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NUMERICAL METHOD BASED ON BOOLE POLYNOMIAL FOR SOLUTION OF GENERAL FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH HYBRID DELAYS

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Abstract. In this paper, the approximate solution of general functional integro differential equaions with hybrid delays is examined using of Boole polynomials and the collocation points. The solution is obtained as a truncated Boole series on a closed interval in the set of real numbers. By using this method, the approximate solutions of the problems are found. In addition, the error functions of the solutions are calculated by using the residual functions. Furthermore, the fundamental properties of the Boole polynomials and their generating functions are studied. Relationships between Boole polynomials and numbers, Stirling numbers and Euler polynomials and numbers are presented.

Keywords: Numerical methods, general functional integro-differential equations,Boole polynomial, the error analysis.

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1. Introduction

Integro-differential equations are used in modelling phenomena in sciences and engineering. The functional integro-differential equations have many applications in areas such as mathematics, engineering, astronomy, biology and economics [9, 27]. In recent years, scientists have examined various applications of these equations and improving the numerical methods for the approximate solutions. The differential equations have been solved by using the numerical methods based on the Bernoulli polynomials [2, 3, 4, 6, 7, 12], the shifted Bernoulli polynomials [5], the Bessel polynomials [21] and the Morgan-Voyce polynomials [22]. Also, for these equations, the Laguerre wavelet collocation method [35] and the Chelyshkov collocation method [23] have been improved. In addition to, the Haar wavelets method [1], the Hybrid Euler-Taylor matrix method [8], a Chebyshev finite difference method [11], the variational Adomian decomposition method [18], the tau method [20], the kernel space method [29], the modified Taylor expansion method [31],

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the Sinc collocation method [46], the Bernoulli matrix-collocation method [14, 15] and the Bernstein polynomials method [19] have been applied for solving the integro-differential equations. The Nystrom method has been used to solve the Fredholm integral equations of the second kind under interval data [25]. A numerical method based on Chelyshkov polynomials has been presented to solve the linear functional integro-differential equations [30]. For the approximate solutions of the pantograph-type Volterra integro-differential equations, a collocation method based on Laguerre polynomials has been improved [44]. The Lucas matrix-collocation technique has been used for the solutions of the functional integro-differential equation with variable delays [17]. The approximate solutions of the delay linear Fredholm integro-differential equations have been gained by a matrix method based on the shifted Legendre polynomials [45]. A matrix method based on the Dickson polynomials has been developed for the numerical solutions of the general integro-differential-difference equations [28]. In this study, the numerical method is improved to obtain the approximate solutions of the general functional integro-differential equations with hybrid delays. The method is based on the Boole polynomials, their derivatives and the collocation points.

For $a \leq x, t \leq b$, the general functional integro-differential equations with hybrid delays is given in the form of

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} p_{kj}(x)y^{(k)}(\alpha_{kj}x+\beta_{kj}) = f(x) + \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{\upsilon_{rs}(x)}^{\nu_{rs}(x)} K_{rs}(x,t)y^{(r)}(\mu_{rs}t+\gamma_{rs})dt, \tag{1}$$

with initial-boundary conditions

$$\sum_{k=0}^{m_1-1}(a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)) = \eta_i, \qquad i=1,2,3,\ldots,m_1-1. \tag{2}$$

where $p_{kj}, K_{rs}, f(x), \nu_{rs}(x), \upsilon_{rs}(x)$ are known function on the $a \leq x, t \leq b$ and $a_{jk}, b_{jk}, \mu_{rs}, \gamma_{rs}, \lambda_{rs}$ are real constants.

The approximate solution is written form as

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n R_n(x) \tag{3}$$

where $R_n(x)$ is Boole polynomials and $a_n, n = 1, 2, \ldots N$ is the unknown Boole coefficients.

2. Boole Polynomials

The special numbers of Boole polynomial form the basis of the developed method in this study. The Euler Polynomials $E_n(x)$ are defined by the following generating function:

$$\frac{2e^{tx}}{1+e^t} = \sum_{n=0}^{\infty}\frac{E_n(x)}{n!}t^n$$

Here, we note that $E_n = E_n(0)$ denotes the Euler numbers. The Stirling numbers of the first kind $S_1(n,k)$ and second kind $S_2(n,k)$ are indicated by the following generating functions, respectively:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty}\frac{S_1(n,k)}{n!}t^n, \quad \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty}\frac{S_2(n,k)}{n!}t^n$$

The general form of Boole polynomials $R_n(x)$ are following

$$R_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^m}{2^m} \binom{x}{n-m}. \tag{4}$$

The generating function of Boole polynomials $R_n(x)$ is:

$$\sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n = \frac{2(1+t)^x}{2+t} . \tag{5}$$

For $x = 0$, generating function of the Boole numbers is

$$\sum_{n=0}^{\infty} \frac{R_n(0)}{n!} t^n = \frac{2}{2+t} . \tag{6}$$

By using (6), we have

$$\sum_{n=0}^{\infty} \frac{R_n(0)}{n!} t^n = \sum_{n=0}^{\infty} (-\frac{1}{2} t)^n.$$

Comparing the coefficients of t^n on both sides of the above equation, the following well-known formula is obtained:

$$\frac{R_n(0)}{n!} = (-\frac{1}{2})^n.$$

Substituting $t = e^t - 1$ into the equation (5), we have

$$\sum_{n=0}^{\infty} \frac{R_n(x)}{n!} (e^t - 1)^n = \frac{2e^{tx}}{1+e^t} .$$

By using equation (5) and equation (6), one also has the following well known formula involving the Stirling numbers of the first kind $S_1(m, n)$ and the Euler numbers and polynomials, and the Boole numbers and polynomials [9, 24, 26, 27, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43]:

$$\sum_{n=0}^{m} E_n(x) S_1(m, n) = R_n(x)$$

and

$$\sum_{n=0}^{m} E_n S_1(m, n) = R_n(0).$$

3. Main Matrix Relations for the Boole Polynomials

In this section, the matrix relation of the Boole polynomial $\mathbf{R}(x)$, the approximate solution of the equation (3), the kernel function $K_{rs}(x)$ and the initial-boundary conditions the equation (2) are obtained.

We firstly introduce the following matrix relation of the Boole polynomial:

$$\mathbf{R}(x) = \mathbf{X}(x)\mathbf{H}^{\mathbf{T}} \tag{7}$$

where

$$\mathbf{R}(x) = \begin{bmatrix} 1 & x - \frac{1}{2} & \dots & R_N(x) \end{bmatrix}, \mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^N \end{bmatrix}$$

and

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -\frac{1}{2} & 1 & 0 & \dots \\ \frac{1}{2} & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, the matrix form of the solution (3) is written as

$$y(x) \cong y_N(x) = \mathbf{R}(x)\mathbf{A} \tag{8}$$

The kth derivative of the equation (8) is given as follows:

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{R}^{(k)}(x)\mathbf{A}, \quad k = 0, 1 \ldots, m_1 \tag{9}$$

Combining the matrix form of the equation (7) with the matrix form the equation (9), we obtain

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{H}^{\mathbf{T}}\mathbf{A} = \mathbf{X}(x)\mathbf{E}^k\mathbf{H}^{\mathbf{T}}\mathbf{A} \tag{10}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}_{1\times(N+1)}$$

[10, 13]. Using the equation (1), $(\alpha_{kj}x + \beta_{kj})$ is written instead of x for the matrix form of term $y^{(k)}(\alpha_{kj}x + \beta_{kj})$. The matrix form is obtained as

$$y^{(k)}(\alpha_{kj}x + \beta_{kj}) \cong y_N^{(k)}(\alpha_{kj}x + \beta_{kj}) = \mathbf{X}(\alpha_{kj}x + \beta_{kj})\mathbf{E}^k\mathbf{H}^{\mathbf{T}}\mathbf{A}, \ k = 0, 1 \ldots, m_1. \tag{11}$$

Here, the matrix relation between $\mathbf{X}(\alpha_{kj}x + \beta_{kj})$ and $\mathbf{X}(x)$ is expressed as

$$\mathbf{X}(\alpha_{kj}x + \beta_{kj}) = \mathbf{X}(x)\mathbf{B}(\alpha_{kj}, \beta_{kj}) \tag{12}$$

where

$$\mathbf{B}(\alpha_{\mathbf{kj}}, \beta_{\mathbf{kj}}) = \begin{bmatrix} \binom{0}{0}\alpha_{kj}^0\beta_{kj}^0 & \binom{1}{0}\alpha_{kj}^0\beta_{kj}^1 & \ldots & \binom{N}{0}\alpha_{kj}^0\beta_{kj}^N \\ 0 & \binom{1}{1}\alpha_{kj}^1\beta_{kj}^0 & \ldots & \binom{N}{1}\alpha_{kj}^1\beta_{kj}^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \binom{N}{N}\alpha_{kj}^N\beta_{kj}^0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

The relation (12) is written in to the equation (11) and the following relation is obtained.

$$y^{(k)}(\alpha_{kj}x + \beta_{kj}) \cong y_N^{(k)}(\alpha_{kj}x + \beta_{kj}) = \mathbf{X}(x)\mathbf{B}(\alpha_{kj}, \beta_{kj})\mathbf{E}^{\mathbf{k}}\mathbf{H}^{\mathbf{T}}\mathbf{A}, \ k = 0, 1 \ldots, m_1 \tag{13}$$

Similarly, using equation (1), we obtain

$$y^{(r)}(\mu_{kj}t + \gamma_{kj}) \cong y_N^{(r)}(\mu_{kj}t + \gamma_{kj}) = \mathbf{X}(t)\mathbf{B}(\mu_{kj}, \gamma_{kj})\mathbf{E}^{\mathbf{r}}\mathbf{H}^{\mathbf{T}}\mathbf{A}, \ k = 0, 1 \ldots, m_3. \tag{14}$$

The kernel function $K_{rs}(x,t)$ is expanded by the Taylor polynomial and the Boole polynomial, respectively,

$${}^tK_{rs}(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} {}^tk_{mn}^{rs}x^m t^n \text{ and } {}^RK_{rs}(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} {}^Rk_{mn}^{rs}R_m(x)R_n(t) \tag{15}$$

where

$${}^tk_{mn}^{rs} = \frac{1}{m!n!}\frac{\partial^{m+n}k_{rs}(0,0)}{\partial x^m \partial t^n} \qquad m, n = 0, 1, 2, \ldots, N.$$

The matrix relations in the series of the equation (15) is given as follows:

$$K_{\mathbf{rs}}(x,t) = \mathbf{X}(x){}^{\mathbf{t}}\mathbf{K}_{rs}\mathbf{X}^{\mathbf{T}}(t), K = [{}^tk_{mn}^{rs}], \qquad m, n = 0, 1, 2, \ldots, N. \tag{16}$$

Thus, by using to the equation (15)

$${}^{\mathbf{R}}\mathbf{K}_{\mathbf{rs}} = (\mathbf{H}^{\mathbf{T}})^{-1}\ {}^{\mathbf{t}}\mathbf{K}_{\mathbf{rs}}\mathbf{H}^{-\mathbf{1}} \Rightarrow {}^{\mathbf{t}}\mathbf{K}_{\mathbf{rs}} = \mathbf{H}^{\mathbf{T}}\ {}^{\mathbf{R}}\mathbf{K}_{\mathbf{rs}}\mathbf{H}. \tag{17}$$

is obtained.

Consequently, combining equation (10) and the equation (2), we get

$$\sum_{k=0}^{m-1}(a_{ik}\mathbf{X}(a)+b_{ik}\mathbf{X}(b))\mathbf{E^kH^TA}=\eta_i, \qquad i=0,1,2,\ldots,m-1. \tag{18}$$

4. Collocation method

In this section, the matrix equations are obtained using the collocation points and the matrix relations presented in Section 3. Firstly, the matrix relations in the equation (13), (14) and (16) are substituted into the equation (1). Therefore, the following matrix equation is obtained:

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}p_{kj}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{E^kH^T} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\mathbf{X}(x)^{\mathbf{t}}\mathbf{K_{rs}Q_{rs}B}(\mu_{kj},\gamma_{kj})\mathbf{E^rH^T}\right\}\mathbf{A}=f(x) \tag{19}$$

where

$$\mathbf{Q_{rs}}(x)=\int_{\upsilon_{rs}(x)}^{\nu_{rs}(x)}\mathbf{X^T}(t)\mathbf{X}(t)\ dt=[q_{mn}^{rs}(x)], r=0,1,\ldots,m_3; s=0,1,\ldots,m_4$$

$$q_{mn}^{rs}(x)=\frac{(\nu_{rs}(x)^{m+n+1}-\upsilon_{rs}(x)^{m+n+1})}{m+n+1}, \qquad m,n=0,1,\ \ldots,N.$$

Lets define the collocation points as follows:

$$x_i=a+\frac{b-a}{N}i, \qquad i=0,1,\ldots,N \tag{20}$$

These points are substituted in the equation (19), the following system of the matrix equation is obtained

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}p_{kj}(x_i)\mathbf{X}(x_i)\mathbf{B}(\alpha_{kj},\beta_{kj})\mathbf{E^kH^T} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\mathbf{X}(x_i)^{\mathbf{t}}\mathbf{K_{rs}Q_{rs}B}(\mu_{kj},\gamma_{kj})\mathbf{E^rH^T}\right\}\mathbf{A}=f(x_i). \tag{21}$$

Hence, fundamental matrix equation can be written as

$$\left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}\mathbf{P_{kj}XB}(\alpha_{kj},\beta_{kj})\mathbf{E^kH^T}-\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\overline{\mathbf{X}}\ \overline{\mathbf{K_{rs}}}\ \overline{\mathbf{Q_{rs}}}\ \overline{\mathbf{B}}(\mu_{kj},\gamma_{kj})\ \overline{\mathbf{E^r}}\ \overline{\mathbf{H^T}}\right\}\mathbf{A}=\mathbf{F} \tag{22}$$

where

$$\mathbf{P_{kj}}=\begin{bmatrix}P_{kj}(x_0) & 0 & \ldots & 0\\ 0 & P_{kj}(x_1) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & P_{kj}(x_N)\end{bmatrix}_{(N+1)\times(N+1)}, \mathbf{F}=\begin{bmatrix}f(x_0)\\ f(x_1)\\ \vdots\\ f(x_N)\end{bmatrix}_{1\times(N+1)},$$

$$\overline{\mathbf{H}^T} = \begin{bmatrix} \mathbf{H^T} \\ \mathbf{H^T} \\ \vdots \\ \mathbf{H^T} \end{bmatrix}_{(N+1)\times(N+1)^2}, \overline{\mathbf{Q_{rs}}} = \begin{bmatrix} \mathbf{Q_{rs}}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{Q_{rs}}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{Q_{rs}}(x_N) \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\overline{\mathbf{E^r}} = \begin{bmatrix} \mathbf{E^r} & 0 & \dots & 0 \\ 0 & \mathbf{E^r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{E^r} \end{bmatrix}_{(N+1)^\times(N+1)r}, \overline{\mathbf{K_{rs}}} = \begin{bmatrix} \mathbf{K_{rs}} & 0 & \dots & 0 \\ 0 & \mathbf{K_{rs}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{K_{rs}} \end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

and

$$\overline{\mathbf{B}}(\mu_{kj}, \gamma_{kj}) = \begin{bmatrix} \mathbf{B}(\mu_{kj}, \gamma_{kj}) & 0 & \dots & 0 \\ 0 & \mathbf{B}(\mu_{kj}, \gamma_{kj}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}(\mu_{kj}, \gamma_{kj}) \end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

The equation (22) is modification of the equation (1). This matrix equation is the matrix equation with a system of $(N+1)$ linear algebraic equations and unknown Boole coefficients $a_0,\ a_1,\ \dots,\ a_N$. Thus, the equation (22) is written as follows:

$$\mathbf{WA} = \mathbf{F} \tag{23}$$

where

$$\mathbf{W} = \sum_{k=0}^{m_1}\sum_{j=0}^{m_2} \mathbf{P_{kj}XB}(\alpha_{kj}, \beta_{kj})\mathbf{E^kH^T} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs}\overline{\mathbf{X}}\ \overline{\mathbf{K_{rs}}}\ \overline{\mathbf{Q_{rs}}}\ \overline{\mathbf{B}}(\mu_{kj}, \gamma_{kj})\ \overline{\mathbf{E^r}}\ \overline{\mathbf{H^T}} \tag{24}$$

The matrix form of conditions of the equation (18) can also be represented as follows:

$$\mathbf{U}_i\mathbf{A} = [\eta_i] \text{ or } [U_i; \eta_i]; \qquad i = 0, 1, 2, \dots, m_1 - 1 \tag{25}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m_1-1} (a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b))\mathbf{E^kH^T} = \begin{bmatrix} u_{i0} & u_{i1} & \dots & u_{iN} \end{bmatrix}, i = 0, 1, \dots, m_1 - 1$$

Finally, the m rows of the augmented matrix form the equation (23) are deleted. The m rows of the conditions the equation (25) are written instead of the deleted m rows. As a result, the following new augmented matrix relation is obtained as

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{F}}] = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \dots & w_{1N} & ; & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{(N-m_1)0} & w_{(N-m_1)1} & \dots & w_{(N-m_1)N} & ; & f(x_{N-m_1}) \\ u_{00} & u_{01} & \dots & u_{0N} & ; & \eta_0 \\ u_{10} & u_{11} & \dots & u_{1N} & ; & \eta_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{(m_1-1)0} & u_{(m_1-1)1} & \dots & u_{(m_1-1)N} & ; & \eta_{m_1-1} \end{bmatrix}. \tag{26}$$

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{F}}] = N + 1$, the unknown Boole polynomials are obtained with $\mathbf{A} = (\widetilde{\mathbf{W}})^{-\mathbf{1}}\widetilde{\mathbf{F}}$.Then, the Boole coefficients obtained are placed in the solution (3). As a result, the solutions of the equation (1) are reached.

5. Residual correction and Error estimation

In this section, the error function of the approximate solution has been given for the presented method. Also, the residual correction of the Boole polynomials solution has been developed. The residual function of the presented method is defined as

$$\Re_N(x) = L[y_N(x)] - f(x) \tag{27}$$

where $y_N(x)$ is the Boole polynomials solution defined by the equation (3) of the problem, is the approximate solution of the problem. The error function $e_N(x)$ is defined as

$$e_N(x) = y(x) - y_N(x) \tag{28}$$

with the exact solution $y(x)$. According to the equation (1), the equation (2), the equation (27) and the equation (28), the error differential equation is obtained as

$$\begin{aligned} &\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} p_{kj}(x) e_N^{(k)}(\alpha_{kj}x+\beta_{kj}) \\ &-\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{\upsilon_{rs}(x)}^{\nu_{rs}(x)} K_{rs}(x,t)e_N^{(k)}(\mu_{rs}t+\gamma_{rs})dt = f(x)+\Re_N(x) \end{aligned} \tag{29}$$

with the homogeneous initial-boundary conditions

$$\sum_{k=0}^{m_1-1}(a_{ik}e_N^{(k)}(a)+b_{ik}e_N^{(k)}(b)) = 0, \qquad i = 1,2,3,\ldots,m_1-1. \tag{30}$$

By using the sum of $y_N(x)$ and $e_{N,M}(x)$, we obtain the corrected the Boole polynomials solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$. In addition, using the Boole error function $e_N(x)$ and the estimated error function $e_{N,M}(x)$, the corrected Boole error function $e_{N,M}(x)$ is given as following

$$E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y_N(x) - y_{N,M}(x).$$

6. Numerical examples

Example 1. For $0 \le x,\ t \le 1$, the equation is given as

$$y''(x) - (x-2)y'(x) + y(2x-\frac{1}{2}) = -2x^2 + \frac{13}{2}x + \frac{5}{2} + \int_{x-1}^{x+1}(x-1)y'(t)dt \tag{31}$$

with initial-boundary conditions $y(0) = 2$ and $y'(0) = -\frac{1}{2}$. The equation (31) is written as

$$\begin{aligned} &y''(x) - (x-2)y'(x) + y(2x-\frac{1}{2}) = -2x^2 + \frac{13}{2}x + \frac{5}{2} \\ &-\int_0^{x-1}(x-1)y'(t)dt + \int_0^{x+1}(x-1)y'(t)dt \end{aligned} \tag{32}$$

where $P_{00} = 1,\ P_{10} = -(x-2),\ P_{20} = 1,\ \alpha_{00} = 2,\ \beta_{00} = -\frac{1}{2}, \lambda_{10} = -1,\ \lambda_{11} = 1,\ \nu_{10}(x) = (x-1),\ \nu_{11}(x) = (x+1),\ \upsilon_{10}(x) = \upsilon_{11}(x) = 0,\ K_{10}(x,t) = K_{11}(x,t) = (x-1)$. For $N = 3$ in interval [0,1], the collocation points are obtained as

$$\Big\{x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = 1,\ x_3 = \frac{2}{3},\ x_4 = 1\Big\}.$$

The fundamental matrix equation of the equation (32) is written as

$$\left\{ \mathbf{P}_{20}\mathbf{X}\mathbf{E}^{\mathbf{2}}\mathbf{H}^{\mathbf{T}} + \mathbf{P_{10}}\mathbf{X}\mathbf{E}\mathbf{H}^{\mathbf{T}} + \mathbf{P_{00}}\mathbf{X}\mathbf{B}(2,-\frac{1}{2})\mathbf{H}^{\mathbf{T}} \right. \\ \left. - \lambda_{10}\overline{\mathbf{X}}\ \overline{K_{\mathbf{10}}}\ \overline{\mathbf{Q_{10}}}\ \overline{\mathbf{E}}\ \overline{\mathbf{H}^{\mathbf{T}}} - \lambda_{11}\overline{\mathbf{X}}\ \overline{\mathbf{K_{11}}}\ \overline{\mathbf{Q_{11}}}\ \overline{\mathbf{E}}\ \overline{\mathbf{H}^{\mathbf{T}}} \right\} \mathbf{A} = \mathbf{F} \tag{33}$$

where

$$\mathbf{P_{00}} = \mathbf{P_{20}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}, \mathbf{P}_{10} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}, \mathbf{H}^{\mathbf{T}} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -\frac{9}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4},$$

$$\overline{\mathbf{B}}(\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}}) = Diag\begin{bmatrix}\mathbf{B}(\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}}) & \mathbf{B}(\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}}) & \mathbf{B}(\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}}) & \mathbf{B}(\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}})\end{bmatrix}_{16\times 16},$$

$$\overline{\mathbf{Q_{10}}} = Diag\begin{bmatrix}\mathbf{Q_{10}}(0) & \mathbf{Q_{10}}(\frac{\mathbf{1}}{\mathbf{3}}) & \mathbf{Q_{10}}(\frac{\mathbf{2}}{\mathbf{3}}) & \mathbf{Q_{10}}(\mathbf{1})\end{bmatrix}_{16\times 16},$$

$$\overline{\mathbf{Q_{11}}} = Diag\begin{bmatrix}\mathbf{Q_{11}}(0) & \mathbf{Q_{11}}(\frac{\mathbf{1}}{\mathbf{3}}) & \mathbf{Q_{11}}(\frac{\mathbf{2}}{\mathbf{3}}) & \mathbf{Q_{11}}(\mathbf{1})\end{bmatrix}_{16\times 16},$$

$$\overline{\mathbf{X}} = Diag\begin{bmatrix}\mathbf{X}(0) & \mathbf{X}(\frac{\mathbf{1}}{\mathbf{3}}) & \mathbf{X}(\frac{\mathbf{2}}{\mathbf{3}}) & \mathbf{X}(\mathbf{1})\end{bmatrix}_{4\times 16}, \overline{\mathbf{H}^{\mathbf{T}}} = Diag\begin{bmatrix}\mathbf{H}^{\mathbf{T}} & \mathbf{H}^{\mathbf{T}} & \mathbf{H}^{\mathbf{T}} & \mathbf{H}^{\mathbf{T}}\end{bmatrix}_{16\times 4},$$

$$\overline{\mathbf{K_{10}}} = Diag\begin{bmatrix}\mathbf{K_{10}} & \mathbf{K_{10}} & \mathbf{K_{10}} & \mathbf{K_{10}}\end{bmatrix}_{16\times 16}, \overline{\mathbf{K_{11}}} = Diag\begin{bmatrix}\mathbf{K_{11}} & \mathbf{K_{11}} & \mathbf{K_{11}} & \mathbf{K_{11}}\end{bmatrix}_{16\times 16},$$

$$\overline{\mathbf{E}} = Diag\begin{bmatrix}\mathbf{E} & \mathbf{E} & \mathbf{E} & \mathbf{E}\end{bmatrix}_{16\times 16}$$

From solution of the equation (32)

$$[\mathbf{W};\mathbf{F}] = \begin{bmatrix} 1 & 3 & -\frac{17}{4} & \frac{17}{2} & ; & \frac{5}{2} \\ 1 & \frac{8}{3} & -\frac{65}{36} & \frac{35}{27} & ; & \frac{40}{9} \\ 1 & \frac{7}{3} & \frac{7}{36} & -\frac{151}{54} & ; & \frac{107}{18} \\ 1 & 2 & \frac{7}{4} & -4 & ; & 7 \end{bmatrix}$$

According to the problem (31), the matrix form of conditions (25)is written as

$$[\mathbf{U_0};\lambda_\mathbf{0}] = \begin{bmatrix}1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & ;2\end{bmatrix} \text{ and} [\mathbf{U_1};\lambda_\mathbf{1}] = \begin{bmatrix}0 & 1 & -2 & 5 & ;-\frac{1}{2}\end{bmatrix}$$

In the augmented matrix, the 3^{rd} and 4^{th} rows are deleted and the matrix form of conditions are written. As a result, the following matrix is gained.

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}] = \begin{bmatrix} 1 & 3 & -\frac{17}{4} & \frac{17}{2} & ; & \frac{5}{2} \\ 1 & \frac{8}{3} & -\frac{65}{36} & \frac{35}{27} & ; & \frac{40}{9} \\ 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & ; & 2 \\ 0 & 1 & -2 & 5 & ; & -\frac{1}{2} \end{bmatrix}$$

This matrix is solved and the unknown Boole coefficients (or the Boole numbers) are found as

$$\mathbf{A} = \begin{bmatrix}\frac{9}{4} & \frac{3}{2} & 1 & 0\end{bmatrix}^T.$$

For $N = 3$, the Boole coefficients obtained are written in the solution (3). The Boole polynomials solution of the equation (31) is obtained as follows

$$y(x) = x^2 - \frac{1}{2}x + 2.$$

This is the exact solution of the given problem (31).

Example 2. The following problem

$$y'(x) = y(x-1) + \int_{x-1}^{x} y(t)dt \tag{34}$$

is given with the initial-boundary conditions $y(0) = 1$ [32, 44]. The exact solution of this equation is $y(x) = e^x$. The exact solution $y(x)$, the Boole solutions $y_N(x)$ and the corrected Boole solutions $y_{N,M}(x)$ of the problem (34) are calculated for the values $N, M = 4, 5$ and $N, M = 8, 9$. These results are shown in Figure 1 for $N, M = 8, 9$. Also, the absolute error function $|e_N|$, the estimated error function $|e_{N,M}|$ and the corrected Boole error function $|E_{N,M}|$ of the problem (34) are obtained for the values $N, M = 4, 5$ and $N, M = 8, 9$. Additionally, the authors in reference [44] are solved the problem (34) using the Laguerre collocation method (LCM). The absolute error functions, the estimated error functions and the corrected error functions of the problem (34) are compared with the presented method and the Laguerre collocation method (LCM), in Table 1 and 2.

Table 1. The comparison of the absolute error functions, the estimated error functions and the corrected error functions of the presented method and the LCM for problem (34).

	Presented Method			Laguerre Collocation Method [44]		
x_i	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$
	$N = 4$	$N, M = 4, 5$	$N, M = 4, 5$	$N = 4$	$N, M = 4, 5$	$N, M = 4, 5$
0	0	0	0	2.4425e-015	1.2212e-014	9.7700e-015
0.2	5.6620e-04	7.2149e-04	1.5529e-04	4.1670e-004	5.5366e-004	1.3696e-004
0.4	1.3027e-03	1.5284e-03	2.2572e-04	1.4750e-003	1.8183e-003	3.4334e-004
0.6	1.7643e-03	1.9855e-03	2.2121e-04	2.4747e-003	2.8991e-003	4.2442e-004
0.8	1.9854e-03	2.1937e-03	2.0835e-04	2.9179e-003	3.2314e-003	3.1354e-004
1.0	3.0124e-03	3.1842e-03	1.7186e-04	3.0410e-003	3.0791e-003	3.8131e-005

Table 2. The comparison of the absolute error functions, the estimated error functions and the corrected error functions of the presented method and the LCM for problem (34).

	Presented Method			Laguerre Collocation Method [44]		
x_i	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$
	$N = 8$	$N, M = 8, 9$	$N, M = 8, 9$	$N = 8$	$N, M = 8, 9$	$N, M = 8, 9$
0	0	0	0	3.1308e-014	1.3840e-012	1.3527e-012
0.2	1.0885e-06	9.7953e-07	1.0902e-07	1.5829e-006	2.4889e-006	9.0600e-007
0.4	1.2423e-06	8.9501e-07	3.4727e-07	2.1689e-006	3.3111e-006	1.1422e-006
0.6	8.4992e-07	2.9388e-07	5.5604e-07	2.0004e-006	2.8536e-006	8.5320e-007
0.8	4.9047e-07	1.9043e-07	6.8090e-07	1.7385e-006	2.1082e-006	3.6970e-007
1.0	4.6580e-07	2.9577e-07	7.6157e-07	1.8455e-006	1.7977e-006	4.7802e-008

Example 3. For $0 \leq x, t \leq 1$, the first order integro-differential equations is given as

$$y'(x) = y(x) - 2y'(x - \frac{1}{2}) + (x - x^2)y(\frac{1}{2}x - 1) + \int_0^x xe^{-t}y(t)dt + \int_0^{\frac{x}{2}} (x^2 - 2t - 2)y'(t)dt + f(x), \quad (35)$$

with initial-boundary conditions $y(0) = y'(0) = 1$ [14, 44]. Here, the exact solution of this equation is $y(x) = e^x$ and $g(x) = -(x-x^2)e^{\frac{x}{2}-1} + 2e^{x-\frac{1}{2}} - x^2e^{\frac{x}{2}} + xe^{\frac{x}{2}}$. For $N, M = 7, 8$, the exact solution $y(x)$, the Boole solutions $y_N(x)$ and the corrected Boole solutions $y_{N,M}(x)$ of the problem (35) are calculated and compared in Figure 2. The error functions $|e_N|$, the estimated error functions $|e_{N,M}|$ and the corrected Boole error functions $|E_{N,M}|$ of the problem (35) are obtained for the values $N, M = 7, 8$. The Bernouli collocation method (BCM) are used by authors in [14] to solve problem (35) for $N, M = 7, 8$. In Table 3, the error functions $|e_N|$, the estimated error functions $|e_{N,M}|$ and the corrected error functions $|E_{N,M}|$ of the problem (35) are compared with the presented method and the Bernoulli collocation method (BCM).

TABLE 3. The comparison of the absolute error functions, the estimated error functions and the corrected error functions of the presented method and the BCM for problem (35).

	Presented Method			Bernoulli Collocation Method [14]		
x_i	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$	$\|e_N\|$	$\|e_{N,M}\|$	$\|E_{N,M}\|$
	$N = 7$	$N, M = 7, 8$	$N, M = 7, 8$	$N = 7$	$N, M = 7, 8$	$N, M = 7, 8$
0	0	0	0	0	0	0
0.2	7.4141e-07	7.5814e-07	1.6721e-08	7.4141e-07	7.5814e-07	1.6721e-08
0.4	1.3555e-06	1.2028e-06	1.5276e-07	1.3555e-06	1.2028e-06	1.5276e-07
0.6	4.1432e-07	5.4210e-08	3.6011e-07	4.1432e-06	5.4210e-08	3.6011e-07
0.8	1.9036e-06	2.1524e-06	2.4881e-07	1.9036e-06	2.1524e-06	2.4881e-07
1.0	3.3473e-06	3.0698e-06	2.7749e-07	3.3473e-06	3.0698e-06	2.7749e-07

Example 4. In this example, the Volterra delay integro-differential equation

$$y'(x) = -(6 + \sin(x))y(x) + y(x - \frac{\pi}{4}) - \int_{x-\frac{\pi}{4}}^{x} \sin(t)y(t)dt, \quad x \geq 0 \tag{36}$$

is considered with initial-boundary conditions $y(0) = e$ [16, 32]. Here, the exact solution of this equation is $y(x) = e^{\cos(x)}$. For $N, M = 13, 14$, the exact solution $y(x)$, the Boole solutions $y_N(x)$ and the corrected Boole solutions $y_{N,M}(x)$ of the problem (36) are calculated. These results are compared in the Figure 3. Also,the Taylor collocation method (TCM) are used by the authors in reference [16] to solve the problem (36). The absolute error functions $|e_N|$ and the estimated error functions $|E_{N,M}|$ of the problem (36) have been calculated by the presented method for $N, M = 4, 5$, $N, M = 9, 10$ and $N, M = 13, 14$. The values of the absolute error functions and the estimated error functions are compared with the Taylor collocation method (TCM) in Table 4 and Table 5, respectively.

TABLE 4. The comparison of the absolute error functions of the presented method and the TCM for problem (36).

	Presented Method			Taylor Collocation Method [16]		
x_i	$\|e_4\|$	$\|e_9\|$	$\|e_{13}\|$	$\|e_4\|$	$\|e_9\|$	$\|e_{13}\|$
0	0	0	7.1054e-15	0	0	0
0.2	5.2753e-04	4.0774e-07	9.1747e-06	0.52753e-3	0.40761e-6	0.89207e-5
0.4	6.2596e-04	3.4248e-05	3.6366e-06	0.62596e-3	0.34247e-4	0.39875e-5
0.6	4.9358e-04	3.1869e-05	8.3322e-06	0.49358e-3	0.31869e-4	0.10439e-5
0.8	1.9473e-04	1.1447e-05	3.6703e-06	0.19473e-3	0.11447e-4	0.10260e-5
1.0	1.2256e-02	1.1688e-06	2.1717e-07	0.12256e-1	0.11688e-5	0.50482e-6

TABLE 5. The comparison of the estimated error functions of the presented method and the TCM for problem (36).

	Presented Method			Taylor Collocation Method [16]		
x_i	$\lvert e_{4,5}\rvert$	$\lvert e_{9,10}\rvert$	$\lvert e_{13,14}\rvert$	$\lvert e_{4,5}\rvert$	$\lvert e_{9,10}\rvert$	$\lvert e_{13,14}\rvert$
0	1.0842e-19	1.3010e-18	1.6653e-16	0	0	0
0.2	6.3375e-04	8.9251e-06	6.7174e-07	0.19423e-6	0.25852e-10	0.30750e-15
0.4	1.6206e-03	1.3963e-06	1.7431e-07	0.11481e-6	0.32028e-10	0.38264e-14
0.6	7.5441e-04	6.0725e-06	5.2337e-07	0.32789e-7	0.16568e-9	0.16943e-13
0.8	2.9467e-04	2.8320e-06	2.3742e-07	0.53322e-6	0.73745e-9	0.92470e-13
1.0	1.0692e-02	7.9573e-07	1.6906e-08	0.23872e-5	0.22349e-8	0.36194e-12

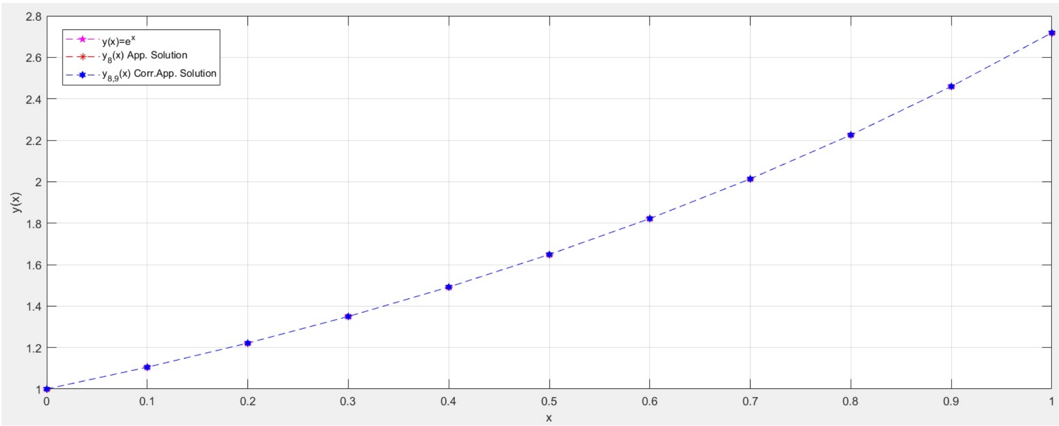

FIGURE 1. The comparison of the exact solutions, the Boole solutions and the corrected Boole solutions of the problem (34) for the values $N, M = 8, 9$.

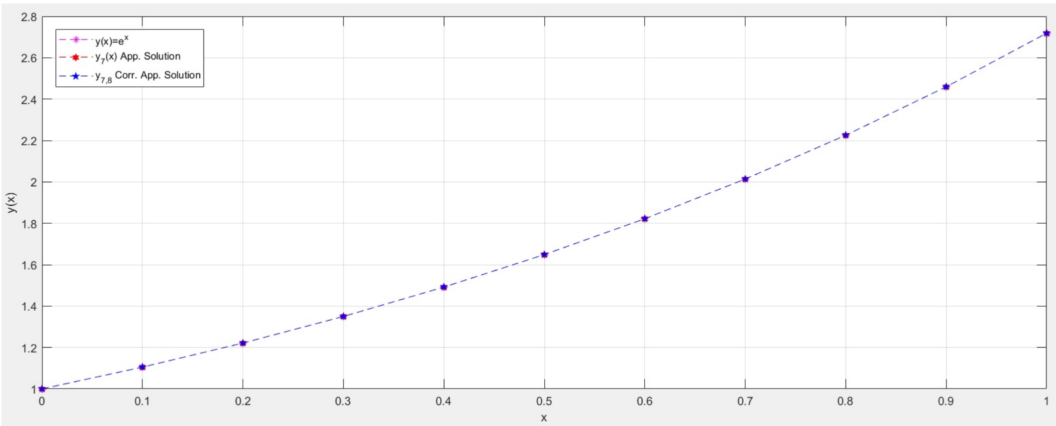

FIGURE 2. The comparison of the exact solutions, the Boole solutions and the corrected Boole solutions of the problem (35) for the values $N, M = 7, 8$.

7. CONCLUSIONS

The Boole matrix method has been developed to find Boole solution of the general functional integro-differential equations with hybrid delay. This method has been used to obtain the approximate solutions and the error estimations based on residual function of

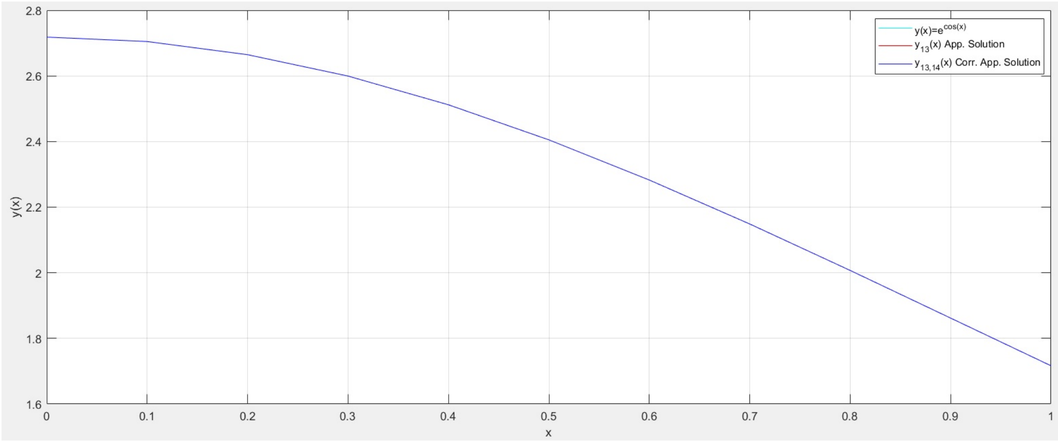

FIGURE 3. The comparison of the exact solutions, the Boole solutions and the corrected Boole solutions of the problem (36) for the values $N, M = 13, 14$.

the problems. The results show that the presented method is both usable and reliable. The presented method has been written in MATLAB program code. In this way, the results have been easily obtained. The presented method was used to obtain the Boole solutions and the error functions of the problem (34) for the values $N, M = 4, 5$ and $N, M = 8, 9$. Additionally, the results of the error functions have been compared with the Laguerre collocation method. For the values $N, M = 7, 8$, the presented method was used to calculate the error functions and the Boole solutions for problem (35). These results were compared with the Bernoulli collocation method. The Boole solutions and error functions of problem (36) was solved for values $N, M = 4, 5$, $N, M = 9, 10$ and $N, M = 13, 14$, using the presented method. The results of the error functions were compared with those obtained using the Taylor collocation method. In future studies, the Boole matrix method can be improved for the approximate solutions the system of integro differential equations, nonlinear integro differential equations, integro-differential–difference equations or different models of these equations.

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