined by $\forall (a_1, a_2)$, $(b_1, b_2) \in L^* : (a_1, a_2) \leq_{L^*} (b_1, b_2) \Leftrightarrow (a_1 \leq b_1$ and $a_2 \geq b_2)$. $(L^*, \leq_{L^*}, \wedge_{L^*}, \vee_{L^*}, 0_{L^*}, 1_{L^*})$ is a complete lattice with*

$$(a_1, a_2) \wedge_{L^*} (b_1, b_2) = (\min(a_1, b_1), \max(a_2, b_2))$$
$$(a_1, a_2) \vee_{L^*} (b_1, b_2) = (\max(a_1, b_1), \min(a_2, b_2))$$

$0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ are the units of L^*.

2.3. **Picture fuzzy sets.**

Definition 2.5. [10] *A* picture fuzzy set *E on $\mathbf{U}$ is defined as the expression $E = \{\langle a, \mu_E(a), \eta_E(a), \nu_E(a)\rangle \mid a \in \mathbf{U}\}$, with $\mu_E(a) + \eta_E(a) + \nu_E(a) \leq 1$, for any $a \in \mathbf{U}$.*

$\mu_E(a) \in [0,1]$, $\eta_E(a) \in [0,1]$ and $\nu_E(a) \in [0,1]$ are called, respectively, the positive, neutral and negative membership degrees of a in E.

The quantity $\pi_E(a) = 1 - (\mu_E(a) + \eta_E(a) + \nu_E(a))$ is said to be the refusal membership degree of a in E.

$PFS(\mathbf{U})$ stands for the set of all picture fuzzy sets on $\mathbf{U}$.

The following section is essential in this paper because it leads to a good understanding of the properties of picture fuzzy sets.

3. Structure of the set $\mathbb{D}^*$

According to [12, 22], we consider the set $\mathbb{D}^*$ defined by:

$$\mathbb{D}^* = \left\{a = (a_1, a_2, a_3) \in [0,1]^3, a_1 + a_2 + a_3 \leq 1\right\}$$

Obviously, any picture fuzzy set: $E = \{\langle a, \mu_E(a), \eta_E(a), \nu_E(a)\rangle \mid a \in \mathbf{U}\}$, corresponds to a $\mathbb{D}^*$-fuzzy subset, i.e., a mapping $E : \mathbf{U} \longrightarrow \mathbb{D}^*$ given by $E(a) = (\mu_E(a), \eta_E(a), \nu_E(a)) \in \mathbb{D}^*$.

We'll suppose that for all $a \in \mathbb{D}^*$, a_1, a_2 and a_3, respectively, refer to the first, second and third components of a, i.e., $a = (a_1, a_2, a_3)$.

3.1. **Order of $\mathbb{D}^*$.**

Definition 3.1. *According to* [22, 23, 13], *take $\preceq$ to be the order relation on $\mathbb{D}^*$ given by*

$$a \preceq b \text{ if and only if } (a_1, a_3) <_{L^*} (b_1, b_3) \text{ or } ((a_1, a_3) = (b_1, b_3) \text{ and } a_2 \leq b_2)$$

i.e., $(a_1 < b_1$ and $a_3 \geq b_3)$ or $(a_1 = b_1$ and $a_3 > b_3)$ or $(a_1 = b_1,\ a_3 = b_3$ and $a_2 \leq b_2)$, for all $a, b \in \mathbb{D}^*$.

Recalling that $(\mathbb{D}^*, \preceq)$ is a bounded lattice [12, 22] *with top element $1_{\mathbb{D}^*} = (1,0,0)$ and bottom element $0_{\mathbb{D}^*} = (0,0,1)$. And for each $a, b \in \mathbb{D}^*$, $a \curlywedge b$ and $a \curlyvee b$ are defined as follows*

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3), & \text{otherwise.} \end{cases}$$

$$a \curlyvee b = \begin{cases} b, & \text{if } a \preceq b, \\ a, & \text{if } b \preceq a, \\ (a_1 \vee b_1, 0, a_3 \wedge b_3), & \text{otherwise.} \end{cases}$$

For two incomparable elements $a, b \in \mathbb{D}^*$, the components of these elements verify ($a_1 > b_1$ and $a_3 > b_3$) or ($a_1 < b_1$ and $a_3 < b_3$), and we write $a \parallel b$.

Remark 3.1. *Let $a, b \in \mathbb{D}^*$.*

- *$a \succ b$ is equivalent to ($a_1 > b_1$ and $a_3 \leq b_3$) or ($a_1 = b_1$ and $a_3 < b_3$) or ($a_1 = b_1$, $a_3 = b_3$ and $a_2 > b_2$).*
- *$a \succ 0_{\mathbb{D}^*}$ is equivalent to $a_1 > 0$ or ($a_1 = 0$ and $a_3 < 1$).*

Next, we give some properties that will be useful in the sequel.

Proposition 3.1. *Let $a, b, c \in \mathbb{D}^*$. Then*

(1) *$a \curlywedge b \preceq a$, $a \curlywedge b \preceq b$.*
(2) *$a \preceq a \curlyvee b$, $b \preceq a \curlyvee b$.*
(3) *$a \curlywedge b \preceq a \curlyvee b$.*
(4) *$a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$ if and only if $a \curlywedge b \succ 0_{\mathbb{D}^*}$.*
(5) *$a \succeq c$ and $b \succeq c$ if and only if $a \curlywedge b \succeq c$.*
(6) *$a \succ 0_{\mathbb{D}^*}$ or $b \succ 0_{\mathbb{D}^*}$ if and only if $a \curlyvee b \succ 0_{\mathbb{D}^*}$.*
(7) *If $a \succeq c$ or $b \succeq c$, then $a \curlyvee b \succeq c$.*
(8) *$a \preceq c$ and $b \preceq c$ if and only if $a \curlyvee b \preceq c$.*
(9) *If $b \preceq c$, then $a \curlyvee b \preceq a \curlyvee c$ and $a \curlywedge b \preceq a \curlywedge c$.*

Proof. Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3) \in \mathbb{D}^*$.

(1) There are three cases to distinguish.
Case 01: $a \preceq b$. It follows that $a \curlywedge b = a$, hence $a \curlywedge b \preceq a$ and $a \curlywedge b \preceq b$.
Case 02: $b \preceq a$. It follows that $a \curlywedge b = b$, hence $a \curlywedge b \preceq a$ and $a \curlywedge b \preceq b$.
Case 03: $a \parallel b$ means ($a_1 > b_1$ and $a_3 > b_3$) or ($a_1 < b_1$ and $a_3 < b_3$).
The first sub-case gives
$a \curlywedge b = (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3) = (b_1, 1 - b_1 - a_3, a_3)$.

$$\text{Then} \begin{cases} a_1 \wedge b_1 = b_1 < a_1 \text{ and } a_3 \vee b_3 = a_3, \text{ hence } a \curlywedge b \preceq a. \\ \text{And} \\ a_1 \wedge b_1 = b_1 \text{ and } a_3 \vee b_3 = a_3 > b_3, \text{ hence } a \curlywedge b \preceq b. \end{cases}$$

Similarly, we obtain the same result in the second sub-case.
Therefore, we conclude that $a \curlywedge b \preceq a$, $a \curlywedge b \preceq b$, for all $a, b \in \mathbb{D}^*$.
(2) Similar to (1).
(3) From (1) and (2).
(4) Suppose that $a \succ 0_{\mathbb{D}^*}$, $b \succ 0_{\mathbb{D}^*}$. Then

$$a \curlywedge b = \begin{cases} a, & \text{if } a \preceq b, \\ b, & \text{if } b \preceq a, \\ (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3), & \text{otherwise.} \end{cases}$$

The result is clear if $a \curlywedge b = a$ or $a \curlywedge b = b$, it remains to prove that the property is true in the case $a \curlywedge b = (a_1 \wedge b_1, 1 - a_1 \wedge b_1 - a_3 \vee b_3, a_3 \vee b_3)$.
Since $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$, it follows that

$$\begin{cases} a_1 > 0 \text{ and } a_3 < 1 & (1) \\ \text{or} & \\ a_1 = 0 \text{ and } a_3 < 1 & (2) \end{cases} \text{ and } \begin{cases} b_1 > 0 \text{ and } b_3 < 1 & (3) \\ \text{or} & \\ b_1 = 0 \text{ and } b_3 < 1 & (4) \end{cases},$$

Then we distinguish four cases:

Case 01: We have (1) and (3) , i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), give $a_1 \wedge b_1 > 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 02: We have (1) and (4) , i.e., ($a_1 > 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), give $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 03: We have (2) and (3) , i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 > 0$ and $b_3 < 1$), give $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Case 04: We have (2) and (4) , i.e., ($a_1 = 0$ and $a_3 < 1$) and ($b_1 = 0$ and $b_3 < 1$), give $a_1 \wedge b_1 = 0$ and $a_3 \vee b_3 < 1$. It follows that $a \curlywedge b \succ 0_{\mathbb{D}^*}$.

Conversely, suppose that $a \curlywedge b \succ 0_{\mathbb{D}^*}$ and $a = 0_{\mathbb{D}^*}$. Then $a \curlywedge b = 0_{\mathbb{D}^*} \curlywedge b = 0_{\mathbb{D}^*}$, for each $b \in \mathbb{D}^*$. This is a contradiction. Thus, if $a \curlywedge b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ and $b \succ 0_{\mathbb{D}^*}$.

(5) Similar to (4).

(6) Suppose that $a \succ 0_{\mathbb{D}^*}$. Since $a \curlyvee b \succeq a$, then $a \curlyvee b \succ 0_{\mathbb{D}^*}$.
Conversely, suppose that $a \curlyvee b \succ 0_{\mathbb{D}^*}$, $a = 0_{\mathbb{D}^*}$ and $b = 0_{\mathbb{D}^*}$. Then $a \curlyvee b = 0_{\mathbb{D}^*} \curlyvee 0_{\mathbb{D}^*} = 0_{\mathbb{D}^*}$. This is a contradiction. Thus, if $a \curlyvee b \succ 0_{\mathbb{D}^*}$ implies $a \succ 0_{\mathbb{D}^*}$ or $b \succ 0_{\mathbb{D}^*}$.

(7) Similar to (6).

(8) Suppose that $a \preceq c$ and $b \preceq c$.
The result is clear if $a \curlyvee b = a$ or $a \curlyvee b = b$.
If $a \curlyvee b = (a_1 \vee b_1, 0, a_3 \wedge b_3)$, then $a_1 \vee b_1 \leq c_1$, $a_3 \wedge b_3 \geq c_3$ and $0 \leq c_2$. Hence $a \curlyvee b \preceq c$.
Conversely, suppose that $a \curlyvee b \preceq c$.
Since $a \preceq a \curlyvee b$ and $b \preceq a \curlyvee b$, then $a \preceq c$ and $b \preceq c$.

(9) Suppose that $b \preceq c$.
Since $a \preceq a \curlyvee c$ and $b \preceq c \preceq a \curlyvee c$, then from (8) $a \curlyvee b \preceq a \curlyvee c$.
Since $a \curlywedge b \preceq a$ and $a \curlywedge b \preceq b \preceq c$, then from (5) $a \curlywedge b \preceq a \curlywedge c$.

□

Remark 3.2. *Generally, the converse implication on* (7) *is not true. Indeed,*
let $a = (0.2, 0.4, 0.3)$, $b = (0.1, 0.3, 0.2)$, $c = (0.2, 0.5, 0.3)$.
$a \curlyvee b = (0.2, 0, 0.2) \succeq (0.2, 0.5, 0.3) = c$, *but* $a \preceq c$ *and* $b \parallel c$.

3.2. **Picture fuzzy negators of $\mathbb{D}^*$.** Picture fuzzy negators of $\mathbb{D}^*$ are a generalization of fuzzy negators and intuitionistic fuzzy negators.

Definition 3.2. [12] *A* picture fuzzy negator *is any non-increasing mapping* $N : \mathbb{D}^* \longrightarrow \mathbb{D}^*$ *satisfying* $N(0_{\mathbb{D}^*}) = 1_{\mathbb{D}^*}$ *and* $N(1_{\mathbb{D}^*}) = 0_{\mathbb{D}^*}$.
N *is called an involutive picture fuzzy negator, if* $N(N(a)) = a$, *for all* $a \in \mathbb{D}^*$.

Proposition 3.2. *Let* $a = (a_1, a_2, a_3) \in \mathbb{D}^*$. *The mappings* N_1 *and* N_S *defined respectively by* $N_1(a) = (a_3, a_2, a_1)$ *and* $N_S(a) = (a_3, 1 - a_1 - a_2 - a_3, a_1)$, *for all* $a \in \mathbb{D}^*$, *are involutive picture fuzzy negators and* N_S *is called the standard picture fuzzy negator.*

4. Picture fuzzy sets operations

4.1. **Picture fuzzy inclusion.**

Definition 4.1. *For $E, F \in PFS(\mathbf{U})$. We say that $E \subseteq F$, if $E(a) \preceq F(a)$, for all $a \in \mathbf{U}$, where $\preceq$ is the order of $\mathbb{D}^*$. In more detail,*

$$E \subseteq F \iff (\mu_E(a), \eta_E(a), \nu_E(a)) \preceq (\mu_F(a), \eta_F(a), \nu_F(a)), \text{ for all } a \in \mathbf{U}.$$

$$\iff \begin{cases} \mu_E(a) < \mu_F(a) \text{ and } \nu_E(a) \geq \nu_F(a), \\ \text{or} \\ \mu_E(a) = \mu_F(a) \text{ and } \nu_E(a) > \nu_F(a), \\ \text{or} \\ \mu_E(a) = \mu_F(a), \nu_E(a) = \nu_F(a) \text{ and } \eta_E(a) \leq \eta_F(a). \end{cases}$$

As seen in Definition 3.1 the picture fuzzy inclusion is an order on $PFS(\mathbf{U})$.

Definition 4.2. *For $E, F \in PFS(\mathbf{U})$. We say that $E = F$, if $E(a) = F(a)$, for all $a \in \mathbf{U}$.*

4.2. **Picture fuzzy intersection and picture fuzzy union.**

Definition 4.3. *Let $E, F \in PFS(\mathbf{U})$. According to Definition 3.1, we define the* picture fuzzy intersection *by $(E \cap F)(a) = E(a) \curlywedge F(a)$. In more detail,*

$$E \cap F = \{\langle a, \mu_{E\cap F}(a), \eta_{E\cap F}(a), \nu_{E\cap F}(a)\rangle \mid a \in \mathbf{U}\}, \text{ where}$$

$$\mu_{E\cap F}(a) = \mu_E(a) \wedge \mu_F(a),$$

$$\eta_{E\cap F}(a) = \begin{cases} \eta_A(a), & \text{if } E(a) \preceq F(a), \\ \eta_F(a), & \text{if } F(a) \preceq E(a), \\ 1 - \mu_E(a) \wedge \mu_F(a) - \nu_E(a) \vee \nu_F(a), & \text{otherwise.} \end{cases}$$

$$\nu_{E\cap F}(a) = \nu_E(a) \vee \nu_F(a).$$

And picture fuzzy union *by $(E \cup F)(a) = E(a) \curlyvee F(a)$. In more detail,*

$$E \cup F = \{\langle a, \mu_{E\cup F}(a), \eta_{E\cup F}(a), \nu_{E\cup F}(a)\rangle \mid a \in \mathbf{U}\}, \text{ where}$$

$$\mu_{E\cup F}(a) = \mu_E(a) \vee \mu_F(a).$$

$$\eta_{E\cup F}(a) = \begin{cases} \eta_F(a), & \text{if } E(a) \preceq F(a), \\ \eta_E(a), & \text{if } F(a) \preceq E(a), \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_{E\cup F}(a) = \nu_E(a) \wedge \nu_F(a).$$

Also, according to Definition 3.1 $(PFS(\mathbf{U}), \subseteq, \cap, \cup, \emptyset, \mathbf{U})$ is a bounded lattice.

Example 4.1. *Let $\mathbf{U} = \{a, b, c\}$ and let $E, F, G \in PFS(\mathbf{U})$, where*
$E = \{\langle a, 0.01, 0.30, 0.52\rangle, \langle b, 0.02, 0.11, 0.36\rangle, \langle c, 0.13, 0.40, 0.32\rangle\}$,
$F = \{\langle a, 0.01, 0.35, 0.52\rangle, \langle b, 0.28, 0.33, 0.15\rangle, \langle c, 0.21, 0.00, 0.09\rangle\}$,
$G = \{\langle a, 0.00, 0.44, 0.21\rangle, \langle b, 0.05, 0.51, 0.27\rangle, \langle c, 0.21, 0.07, 0.53\rangle\}$.

Note that for all $x \in \mathbf{U}$, $E(x) \preceq F(x)$, then $E \subseteq F$. Moreover, $E \cap F = E$ and $E \cup F = F$. On the other hand, there exists $a \in \mathbf{U}$ such that $E(a) \| G(a)$, then neither $E \subseteq G$ nor $G \subseteq E$. Moreover, $E \cap G = \{\langle a, 0.00, 0.48, 0.52\rangle, \langle b, 0.02, 0.11, 0.36\rangle, \langle c, 0.13, 0.34, 0.53\rangle\}$ and $E \cup G = \{\langle a, 0.01, 0.00, 0.21\rangle, \langle b, 0.05, 0.51, 0.27\rangle, \langle c, 0.21, 0.00, 0.32\rangle\}$.

Proposition 4.1. *Let $E, F, G \in PFS(\mathbf{U})$. As in classical set theory, the definitions we have just given lead to the following properties:*

1. $E \cap (F \cap G) = (E \cap F) \cap G$, $E \cup (F \cup G) = (E \cup F) \cup G$.
2. $E \cap F = F \cap E$, $E \cup F = F \cup E$.
3. $E \cap (E \cup F) = E$, $E \cup (E \cap F) = E$.
4. $E \cap E = E$, $E \cup E = E$.

(5) $E \cup F \supseteq E \supseteq E \cap F$, $E \cup F \supseteq F \supseteq E \cap F$.
(6) $E \cup \emptyset = E$, $E \cup \mathbf{U} = \mathbf{U}$, $\forall E \in PFS(\mathbf{U})$.
(7) $E \cap \emptyset = \emptyset$, $E \cap \mathbf{U} = E$, $\forall E \in PFS(\mathbf{U})$.

Proof. Using the properties in Proposition 3.1, the proofs are straightforward. □

4.3. **Picture fuzzy complement of a picture fuzzy set.**

Definition 4.4. *Let $E \in PFS(\mathbf{U})$. Using the negators N_S and N_1 in Proposition 3.2, we define $E^{\mathcal{C}}$ and $\overline{E}$ respectively by*
$E^{\mathcal{C}} = N_S(E) = \{\langle a, \nu_E(a), 1 - \mu_E(a) - \eta_E(a) - \nu_E(a), \mu_E(a)\rangle \mid a \in \mathbf{U}\}$,
$\overline{E} = N_1(E) = \{\langle a, \nu_E(a), \eta_E(a), \mu_E(a)\rangle \mid a \in \mathbf{U}\}$.

Example 4.2. *The complement of the picture fuzzy sets given in Example 4.1 is*
$E^{\mathcal{C}} = \{\langle a, 0.52, 0.17, 0.01\rangle, \langle b, 0.36, 0.51, 0.02\rangle, \langle c, 0.32, 0.15, 0.13\rangle\}$,
$F^{\mathcal{C}} = \{\langle a, 0.52, 0.12, 0.01\rangle, \langle b, 0.15, 0.24, 0.28\rangle, \langle c, 0.09, 0.70, 0.21\rangle\}$,
$G^{\mathcal{C}} = \{\langle a, 0.21, 0.35, 0.00\rangle, \langle b, 0.27, 0.17, 0.05\rangle, \langle c, 0.53, 0.19, 0.21\rangle\}$.

Unlike classical subsets, a picture fuzzy set E usually satisfies $E^{\mathcal{C}} \cap E \neq \emptyset$ and $E^{\mathcal{C}} \cup E \neq \mathbf{U}$, however, some other properties of classical set theory are satisfied, such as:

Proposition 4.2. *For $E, F \in PFS(\mathbf{U})$, the complement of picture fuzzy sets $\mathcal{C}$ verifies the following properties:*
(1) $(E \cap F)^{\mathcal{C}} = E^{\mathcal{C}} \cup F^{\mathcal{C}}$, $(E \cup F)^{\mathcal{C}} = E^{\mathcal{C}} \cap F^{\mathcal{C}}$.
(2) $(\emptyset)^{\mathcal{C}} = \mathbf{U}$, $(\mathbf{U})^{\mathcal{C}} = \emptyset$.
(3) $\left(E^{\mathcal{C}}\right)^{\mathcal{C}} = E$.
(4) $E \subseteq F$ *implies* $F^{\mathcal{C}} \subseteq E^{\mathcal{C}}$.

Proof. (1) Let $a \in \mathbf{U}$

$$(E \cap F)(a) = \begin{cases} E(a), & \text{if } E(a) \preceq F(a). \\ F(a), & \text{if } F(a) \preceq E(a). \\ (\mu_{E\cap F}(a), \eta_{E\cap F}(a), \nu_{E\cap F}(a)), & \text{otherwise.} \end{cases}$$

Where
$$\begin{aligned} \mu_{E\cap F}(a) &= \mu_E(a) \wedge \mu_F(a), \\ \eta_{E\cap F}(a) &= 1 - \mu_E(a) \wedge \mu_F(a) - \nu_E(a) \vee \nu_F(a), \\ \nu_{E\cap F}(a) &= \nu_E(a) \vee \nu_F(a). \end{aligned}$$

Hence,
$$(E \cap F)^{\mathcal{C}}(a) = \begin{cases} E^{\mathcal{C}}(a), & \text{if } F^{\mathcal{C}}(a) \preceq E^{\mathcal{C}}(a). \\ F^{\mathcal{C}}(a), & \text{if } E^{\mathcal{C}}(a) \preceq F^{\mathcal{C}}(a). \\ \{\langle a, \nu_E(a) \vee \nu_F(a), 0, \mu_E(a) \wedge \mu_F(a)\rangle \mid a \in \mathbf{U}\}, & \text{otherwise.} \end{cases}$$

In other hand,
$E^{\mathcal{C}} = \{\langle a, \nu_E(a), 1 - \mu_E(a) - \eta_E(a) - \nu_E(a), \mu_E(a)\rangle \mid a \in \mathbf{U}\}$,
$F^{\mathcal{C}} = \{\langle a, \nu_F(a), 1 - \mu_F(a) - \eta_F(a) - \nu_F(a), \mu_F(a)\rangle \mid a \in \mathbf{U}\}$.

Hence,
$$\left(E^{\mathcal{C}} \cup F^{\mathcal{C}}\right)(a) = \begin{cases} E^{\mathcal{C}}(a), & \text{if } F^{\mathcal{C}}(a) \preceq E^{\mathcal{C}}(a). \\ F^{\mathcal{C}}(a), & \text{if } E^{\mathcal{C}}(a) \preceq F^{\mathcal{C}}(a). \\ \{\langle a, \nu_E(a) \vee \nu_F(a), 0, \mu_E(a) \wedge \mu_F(a)\rangle \mid a \in \mathbf{U}\}, & \text{otherwise.} \end{cases}$$

Therefore $(E \cap F)^{\mathcal{C}} = E^{\mathcal{C}} \cup F^{\mathcal{C}}$.
In a similar way, it can be shown that $(E \cup F)^{\mathcal{C}} = E^{\mathcal{C}} \cap F^{\mathcal{C}}$.

(2) For all $a \in \mathbf{U}$, $(\emptyset)^{\mathcal{C}}(a) = (\emptyset(a))^{\mathcal{C}} = (0,0,1)^{\mathcal{C}} = (1,0,0) = 1_{\mathbb{D}^*}$.
Hence $(\emptyset)^{\mathcal{C}} = \{\langle a,1,0,0\rangle \mid a \in \mathbf{U}\} = \mathbf{U}$.
Dually, $(\mathbf{U})^{\mathcal{C}} = \emptyset$.

Finally, (3) and (4) are easy to verify. □

In what follows we denote by $\mathbb{D}_0^* = \mathbb{D}^* - \{0_{\mathbb{D}^*}\}, \mathbb{D}_1^* = \mathbb{D}^* - \{1_{\mathbb{D}^*}\}$.

5. Characteristic sets of a picture fuzzy set

Among the crucial notions in fuzzy set theory, are the notions of support, kernel, cuts and fuzzy line of degree α of a fuzzy set, where $\alpha \in \mathbb{D}^*$. In the sequel, we generalize these notions to the notions of a picture fuzzy set with respect to the order $\preceq$ in Definition 3.1.

5.1. **Support of a picture fuzzy set.**

Definition 5.1. *Let $E \in PFS(\mathbf{U})$.* The support of E *is the classical subset $S(E)$ on $\mathbf{U}$ given by*

$$S(E) = \{a \in \mathbf{U} \mid E(a) \succ 0_{\mathbb{D}^*}\}.$$

According to Remark 3.1, $S(E) = \{a \in \mathbf{U} \mid \mu_E(a) > 0 \text{ or } (\mu_E(a) = 0 \text{ and } \nu_E(a) < 1)\}$.

5.2. **Kernel of a picture fuzzy set.**

Definition 5.2. *Let $E \in PFS(\mathbf{U})$.* The kernel of E *is the classical subset $Ker(E)$ on $\mathbf{U}$ given by*

$$Ker(E) = \{a \in \mathbf{U} \mid E(a) = 1_{\mathbb{D}^*}\}.$$

5.3. **α-cuts and Strong α-cuts of a picture fuzzy set.**

Definition 5.3. *Let $E \in PFS(\mathbf{U})$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_0^*$,* the α-cut of E *is the classical subset E_α on $\mathbf{U}$ given by*

$$E_\alpha = \{a \in \mathbf{U} \mid E(a) \succeq \alpha\}.$$

According to Definition 3.1, $E_\alpha = \{a \in \mathbf{U} | (\mu_E(a) > \alpha_1 \text{ and } \nu_E(a) \leq \alpha_3) \text{ or } (\mu_E(a) = \alpha_1 \text{ and } \nu_E(a) < \alpha_3) \text{ or } (\mu_E(a) = \alpha_1, \ \nu_E(a) = \alpha_3 \text{ and } \eta_E(a) \geq \alpha_2)\}$.

Definition 5.4. *Let $E \in PFS(\mathbf{U})$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_1^*$,* the strong α-cut of E *is the classical subset E_α^+ on $\mathbf{U}$ given by*

$$E_\alpha^+ = \{a \in \mathbf{U} \mid E(a) \succ \alpha\}.$$

According to Remark 3.1, $E_\alpha^+ = \{a \in \mathbf{U} \mid (\mu_E(a) > \alpha_1 \text{ and } \nu_E(a) \leq \alpha_3) \text{ or } (\mu_E(a) = \alpha_1 \text{ and } \nu_E(a) < \alpha_3) \text{ or } (\mu_E(a) = \alpha_1, \ \nu_E(a) = \alpha_3 \text{ and } \eta_E(a) > \alpha_2)\}$.

Proposition 5.1. *Let $E, F \in PFS(\mathbf{U})$. For any $\alpha, \beta \in \mathbb{D}^*$, we have*

(1) $E_\alpha^+ \subseteq E_\alpha$.
(2) $E \subseteq F$ *if and only if* $E_\alpha \subseteq F_\alpha$, *for all* $\alpha \in \mathbb{D}_0^*$.
(3) $E \subseteq F$ *if and only if* $E_\alpha^+ \subseteq F_\alpha^+$, *for all* $\alpha \in \mathbb{D}_1^*$.
(4) $\alpha \preceq \beta$ *implies* $E_\alpha \supseteq E_\beta$, *for all* $\alpha, \beta \in \mathbb{D}_0^*$.
(5) $\alpha \preceq \beta$ *implies* $E_\alpha^+ \supseteq E_\beta^+$, *for all* $\alpha, \beta \in \mathbb{D}_1^*$.
(6) $(E \cap F)_\alpha = E_\alpha \cap F_\alpha$.
(7) $(E \cup F)_\alpha \supseteq E_\alpha \cup F_\alpha$.

Proof. Let $a \in \mathbf{U}$.

(1) Clear.

(2) Assume $E \subseteq F$ and suppose that $a \in E_\alpha$, then $E(a) \succeq \alpha$. Since $F(a) \succeq E(a)$ for all $a \in \mathbf{U}$, it follows that $F(a) \succeq \alpha$. Thus $a \in F_\alpha$.
Conversely, assume $E_\alpha \subseteq F_\alpha$.
Put $E(a) = \alpha$. It is clear that if $\alpha = 0_{\mathbb{D}^*}$, $F(a) \succeq 0_{\mathbb{D}^*}$.
If $\alpha \neq 0_{\mathbb{D}^*}$, then for all $a \in E_\alpha$ implies $a \in F_\alpha$. Thus $F(a) \succeq \alpha = E(a)$, for all $a \in \mathbf{U}$. Hence $E \subseteq F$.

(3) The direct implication is similar to the previous proof.
For the converse implication, suppose for a contradiction that $E_\alpha^+ \subseteq F_\alpha^+$, for all $\alpha \in \mathbb{D}_1^*$ but that $E \nsubseteq F$.
Then there exists $a \in \mathbf{U}$ such that $E(a) \succ F(a)$ or $E(a) \| F(a)$.
If $E(a) \succ F(a)$, we can take α between $E(a)$ and $F(a)$ i.e., $E(a) \succ \alpha \succ F(a)$.
This contradicts the fact that $E_\alpha^+ \subseteq F_\alpha^+$.
If $E(a) \| F(a)$, we have two cases: $\mu_E(a) < \mu_F(a)$ and $\nu_E(a) < \nu_F(a)$ or $\mu_F(a) < \mu_E(a)$ and $\nu_F(a) < \nu_E(a)$.
For the first case, take $\lambda = \left(\mu_E(a), \frac{\eta_E(a)+\eta_F(a)}{2}, \frac{\nu_E(a)+\nu_F(a)}{2}\right)$, it is clear that $\lambda < E(a)$ and $\lambda \| F(a)$. This is also a contradiction.
Similarly, we obtain the same result in the second case.

(4) Assume $\alpha \preceq \beta$ and suppose that $a \in E_\beta$. Then $E(a) \succeq \beta \succeq \alpha$. Thus $a \in E_\alpha$.

(5) Similar to (4).

(6)
$$\begin{aligned} (E \cap F)_\alpha &= \{a \in \mathbf{U} \mid (E \cap F)(a) \succeq \alpha\} \\ &= \{a \in \mathbf{U} \mid E(a) \curlywedge F(a) \succeq \alpha\} \\ &= \{a \in \mathbf{U} \mid E(a) \succeq \alpha \text{ and } F(a) \succeq \alpha\} \\ &= \{a \in \mathbf{U} \mid E(a) \succeq \alpha\} \cap \{a \in \mathbf{U} \mid F(a) \succeq \alpha\} \\ &= E_\alpha \cap F_\alpha. \end{aligned}$$

(7) Similar to (6).

□

Remark 5.1. *Concerning this proposition, it is important to note the following details:*

- *Generally, the converse of* (4) *and* (5) *is not true. Indeed,*
  Let $\mathbf{U} = \{a, b\}$ *and let* $E, F \in PFS(\mathbf{U})$ *given by*
  $E = \{\langle a, 1, 0, 0\rangle, \langle b, 0.55, 0.23, 0.11\rangle\}$, $F = \{\langle a, 0.75, 0.12, 0.02\rangle, \langle b, 1, 0, 0\rangle\}$.
  And take $\alpha = (0.61, 0.01, 0.21), \beta = (0.62, 0.01, 0.22)$. *It is easy to observe that* $E_\alpha = \{a\} \subseteq E_\beta = \{a\}$ *and* $E_\alpha^+ = \{a\} \subseteq E_\beta^+ = \{a\}$ *but* $\alpha \| \beta$.
- *The converse of* (4) *and* (5) *is true when* α *and* β *are comparable.*
- *As seen in Remark 3.2, the converse of* (7) *is not true.*

Corollary 5.1. *For all* $E, F \in PFS(\mathbf{U})$, *we have*

(1) $E = F$ *if and only if* $E_\alpha = F_\alpha$, *for all* $\alpha \in \mathbb{D}_0^*$.
(2) $E = F$ *if and only if* $E_\alpha^+ = F_\alpha^+$, *for all* $\alpha \in \mathbb{D}_1^*$.
(3) *If* E *is a crisp subset on* $\mathbf{U}$, *then* $E_\alpha = E$, *for all* $\alpha \in \mathbb{D}_0^*$
(4) *if* $\alpha = 0_{\mathbb{D}^*}$, *then* $E_\alpha^+ = S(E)$ *and* $E_\alpha = \mathbf{U}$.
(5) *if* $\alpha = 1_{\mathbb{D}^*}$, *then* E_α *is the kernel of* E.

5.4. Picture fuzzy line of degree α of a picture fuzzy set.

Definition 5.5. *Let* $E \in PFS(\mathbf{U})$. *For* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}^*$, the picture fuzzy line of degree α of E *is the classical subset* $L_\alpha(E)$ *on* $\mathbf{U}$ *given by*

$$L_\alpha(E) = \{a \in \mathbf{U} \mid E(a) = \alpha\}.$$

Proposition 5.2. *Let* $E, F \in PFS(\mathbf{U})$. *For all* $\alpha, \beta \in \mathbb{D}^*$, *we have*

(1) $L_\alpha(E) \subseteq E_\alpha$, $L_{1_{\mathbb{D}^*}}(E) = Ker(E)$.
(2) $E = F$ *if and only if* $L_\alpha(E) = L_\alpha(F)$.
(3) *If* $\alpha \neq \beta$, *then* $L_\alpha(E) \cap L_\beta(E) = \emptyset$.
(4) $L_\alpha(E) \cap L_\alpha(F) \subseteq L_\alpha(E \cap F)$.
(5) $L_\alpha(E) \cup L_\beta(F) \subseteq L_{\alpha \curlyvee \beta}(E \cup F)$.
(6) $L_\alpha(E \cap F) \subseteq E_\alpha \cap F_\alpha$.

Proof. Let $a \in \mathbf{U}$ and let $\alpha, \beta \in \mathbb{D}^*$.

(1) Clear.
(2) Assume $E = F$ and suppose that $a \in L_\alpha(E)$. Then $F(a) = E(a) = \alpha$, hence $a \in L_\alpha(F)$.
Conversely, suppose that $L_\alpha(E) = L_\alpha(F)$.
Put $E(a) = \alpha$, that is, $a \in L_\alpha(E)$. This equivalent to $a \in L_\alpha(F)$, i.e., $F(a) = \alpha$. Therefore, $E = F$.
(3) Suppose that $\alpha \neq \beta$ and $a \in L_\alpha(E)$, that is, $E(a) = \alpha \neq \beta$, then $a \notin L_\beta(E)$. Thus $L_\alpha(E) \cap L_\beta(E) = \emptyset$.
(4) Suppose that $a \in L_\alpha(E) \cap L_\alpha(F)$. Then $a \in L_\alpha(E)$ and $a \in L_\alpha(F)$, imply that $E(a) = \alpha$ and $F(a) = \alpha$. Hence $E(a) \curlywedge F(a) = \alpha$, this gives $(E \cap F)(a) = \alpha$. Consequently $a \in L_\alpha(E \cap F)$.
(5) Similar to (4).
(6) Suppose that $a \in L_\alpha(E \cap F)$. Then $(E \cap F)(a) = \alpha$, that is, $E(a) \curlywedge F(a) = \alpha$, it follows that $E(a) \succeq \alpha$ and $F(a) \succeq \alpha$. Thus $a \in E_\alpha$ and $a \in F_\alpha$. Hence $a \in E_\alpha \cap F_\alpha$.

□

Remark 5.2. *Concerning this proposition, it is important to note the following details:*

- *The converse of* (3) *holds if* $L_\alpha(E) \neq \emptyset$ *or* $L_\beta(E) \neq \emptyset$.
- *Generally, the converse inclusion of* (4) *and* (5) *is not true. Indeed,*
  Let $\mathbf{U} = \{a, b\}$ *and let* $E, F \in PFS(\mathbf{U})$ *given by*
  $E = \{\langle a, 0.10, 0.30, 0.40\rangle, \langle b, 0.30, 0.20, 0.10\rangle\}$,
  $F = \{\langle a, 0.20, 0.20, 0.50\rangle, \langle b, 0.20, 0.40, 0.30\rangle\}$.
  Then, $E \cap F = \{\langle a, 0.10, 0.40, 0.50\rangle, \langle b, 0.20, 0.40, 0.30\rangle\}$.
  Take $\alpha = (0.10, 0.40, 0.50)$, *hence* $L_\alpha(E \cap F) = \{a\}$ *and* $L_\alpha(E) = L_\alpha(F) = \emptyset$
  Note that $L_\alpha(E \cap F) \nsubseteq L_\alpha(E) \cap L_\alpha(F)$.
  In the same way, take $\alpha = (0.20, 0.10, 0.50)$ *and* $\beta = (0.08, 0.31, 0.40)$, *It is easy to see that* $L_{\alpha \curlyvee \beta}(E \cup F) = \{a\}$, $L_\alpha(E) = \emptyset$ *and* $L_\beta(F) = \emptyset$.
  Hence, $L_{\alpha \curlyvee \beta}(E \cup F) \nsubseteq L_\alpha(E) \cup L_\beta(F)$.

5.5. **Some decomposition theorems of a picture fuzzy set.** These theorems permit to express any picture fuzzy subset on $\mathbf{U}$ in terms of its α-cuts, strong α-cut and picture fuzzy line of degree α.

Theorem 5.1. *Let* $E \in PFS(\mathbf{U})$. *Then*

$$E(a) = \curlyvee_{\alpha \in \mathbb{D}^*} \alpha E_\alpha(a), \text{ for all } a \in \mathbf{U}.$$

Where

$$E_\alpha(a) = \begin{cases} 1 & \text{if } a \in E_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $a \in \mathbf{U}$.

Put $E(a) = \lambda$, where $\lambda \in \mathbb{D}_0^*$, then $E_\lambda(a) = 1$.

We can express $\mathbb{D}^*$ as the union of three sets D_1, D_2, D_3, where $D_1 = \{\alpha \in \mathbb{D}^* \mid \alpha \preceq \lambda\}$, $D_2 = \{\alpha \in \mathbb{D}^* \mid \alpha \succ \lambda\}$ and $D_3 = \{\alpha \in \mathbb{D}^* \mid \alpha \parallel \lambda\}$.

It holds that
$\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha E_\alpha(a)=\left(\underset{\alpha\in D_1}{\curlyvee}\{\alpha E_\alpha(a)\}\right)\curlyvee\left(\underset{\alpha\in D_2}{\curlyvee}\{\alpha E_\alpha(a)\}\right)\curlyvee\left(\underset{\alpha\in D_3}{\curlyvee}\{\alpha E_\alpha(a)\}\right).$

If $\alpha\in D_1$, then $E_\alpha(a)=1$. Hence $\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha E_\alpha(a)=\underset{\alpha\in D_1}{\curlyvee}\alpha E_\alpha(a)=\underset{\alpha\in D_1}{\curlyvee}\alpha=\lambda.$

Otherwise, $E_\alpha(a)=0$. Then $\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha E_\alpha(a)=0.$

Therefore, $\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha E_\alpha(a)=\lambda\curlyvee 0\curlyvee 0=\lambda=E(a)$. ☐

Theorem 5.2. *Let $E\in PFS(\mathbf{U})$. Then*

$$E(a)=\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha E_\alpha^+(a)\,,\ \textit{for all } a\in\mathbf{U}.$$

Proof. Similar to the previous proof, it suffices to take $D_1=\{\alpha\in\mathbb{D}^*\mid\alpha\prec\lambda\}$, $D_2=\{\alpha\in\mathbb{D}^*\mid\alpha\succeq\lambda\}$ and $D_3=\{\alpha\in\mathbb{D}^*\mid\alpha\parallel\lambda\}$. ☐

Theorem 5.3. *Let $E\in PFS(\mathbf{U})$. Then*

$$E(a)=\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha L_\alpha(E)(a)\,,\ \textit{for all } a\in\mathbf{U}.$$

Proof. Let $a\in\mathbf{U}$.

Put $E(a)=\lambda$, where $\lambda\in\mathbb{D}^*$, then $L_\lambda(E)(a)=1$.

$$\underset{\alpha\in\mathbb{D}^*}{\curlyvee}\alpha L_\alpha(E)(a)=\left(\underset{\alpha=\lambda}{\curlyvee}\{\alpha L_\alpha(E)(a)\}\right)\curlyvee\left(\underset{\alpha\neq\lambda}{\curlyvee}\{\alpha L_\alpha(E)(a)\}\right)=\lambda\curlyvee 0=\lambda=E(a).$$
☐

Proposition 5.3. *Let $E\in PFS(\mathbf{U})$. Then for all $\alpha,\lambda\in\mathbb{D}^*$,*

$$E_\alpha=\underset{\alpha\preceq\lambda}{\cup}E_\lambda.$$

Proof. Direct. ☐

6. Modal operators defined on picture fuzzy sets

In this section, we extend some of Atanassov's modal operators to the picture fuzzy set case.

6.1. **Necessity and Possibility operators.** Now, we define two operators on the set of picture fuzzy sets that transform every picture fuzzy set into an intuitionistic fuzzy set. These operators extend Atanasov's operators ([3], [5]) "necessity" and "possibility" defined in certain modal logics.

Definition 6.1. *For $E\in PFS(\mathbf{U})$, the following associated picture fuzzy sets $\Box E$* (necessity) *and $\Diamond E$* (possibility) *on $\mathbf{U}$ are given by:*

$\Box E=\{\langle a,\mu_E(a),\eta_E(a),1-\eta_E(a)-\mu_E(a)\rangle\mid a\in\mathbf{U}\}$.

$\Diamond E=\{\langle a,1-\eta_E(a)-\nu_E(a),\eta_E(a),\nu_E(a)\rangle\mid a\in\mathbf{U}\}$.

With the involutive negator N_1 in Proposition 3.2, the operators $\Box,\Diamond$ verify a similar relation like that seen between modal operators on Łukasiewicz-Moisil algebras.

Recalling that $\overline{E}=N_1(E)=\{\langle a,\nu_E(a),\eta_E(a),\mu_E(a)\rangle\mid a\in\mathbf{U}\}$.

Proposition 6.1. *Let $E\in PFS(\mathbf{U})$. Then*

(1) $\overline{\Box\overline{E}}=\Diamond E$.

(2) $\overline{\Diamond\overline{E}}=\Box E$.

Proof. Let $a\in\mathbf{U}$,

(1) $$\begin{aligned}\overline{\Box\overline{E}} &= \overline{\Box\overline{\{\langle a,\mu_E(a),\eta_E(a),\nu_E(a)\rangle \mid a\in\mathbf{U}\}}},\\ &= \overline{\Box\{\langle a,\nu_E(a),\eta_E(a),\mu_E(a)\rangle \mid a\in\mathbf{U}\}},\\ &= \overline{\{\langle a,\nu_E(a),\eta_E(a),1-\nu_E(a)-\eta_E(a)\rangle \mid a\in\mathbf{U}\}},\\ &= \{\langle a,1-\nu_E(a)-\eta_E(a),\eta_E(a),\nu_E(a)\rangle \mid a\in\mathbf{U}\},\\ &= \Diamond E.\end{aligned}$$

(2) Is obtained dually.

$\Box$

Proposition 6.2. *For any $E \in PFS(\mathbf{U})$, the operators $\Box$ and $\Diamond$ verify the following proprieties.*

(1) *$\Diamond$ is extensive and $\Box$ is retractive (i.e., $\Box E \subseteq E \subseteq \Diamond E$).*
(2) *$\Diamond, \Box$ are idempotent (i.e., $\Diamond\Diamond E = \Diamond E$ and $\Box\Box E = \Box E$).*
(3) *$\Box\Diamond E = \Diamond E$.*
(4) *$\Diamond\Box E = \Box E$.*

Proof. Direct. $\Box$

Remark 6.1. *Obviously, if E is a crisp set, then $\Box E = E = \Diamond E$.*

Proposition 6.3. *For any $E \in PFS(\mathbf{U})$, we have*

(1) $S(\Box E) \subseteq S(E)$.
(2) $S(\Diamond E) = S(E)$.
(3) $Ker(E) = Ker(\Box E)$.
(4) $Ker(E) \subseteq Ker(\Diamond E)$.

Proof. Let $a \in \mathbf{U}$.

(1) Suppose that $a \in S(\Box E)$ i.e., $\Box E(a) \succ 0_{\mathbb{D}^*}$, which means $\mu_{\Box E}(a) = \mu_E(a) > 0$ or $\mu_{\Box E}(a) = \mu_E(a) = 0$ and $\nu_{\Box E}(a) = 1 - \mu_E(a) - \eta_E(a) < 1$. Thus $\mu_E(a) > 0$ or $\mu_E(a) = 0$ and $\eta_E(a) > 0$. Since $\mu_E(a) + \eta_E(a) + \nu_E(a) \leq 1$, it follows that $\mu_E(a) > 0$ or $\mu_E(a) = 0$ End $\nu_E(a) < 1$, that is, $E(a) \succ 0$. Hence $a \in S(E)$.
(2) The direct inclusion is similar to the previous proof. It remains to show that, $S(E) \subseteq S(\Diamond E)$.
Suppose that $a \in S(E)$ i.e., $E(a) \succ 0_{\mathbb{D}^*}$, means $\mu_E(a) > 0$ or $\mu_E(a) = 0$ and $\nu_E(a) < 1$. We discuss two cases:
If $\mu_E(a) > 0$, then $\mu_{\Diamond E}(a) = 1 - \eta_E(a) - \nu_E(a) \geq \mu_E(a) > 0$. Thus $a \in S(\Diamond E)$.
If $\mu_E(a) = 0$ and $\nu_E(a) < 1$, then $\mu_{\Diamond E}(a) = 1 - \eta_E(a) - \nu_E(a) \geq \mu_E(a) = 0$ and $\nu_{\Diamond E}(a) = \nu_E(a) < 1$. Thus $a \in S(\Diamond E)$.
(3) Suppose that $a \in Ker(E)$ i.e., $E(a) = 1_{\mathbb{D}^*}$, that is, $\mu_E(a) = 1$, $\eta_E(a) = 0$ and $\nu_E(a) = 0$. Which is equivalent to $\mu_{\Box E}(a) = \mu_E(a) = 1$, $\eta_{\Box E}(a) = \eta_E(a) = 0$ and $\nu_{\Box E}(a) = 1 - \mu_E(a) - \eta_E(a) = 0$. Hence $a \in Ker(\Box E)$.
(4) Suppose that $a \in Ker(E)$ i.e., $E(a) = 1_{\mathbb{D}^*}$, that is, $\mu_E(a) = 1$, $\eta_E(a) = 0$ and $\nu_E(a) = 0$. This implies that, $\mu_{\Diamond E}(a) = 1 - \eta_E(a) - \nu_E(a) = 1$, $\eta_{\Diamond E}(a) = \eta_E(a) = 0$ and $\nu_{\Diamond E}(a) = \nu_E(a) = 0$. Hence $a \in Ker(\Diamond E)$.

$\Box$

From the operators $\Box, \Diamond$, two new relations are defined as follows:
$E \subseteq_{\Box} F \Longleftrightarrow \mu_E(a) \leq \mu_F(a)$ and $\eta_E(a) \leq \eta_F(a)$, for all $a \in \mathbf{U}$.
$E \subseteq_{\Diamond} F \Longleftrightarrow \eta_E(a) \geq \eta_F(a)$ and $\nu_E(a) \geq \nu_E(a)$, for all $a \in \mathbf{U}$.
These two new relations lead to the following results, which are straightforward.

Proposition 6.4. *Let $E, F \in PFS(\mathbf{U})$. Then*

(1) *$E \subseteq_{\Box} F$ if and only if $\Box E \subseteq \Box F$.*
(2) *$E \subseteq_{\Diamond} F$ if and only if $\Diamond E \subseteq \Diamond F$.*
(3) *$E \subseteq_{\Box} F$ and $E \subseteq_{\Diamond} F$ implies $E \subseteq F$.*

6.2. **Operators $\mathcal{D}_\lambda$ and $\mathcal{F}_\alpha$.** In the following, two other Atanassov's modal operators $\mathcal{D}_\lambda$, $\mathcal{F}_\alpha$ (see [5]) will be extended to the picture fuzzy set case.

These operators are extensions of the operators $\Box$ and $\Diamond$.

Definition 6.2. *Let $E \in PFS(\mathbf{U})$. Define the operator $\mathcal{D}_\lambda$ by*

$$\mathcal{D}_\lambda(E) = \{\langle a, \mu_E(a) + \lambda.\pi_E(a), \eta_E(a), \nu_E(a) + (1-\lambda).\pi_E(a)\rangle \mid a \in \mathbf{U}\},$$

where λ is a fixed number in $[0,1]$.

Proposition 6.5. *Let $E \in PFS(\mathbf{U})$. Then for all $\lambda_1, \lambda_2 \in [0,1]$, the following properties hold:*

(1) *If $\lambda_1 \leq \lambda_2$, then $\mathcal{D}_{\lambda_1}(E) \subseteq \mathcal{D}_{\lambda_1}(E)$.*
(2) *$\mathcal{D}_0(E) = \Box E$.*
(3) *$\mathcal{D}_1(E) = \Diamond E$.*

Proof. Direct. $\Box$

Definition 6.3. *Let $E \in PFS(\mathbf{U})$. For all $\alpha \in \mathbb{D}^*$, define the operator $\mathcal{F}_\alpha$ by*

$$\mathcal{F}_\alpha(E) = \{\langle a, \mu_E(a) + \alpha_1.\pi_E(a), \eta_E(a) + \alpha_2.\pi_E(a), \nu_E(a) + \alpha_3.\pi_E(a))\rangle \,| a \in \mathbf{U}\}.$$

Proposition 6.6. *Let $E \in PFS(\mathbf{U})$ and let $\alpha, \beta \in \mathbb{D}^*$. Then the following properties hold:*

(1) *If $\alpha \leq \beta$, then $\mathcal{F}_\alpha(E) \subseteq \mathcal{F}_\beta(E)$.*
(2) *$\mathcal{F}_{0_{\mathbb{D}^*}}(E) = \Box E$.*
(3) *$\mathcal{F}_{1_{\mathbb{D}^*}}(E) = \Diamond E$.*
(4) *$\mathcal{D}_\lambda(E) = \mathcal{F}_{(\lambda,0,1-\lambda)}(E)$.*
(5) *$\overline{\mathcal{F}_\alpha\left(\overline{E}\right)} = \mathcal{F}_{\overline{\alpha}}(E)$.*

Proof. The statements (1), (2), (3) and (4) are easy to check. Using the negator N_1, the proof of (5) is direct. $\Box$

Theorem 6.1. *Let $E \in PFS(\mathbf{U})$ and let $\alpha, \beta \in \mathbb{D}^*$. Then the following property holds:*

$$\mathcal{F}_\alpha(\mathcal{F}_\beta(E)) = \mathcal{F}_\gamma(E), \text{ where}$$

$$\gamma = (\beta_1 + \alpha_1.(1-\beta_1-\beta_2-\beta_3), \beta_2 + \alpha_2.(1-\beta_1-\beta_2-\beta_3), \beta_3 + \alpha_3.(1-\beta_1-\beta_2-\beta_3)).$$

Proof. Let $\alpha, \beta \in \mathbb{D}^*$. For all $a \in \mathbf{U}$,

$\mathcal{F}_\beta(E) = \{\langle a, \mu_E(a) + \beta_1.\pi_E(a), \eta_E(a) + \beta_2.\pi_E(a), \nu_E(a) + \beta_3.\pi_E(a)\rangle \,| a \in \mathbf{U}\}$ Then $\mathcal{F}_\alpha(\mathcal{F}_\beta(E)) = \left\{\left\langle a, \mu_{\mathcal{F}_\alpha(\mathcal{F}_\beta(E))}(a), \eta_{\mathcal{F}_\alpha(\mathcal{F}_\beta(E))}(a), \nu_{\mathcal{F}_\alpha(\mathcal{F}_\beta(E))}(a)\right\rangle \mid a \in \mathbf{U}\right\}$, where

$$\begin{aligned}
\mu_{\mathcal{F}_\alpha(\mathcal{F}_\beta(E))}(a) &= (\mu_E(a) + \beta_1.\pi_E(a)) + \alpha_1.\pi_{\mathcal{F}_\beta(E)}(a), \\
&= (\mu_E(a) + \beta_1.\pi_E(a)) + \alpha_1.(1 - \mu_E(a) - \beta_1.\pi_E(a) - \\
&\quad \eta_E(a) - \beta_2.\pi_E(a) - \nu_E(a) - \beta_3.\pi_E(a)), \\
&= \mu_E(a) + (\beta_1 + \alpha_1 - \alpha_1.\beta_1 - \alpha_1.\beta_2 - \alpha_1.\beta_3).\pi_E(a). \\
&= \mu_E(a) + (\beta_1 + \alpha_1.(1-\beta_1-\beta_2-\beta_3)).\pi_E(a).
\end{aligned}$$

$$\begin{aligned}
\eta_{\mathcal{F}_\alpha\left(\mathcal{F}_\beta(E)\right)}(a) &= \left(\eta_E(a)+\beta_2.\pi_E(a)\right)+\alpha_2.\pi_{\mathcal{F}_\beta(E)}(a),\\
&= \ \left(\eta_E(a)+\beta_2.\pi_E(a)\right)+\alpha_2.\left(1-\mu_E(a)-\beta_1.\pi_E(a)-\right.\\
&\quad \left.\eta_E(a)-\beta_2.\pi_E(a)-\nu_E(a)-\beta_3.\pi_E(a)\right),\\
&= \eta_E(a)+\left(\beta_2+\alpha_2-\alpha_2.\beta_1-\alpha_2.\beta_2-\alpha_3.\beta_3\right).\pi_E(a),\\
&= \eta_E(a)+\left(\beta_2+\alpha_2.\left(1-\beta_1-\beta_2-\beta_3\right)\right).\pi_E(a).
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathcal{F}_\alpha\left(\mathcal{F}_\beta(E)\right)}(a) &= \left(\nu_E(a)+\beta_3.\pi_E(a)\right)+\alpha_3.\pi_{\mathcal{F}_\beta(E)}(a), \qquad \square\\
&= \ \left(\nu_E(a)+\beta_3.\pi_E(a)\right)+\alpha_3.\left(1-\mu_E(a)-\beta_1.\pi_E(a)-\right.\\
&\quad \left.\eta_E(a)-\beta_2.\pi_E(a)-\nu_E(a)-\beta_3.\pi_E(a)\right),\\
&= \nu_E(a)+\left(\beta_3+\alpha_3-\alpha_3.\beta_1-\alpha_3.\beta_2-\alpha_3.\beta_3\right).\pi_E(a),\\
&= \nu_E(a)+\left(\beta_3+\alpha_3.\left(1-\beta_1-\beta_2-\beta_3\right)\right).\pi_E(a).
\end{aligned}$$

7. Conclusions

In this paper, we have established a number of properties and decompositions for picture fuzzy sets. More precisely, the algebraic structure of $\mathbb{D}^*$ was investigated, in particular its properties that we have extensively used in the work. Next, we have defined some picture fuzzy set operations, with respect to the order of $\mathbb{D}^*$ noted $\preceq$ and the two corresponding laws $\curlywedge$ and $\curlyvee$ of $\mathbb{D}^*$. Chunxin Bo and Xiaohong Zhang [6] defined the intersection and the union in almost the same way, and the difference is only in the definitional conditions, which were on the sets for them, while they were on the elements of the sets in our definition. Support, kernel, α-cut, strong α-cut and picture fuzzy line of degree α of a picture fuzzy set have been defined for all $\alpha \in \mathbb{D}^*$ with respect also the order $\preceq$ of $\mathbb{D}^*$, some properties of these notions have been established as well as some decomposition theorems of a picture fuzzy set. At last, we have extended some of Attanassov's modal operators to the picture fuzzy set case, these operators transform every picture fuzzy set into an intuitionistic fuzzy set.

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