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SOME FIXED POINT THEOREMS UNDER CONTRACTION CONDITIONS USING LINEAR AND RATIONAL EXPRESSION IN DISLOCATED METRIC SPACE

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Abstract. In this paper, we have established some fixed point results in dislocated metric space. These results are proved for different types of contraction conditions using linear and rational expression for single as well as for pair of mappings which generalize some of the well known fixed point results of the literature available in fixed point theory. For pair of mappings we have established common fixed point result. Suitable examples for established results are also given. Few more related results are derived in the end.

Keywords: Fixed Point, Complete Metric Space, Dislocated Metric Space, Convergent Sequence, Contraction Mapping.

AMS Subject Classification: Primary 47H10, 54H25.

1. Introduction

Fixed point theory is one of the most dynamic and evolving research area in nonlinear analysis. Fixed point theory is related to existence and uniqueness of fixed point. In this field, the first important result was given by Banach in 1922 [3] for a contraction mapping in a complete metric space. It is well known as a Banach fixed point theorem. Kannan [15] established a fixed point theorem for new types of discontinuous contraction mappings known as Kannan mappings in a complete metric space. Further, Chatterjea [4] proved a result for discontinuous mapping which is a kind of dual of Kannan mapping. Dass and Gupta [6] generalized the Banach contraction principle for some rational type contraction conditions in metric space. Ciric [5] gave a generalization of Banach contraction principle in metric space. Hitzler and Seda [7] introduced the concept of dislocated metric space

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in the year 2000 and generalized the Banach contraction principle in such spaces. In dislocated metric space distance of a point from itself is not necessarily equal to zero. This concept was not new as it was studied in the context of domain theory [17] under the name metric domains.
Zeyada et al. [25] introduced the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda in dislocated quasi-metric space. The study in such spaces is followed by Isufati [12], Aage and Salunke [1], Rahman and Sarwar [19], Shrivastava et al. [20]. Recently many authors have given several results and theories related to fixed point theory in different metric spaces such as complex valued dislocated metric space [18], fuzzy metric space [11, 22], neutrosophic metric space [2, 10], modular metric space [23] and so on by using different contractions and mappins.
In the current research paper, we establish some fixed point results for single and a pair of continuous self-mapping in the context of dislocated metric space.

2. Preliminaries

The intention of this section is to introduce the basic concepts and definitions used through out this research article which will further help to understand next section. This section includes example along with the definitions.

Definition 2.1.[25] Let K be a nonempty set and let $d : K \times K \to [0, +\infty)$ be a function satisfying the following conditions:

(d_1) $d(m, m) = 0$;
(d_2) $d(m, n) = d(n, m) = 0$ implies $m = n$;
(d_3) $d(m, n) = d(n, m)$ for all $m, n \in K$;
(d_3) $d(m, n) \leq d(m, l) + d(l, n)$ for all $l, m, n \in K$.

If d satisfies the conditions from (d_1) to (d_4) then it is called a metric on K. If it satisfies the conditions (d_1), (d_2) and (d_4), it is called a quasi-metric on K. If d satisfies conditions (d_2) to (d_4) then it is called a dislocated metric on K (or simply $d - metric$).

Example 2.1.[19] Let $K = [0, \infty)$ and define a function $d : K \times K \to [0, +\infty)$ by

$$d(a_1, a_2) = max\{a_1, a_2\}, for\, all\, a_1, a_2 \in K.$$

d is a dislocated metric on K but not a metric on K.

Definition 2.2.[9] Let $\{a_j\}$ be a sequence in dislocated metric space (K, d) then

(1) $\{a_j\}$ is known as convergent to $a \in K$ if $\lim_{j\to+\infty} d(a_j, a) = 0$.
In this case a is called d-limit of $\{a_j\}$ and we write $\lim_{j\to+\infty} a_j = a$ or $a_j \to a$.

(2) $\{a_j\}$ is known as Cauchy sequence in (K, d) if for given $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, $d(a_m, a_n) < \epsilon$ that is, $\lim_{m,n\to+\infty} d(a_m, a_n) = 0$.

(3) A dislocated metric space (K, d) is called complete if every Cauchy sequence of points in K converges to a point in K.

Lemma 2.1. *Let $\{x_j\}$ be a sequence in a dislocated metric space (K, d) such that*

$$d(x_j, x_{j+1}) \leq hd(x_{j-1}, x_j),$$

where, $0 \leq h < 1$ and $j = 1, 2, 3, \cdots$.
Then $\{x_j\}$ is a Cauchy sequence in (K, d).

Proof. Let $i > j \geq 1$, it follows that

$$\begin{aligned} d(x_j, x_i) &\leq d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) + \cdots\cdots + d(x_{i-1}, x_i) \\ &\leq (h^j + h^{j+1} + \cdots + h^{i-1}) d(x_0, x_1) \\ &= h^j (1 + h + h^2 + \cdots + h^{i-j-1}) d(x_0, x_1) \\ &\leq \frac{h^j}{1-h} d(x_0, x_1). \end{aligned}$$

Since $0 \leq h < 1, h^j \to 0$ as $j \to +\infty$.
Therefore, $\{x_j\}$ is a Cauchy sequence in (K, d).
Also $d(x_0, x_1) = 0$ then $d(x_j, x_i) = 0$ for all $i > j$.
Therefore $\{x_j\}$ is a Cauchy sequence in (K, d). □

Definition 2.3.[17] Let (K, d) be a dislocated metric space. A function $f : K \to K$ is called a contraction if there exists $0 \leq \lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y), \; for\, all\, x, y \in K.$$

Definition 2.4.[5] Let (K, d) be a metric space, a self mapping $T : K \to K$ is called a generalized contraction if and only if for every $x, y \in K$, there exist c_1, c_2, c_3, c_4 such that $sup\{c_1 + c_2 + c_3 + 2c_4 : x, y \in K\} < 1$ and

$$d(Tx, Ty) \leq c_1 d(x, y) + c_2 d(x, Tx) + c_3 d(y, Ty) + c_4[d(x, Ty) + d(y, Tx)]. \tag{2.1}$$

Ciric established a unique fixed point theorem for a mapping which satisfies condition (2.1) in metric spaces.

3. Main Results

In this section, we establish some fixed point theorems with examples for single and pair of continuous self mappings under different contraction conditions using linear and rational expression in the context of dislocated metric spaces, which generalizes the results of Ciric .

Theorem 3.1. *Let (K, d) be a complete dislocated metric space and $T : K \to K$ be a continuous function satisfying the following condition:*

$$\begin{aligned} d(Tl, Tm) \leq \alpha_1 d(l, m) + \alpha_2[d(l, Tl) + d(m, Tm)] + \alpha_3[d(l, Tm) + d(m, Tl)] \\ + \alpha_4 \left[\frac{d(l, m) d(l, Tm)}{d(l, m) + d(m, Tm)}\right] + \alpha_5 \left[\frac{d(l, Tm) d(m, Tm)}{d(l, m) + d(m, Tm)}\right] \end{aligned} \tag{3.1}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$ with $\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$ for all $l, m \in K$, then T has a unique fixed point.

Proof. Let $x_0 \in K$ be arbitrary, define the iterative sequence $\{x_n\}$ in K by:
$x_1 = T(x_0), x_2 = T(x_1), \cdots, x_{n+1} = T(x_n), \cdots,$
if for any n, $x_{n+1} = x_n$, then x_n is a fixed point. Therefore, there is no need to go further.

Assert $x_{n+1} \neq x_n$ for any n. Now using (3.1) we have

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \alpha_1 d(x_n, x_{n+1}) + \alpha_2[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\
&+ \alpha_3[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] + \alpha_4 \left[\frac{d(x_n, x_{n+1})d(x_n, Tx_{n+1})}{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})}\right] \\
&+ \alpha_5 \left[\frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_{n+1})}{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})}\right] \\
&\leq \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) \\
&+ \alpha_3[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})] \\
&+ \alpha_4 d(x_n, x_{n+1}) + \alpha_5 d(x_{n+1}, x_{n+2}) \\
d(x_{n+1}, x_{n+2}) &\leq \frac{[\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4]}{[1 - (\alpha_2 + 2\alpha_3 + \alpha_5)]} d(x_n, x_{n+1}).
\end{aligned}$$

Let $h = \frac{[\alpha_1+\alpha_2+2\alpha_3+\alpha_4]}{[1-(\alpha_2+2\alpha_3+\alpha_5)]}$
Note that $0 \leq h < 1$ since $\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$.
Therefore, $d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1})$.
Similarly, $d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n)$, so $d(x_{n+1}, x_{n+2}) \leq h^2 d(x_{n-1}, x_n)$.
In this way, we get $d(x_{n+1}, x_{n+2}) \leq h^{n+1} d(x_0, x_1)$.
Since $0 \leq h < 1 \Rightarrow h^{n+1} \to 0$ as $n \to +\infty$.

By Lemma 2.1, $\{x_n\}$ is a Cauchy sequence in CDMS K. So there exists $\mu \in K$ such that

$$x_n \to \mu \, as \, n \to +\infty.$$

Now we will prove μ is a fixed point of T.
As T is continuous, so $\lim_{n\to+\infty} Tx_n = T\mu$
$\Rightarrow \lim_{n\to+\infty} x_{n+1} = T\mu$
Thus, $T\mu = \mu$.
Hence μ is a fixed point of T.

Uniqueness: If $\mu \in K$ is a fixed point of T then by condition (3.1), we have

$$\begin{aligned}
d(\mu, \mu) &= d(T\mu, T\mu) \\
&\leq \alpha_1 d(\mu, \mu) + \alpha_2[d(\mu, \mu) + d(\mu, \mu)] + \alpha_3[d(\mu, \mu) + d(\mu, \mu)] \\
&+ \alpha_4 \left[\frac{d(\mu, \mu)d(\mu, \mu)}{d(\mu, \mu) + d(\mu, \mu)}\right] + \alpha_5 \left[\frac{d(\mu, \mu)d(\mu, \mu)}{d(\mu, \mu) + d(\mu, \mu)}\right] \\
d(\mu, \mu) &\leq (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \frac{\alpha_4}{2} + \frac{\alpha_5}{2}) d(\mu, \mu)
\end{aligned}$$

Since $0 \leq (\alpha_1 + 2\alpha_2 + 2\alpha_3 + \frac{\alpha_4}{2} + \frac{\alpha_5}{2}) < 1$ and $d(\mu, \mu) \geq 0$, we have, $d(\mu, \mu) = 0$.
Thus, $d(\mu, \mu) = 0$, if μ is a fixed point of T.
Now let μ and ρ be two fixed points of $T(\mu \neq \rho)$, that is, $\mu = T\mu$ and $\rho = T\rho$.

We have,

$$\begin{aligned}
d(\mu,\rho) &= d(T\mu,T\rho)\\
&\leq \alpha_1 d(\mu,\rho)+\alpha_2[d(\mu,T\mu)+d(\rho,T\rho)]+\alpha_3[d(\mu,T\rho)+d(\rho,T\mu)]\\
&+\alpha_4\left[\frac{d(\mu,\rho)d(\mu,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)}\right]+\alpha_5\left[\frac{d(\mu,T\rho)d(\rho,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)}\right]\\
d(\mu,\rho) &\leq [\alpha_1+2\alpha_3+\alpha_4]d(\mu,\rho),\ \textit{since}\, d(\mu,\mu)=d(\rho,\rho)=0.
\end{aligned}$$

which gives $d(\mu,\rho)=0$, since $0\leq \alpha_1+2\alpha_3+\alpha_4<1$ and $d(\mu,\rho)\geq 0$. Similarly $d(\rho,\mu)=0$. Hence $\mu=\rho$ and T has a unique fixed point.

Example 3.1. Let $K=[0,1]$ with a complete dislocated metric defined by $d(l,m)=l+m$ for all $l,m\in K$. Define the continuous self mapping $T:K\to K$ by $Tl=\frac{l}{2}\, for\, all\, l\in K$. Let $\alpha_1=\frac{2}{5},\alpha_2=\frac{1}{20},\alpha_3=\frac{1}{12},\alpha_4=\frac{1}{15},\alpha_5=\frac{1}{20}$.
Then T satisfies all the conditions of Theorem 3.1, as $\alpha_1+2\alpha_2+4\alpha_3+\alpha_4+\alpha_5=\frac{19}{20}<1$ and $d(Tl,Tm)=d(\frac{l}{2},\frac{m}{2})=\frac{l+m}{2}\leq\frac{6}{10}(l+m)+\frac{(l+\frac{m}{2})(4l+\frac{17m}{2})}{60(l+\frac{5m}{2})}$. Thus T has a unique fixed point in K which is $\mu=0$.

Theorem 3.2. *Let (K,d) be a complete dislocated metric space and $T:K\to K$ be a continuous function satisfying the following condition:*

$$\begin{aligned}
d(Tl,Tm) &\leq \alpha_1 d(l,m)+\alpha_2[d(l,Tl)+d(m,Tm)]\left[\frac{d(l,Tm)}{d(l,m)+d(m,Tm)}\right]\\
&+\alpha_3[d(l,Tm)+d(m,Tl)]\left[\frac{d(l,Tm)}{d(l,m)+d(m,Tm)+d(l,Tm)}\right],
\end{aligned}\tag{3.2}$$

where $\alpha_1,\alpha_2,\alpha_3\geq 0$ with $\alpha_1+2\alpha_2+4\alpha_3<1$ for all $l,m\in K$. Then T has a unique fixed point.

*Proof.*Let $x_0\in K$ be arbitrary, define the iterative sequence $\{x_n\}$ in K by:
$x_1=T(x_0),x_2=T(x_1),\cdots,x_{n+1}=T(x_n),\cdots,$
if for any n, $x_{n+1}=x_n$, then x_n is a fixed point. Therefore, there is no need to go further.
Assert $x_{n+1}\neq x_n$ for any n.
Now using (3.2), we have

$$\begin{aligned}
d(x_{n+1},x_{n+2}) &= d(Tx_n,Tx_{n+1})\\
&\leq \alpha_1 d(x_n,x_{n+1})+\alpha_2[d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})]\\
&\left[\frac{d(x_n,Tx_{n+1})}{d(x_n,x_{n+1})+d(x_{n+1},Tx_{n+1})}\right]+\alpha_3[d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)]\\
&\left[\frac{d(x_n,Tx_{n+1})}{d(x_n,x_{n+1})+d(x_{n+1},Tx_{n+1})+d(x_n,Tx_{n+1})}\right]\\
d(x_{n+1},x_{n+2}) &\leq \alpha_1 d(x_n,x_{n+1})+\alpha_2[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]\\
&+\alpha_3[d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})]\\
&\leq \alpha_1 d(x_n,x_{n+1})+\alpha_2 d(x_n,x_{n+1})+\alpha_2 d(x_{n+1},x_{n+2})+\alpha_3 d(x_n,x_{n+1})\\
&+\alpha_3 d(x_{n+1},x_{n+2})+\alpha_3 d(x_{n+1},x_{n+2})+\alpha_3 d(x_n,x_{n+1})\\
d(x_{n+1},x_{n+2}) &\leq \frac{[\alpha_1+\alpha_2+2\alpha_3]}{[1-(\alpha_2+2\alpha_3)]}d(x_n,x_{n+1}).
\end{aligned}$$

Let $h = \frac{[\alpha_1+\alpha_2+2\alpha_3]}{[1-(\alpha_2+2\alpha_3)]}$
Observe that $0 \leq h < 1$, since $\alpha_1 + 2\alpha_2 + 4\alpha_3 < 1$.
Therefore, $d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1})$.
Similarly, $d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n)$, so we get $d(x_{n+1}, x_{n+2}) \leq h^2 d(x_{n-1}, x_n)$.
Proceeding like this, we get $d(x_{n+1}, x_{n+2}) \leq h^{n+1} d(x_0, x_1)$.
where $0 \leq h < 1 \Rightarrow h^{n+1} \to 0$ as $n \to +\infty$.
By Lemma 2.1, $\{x_n\}$ is a Cauchy sequence in CDMS K. So there exists $\mu \in K$ such that $x_n \to \mu$.
Now we will prove μ is a fixed point of T.
Since T is continuous, so $\lim_{n\to+\infty} Tx_n = T\mu$
$\Rightarrow \lim_{n\to+\infty} x_{n+1} = T\mu$
Thus, $T\mu = \mu$.
Hence μ is a fixed point of T.

Uniqueness: If $\mu \in K$ is a fixed point of T then by condition (3.2), we have

$$\begin{aligned} d(\mu,\mu) &= d(T\mu, T\mu) \\ &\leq \alpha_1 d(\mu,\mu) + \alpha_2[d(\mu,T\mu) + d(\mu,T\mu)]\left[\frac{d(\mu,T\mu)}{d(\mu,\mu)+d(\mu,T\mu)}\right] \\ &+ \alpha_3[d(\mu,T\mu)+d(\mu,T\mu)]\left[\frac{d(\mu,T\mu)}{d(\mu,\mu)+d(\mu,T\mu)+d(\mu,T\mu)}\right] \\ d(\mu,\mu) &\leq \left(\alpha_1+\alpha_2+\frac{2}{3}\alpha_3\right) d(\mu,\mu). \end{aligned}$$

Since $0 \leq (\alpha_1 + \alpha_2 + \frac{2}{3}\alpha_3) < 1$ and $d(\mu,\mu) \geq 0$, we have, $d(\mu,\mu) = 0$.
Thus, $d(\mu,\mu) = 0$, if μ is a fixed point of T.
Suppose that μ and ρ are two fixed points of $T(\mu \neq \rho)$, that is, $\mu = T\mu$ and $\rho = T\rho$. We have,

$$\begin{aligned} d(\mu,\rho) &= d(T\mu, T\rho) \\ &\leq \alpha_1 d(\mu,\rho) + \alpha_2[d(\mu,T\mu) + d(\rho,T\rho)]\left[\frac{d(\mu,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)}\right] \\ &+ \alpha_3[d(\mu,T\rho)+d(\rho,T\mu)]\left[\frac{d(\mu,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)+d(\mu,T\rho)}\right] \end{aligned}$$

$$d(\mu,\rho) \quad \leq [\alpha_1 + \alpha_3] d(\mu,\rho), since\, d(\mu,\mu) = d(\rho,\rho) = 0$$

which gives $d(\mu,\rho) = 0$, since $0 \leq \alpha_1 + \alpha_3 < 1$ and $d(\mu,\rho) \geq 0$. Similarly $d(\rho,\mu) = 0$.
Hence $\mu = \rho$ and T has a unique fixed point.

Example 3.2. Let $K = [0,1]$ with complete dislocated metric defined by $d(l,m) = l + m$ for all $l, m \in K$. Define the continuous self mapping $T : K \to K$ s.t. $Tl = \frac{l}{2}$. Let $\alpha_1 = \frac{11}{20}, \alpha_2 = \frac{1}{20}, \alpha_3 = \frac{1}{12}$. Then T satisfies all the conditions of Theorem 3.2, and $\mu = 0$ is the unique fixed point of T in K.

Theorem 3.3. *Let (K, d) be a complete dislocated metric space. Let S and T be two continuous self mapping $S, T : K \to K$ satisfying:*

$$\begin{aligned} d(Sl, Tm) \leq \alpha_1 d(l, m) + \alpha_2[d(l, Sl) + d(m, Tm)]\left[\frac{d(l, Tm)}{d(l, m) + d(m, Tm)}\right] \\ + \alpha_3[d(l, Tm) + d(m, Sl)]\left[\frac{d(l, Tm)}{d(l, m) + d(m, Tm) + d(l, Sl)}\right], \end{aligned} \tag{3.3}$$

where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + 2\alpha_2 + 4\alpha_3 < 1$ for all $l, m \in K$. Then S, T have a unique common fixed point.

Proof. Let $x_0 \in K$ be arbitrary, define the iterative sequence $\{x_n\}$ in K as follows: $x_1 = S(x_0)$, $x_2 = T(x_1)$, $x_3 = S(x_2), \cdots$, $x_{2n} = T(x_{2n-1})$, $x_{2n+1} = S(x_{2n}), \cdots$ for all $n \in N$

We will show that $\{x_n\}$ is a Cauchy sequence in K. From (3.3) we have,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha_1 d(x_{2n}, x_{2n+1}) + \alpha_2[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] \\ &\left[\frac{d(x_{2n}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})}\right] + \alpha_3[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})] \\ &\left[\frac{d(x_{2n}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1}) + d(x_{2n}, Sx_{2n})}\right] \\ &\leq \alpha_1 d(x_{2n}, x_{2n+1}) + \alpha_2[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &+ \alpha_3[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})] \\ d(x_{2n+1}, x_{2n+2}) &\leq \frac{[\alpha_1 + \alpha_2 + 2\alpha_3]}{[1 - (\alpha_2 + 2\alpha_3)]} d(x_{2n}, x_{2n+1}). \end{aligned}$$

Let $h = \frac{[\alpha_1+\alpha_2+2\alpha_3]}{[1-(\alpha_2+2\alpha_3)]}$

Note that $0 \leq h < 1$, since $\alpha_1 + 2\alpha_2 + 4\alpha_3 < 1$.

Therefore, $d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1})$.

Similarly, $d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n})$, so we have $d(x_{2n+1}, x_{2n+2}) \leq h^2 d(x_{2n-1}, x_{2n})$

In this way, we get $d(x_{2n+1}, x_{2n+2}) \leq h^{2n+1} d(x_0, x_1)$.

Since $0 \leq h < 1 \Rightarrow h^{2n+1} \to 0$ as $n \to +\infty$.

By Lemma 2.1, $\{x_n\}$ is a Cauchy sequence in CDMS K. So there is a point $p \in K$ such that $x_n \to p$.

Further the subsequences $\{S(x_{2n})\} \to p$ and $\{T(x_{2n-1})\} \to p$.

Since $S, T : K \to K$ are continuous, we get $S(p) = p$ and $T(p) = p$.

Thus, p is a fixed point S and T.

Uniqueness of common fixed point: If $\mu \in K$ is a fixed point of S and T then by condition (3.3), we have

$$\begin{aligned} d(\mu,\mu) &= d(S\mu, T\mu) \\ &\leq \alpha_1 d(\mu,\mu) + \alpha_2[d(\mu,S\mu) + d(\mu,T\mu)]\left[\frac{d(\mu,T\mu)}{d(\mu,\mu)+d(\mu,T\mu)}\right] \\ &+ \alpha_3[d(\mu,T\mu)+d(\mu,S\mu)]\left[\frac{d(\mu,T\mu)}{d(\mu,\mu)+d(\mu,T\mu)+d(\mu,S\mu)}\right] \\ d(\mu,\mu) &\leq \left(\alpha_1+\alpha_2+\frac{2}{3}\alpha_3\right)d(\mu,\mu). \end{aligned}$$

Since $0 \leq (\alpha_1 + \alpha_2 + \frac{2}{3}\alpha_3) < 1$ and $d(\mu,\mu) \geq 0$, we have, $d(\mu,\mu) = 0$.
Thus, $d(\mu,\mu) = 0$, if μ is a fixed point of T.
Now let μ and ρ be two fixed points of S and T $(\mu \neq \rho)$, then we have
We have,

$$\begin{aligned} d(\mu,\rho) &= d(S\mu, T\rho) \\ &\leq \alpha_1 d(\mu,\rho) + \alpha_2[d(\mu,S\mu) + d(\rho,T\rho)]\left[\frac{d(\mu,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)}\right] \\ &+ \alpha_3[d(\mu,T\rho)+d(\rho,S\mu)]\left[\frac{d(\mu,T\rho)}{d(\mu,\rho)+d(\rho,T\rho)+d(\mu,S\mu)}\right] \end{aligned}$$

$$d(\mu,\rho) \quad \leq [\alpha_1+\alpha_3]d(\mu,\rho), since\, d(\mu,\mu) = d(\rho,\rho) = 0,$$

which gives $d(\mu,\rho) = 0$, since $0 \leq \alpha_1 + \alpha_3 < 1$ and $d(\mu,\rho) \geq 0$. Similarly $d(\rho,\mu) = 0$. Hence $\mu = \rho$ and thus S, T have a unique common fixed point.

Example 3.3. Let $K = [0,1]$ with a complete dislocated metric defined by $d(l,m) = l+m$ for all $l, m \in K$, where the continuous self mappings $S, T : K \to K$ are defined by $S(l) = \frac{l}{5}$ and $T(l) = 0$ for all $l \in K$. Let$\alpha_1 = \frac{21}{50}, \alpha_2 = \frac{1}{10}, \alpha_3 = \frac{1}{12}$. Then S and T satisfies all the conditions of Theorem 3.3 and $\mu = 0$ is the unique common fixed point of S and T in K.

4. Some More Results

This section deals with much more interesting situation, when a sequence of continuous functions converging pointwise to a continuous limit function is shown to have unique fixed point which is the limit of the fixed points of the sequence of functions.

Theorem 4.1. *Let (K,d) be a complete dislocated metric space and $\{T_i\}$ be a sequence of continuous mappings of K into itself converging pointwise to a continuous function $T : K \to K$. Let*

$$\begin{aligned} d(T_il, T_im) &\leq \alpha_1 d(l,m) + \alpha_2[d(l,T_il) + d(m,T_im)] + \alpha_3[d(l,T_im) + d(m,T_il)] \\ &+ \alpha_4\left[\frac{d(l,m)d(l,T_im)}{d(l,m)+d(m,T_im)}\right] + \alpha_5\left[\frac{d(l,T_im)d(m,T_im)}{d(l,m)+d(m,T_im)}\right] \end{aligned} \tag{4.1}$$

for all $l, m \in K$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$, $\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$, $i = 1, 2, 3, \cdots$. If each T_i has a fixed point μ_i and T has a fixed point μ, where $\mu_i \neq \mu$ for all i then the sequence $\{\mu_n\}$ converges to μ.

Proof. Since μ_i is a fixed point of T_i, we have

$$d(\mu,\mu_n) = d(T\mu, T_n\mu_n) \leq d(T\mu, T_n\mu) + d(T_n\mu, T_n\mu_n)$$

Now using (4.1), we have

$$\begin{aligned} d(\mu,\mu_n) &\leq d(T\mu,T_n\mu) + \alpha_1 d(\mu,\mu_n) + \alpha_2 d(T\mu,T_n\mu) \\ &\quad + \alpha_3 d(\mu,\mu_n) + \alpha_3 d(\mu_n,T_n\mu) + \alpha_4 d(\mu,\mu_n),\ since\, d(\mu_n,\mu_n)=0 \\ &\leq d(T\mu,T_n\mu) + (\alpha_1+\alpha_3+\alpha_4)d(\mu,\mu_n) + \alpha_3[d(\mu_n,\mu)+d(\mu,T_n\mu)] + \alpha_2 d(T\mu,T_n\mu) \\ &= (1+\alpha_2+\alpha_3)d(T\mu,T_n\mu) + (\alpha_1+2\alpha_3+\alpha_4)d(\mu,\mu_n) \end{aligned}$$

$$d(\mu,\mu_n) \leq \frac{(1+\alpha_2+\alpha_3)d(T\mu,T_n\mu)}{(1-(\alpha_1+2\alpha_3+\alpha_4))}.$$

$\to 0\, as\, T_n\mu \to T\mu\, as\, n \to +\infty$.
Thus, $\{\mu_n\}$ converges to μ. □

Theorem 4.2. *Let (K,d) be a complete dislocated metric space and $\{T_i\}$ be a sequence of continuous mappings of K into itself converging pointwise to a continuous function $T : K \to K$. Let*

$$\begin{aligned} d(T_i l, T_i m) &\leq \alpha_1 d(l,m) + \alpha_2[d(l,T_i l) + d(m,T_i m)]\left[\frac{d(l,T_i m)}{d(l,m)+d(m,T_i m)}\right] \\ &\quad + \alpha_3[d(l,T_i m)+d(m,T_i l)]\left[\frac{d(l,T_i m)}{d(l,m)+d(m,T_i m)+d(l,T_i m)}\right], \end{aligned} \tag{4.2}$$

for all $l, m \in K$ and $\alpha_1,\alpha_2,\alpha_3 \geq 0$, $\alpha_1+2\alpha_2+4\alpha_3 < 1$, $i = 1,2,3,\cdots$. If each T_i has a fixed point μ_i and T has a fixed point μ, where $\mu_i \neq \mu$ for all i then the sequence $\{\mu_n\}$ converges to μ.

Proof. Since μ_i is a fixed point of T_i, we have

$$d(\mu,\mu_n) = d(T\mu, T_n\mu_n) \leq d(T\mu, T_n\mu) + d(T_n\mu, T_n\mu_n)$$

Now using (4.2), we have

$$\begin{aligned} d(\mu,\mu_n) &\leq d(T\mu,T_n\mu) + \alpha_1 d(\mu,\mu_n) + \alpha_2[d(\mu,T_n\mu) + d(\mu_n,T_n\mu_n)] \\ &\quad \left[\frac{d(\mu,T_n\mu_n)}{d(\mu,\mu_n)+d(\mu_n,T_n\mu_n)}\right] + \alpha_3[d(\mu,T_n\mu_n)+d(\mu_n,T_n\mu)] \\ &\quad \left[\frac{d(\mu,T_n\mu_n)}{d(\mu,\mu_n)+d(\mu_n,T_n\mu_n)+d(\mu,T_n\mu_n)}\right] \\ &= d(T\mu,T_n\mu) + \alpha_1 d(\mu,\mu_n) + \alpha_2[d(\mu,T_n\mu)]\left[\frac{d(\mu,\mu_n)}{d(\mu,\mu_n)}\right] + \\ &\quad \alpha_3[d(\mu,\mu_n)+d(\mu_n,T_n\mu)]\left[\frac{d(\mu,\mu_n)}{2d(\mu,\mu_n)}\right] \\ &\leq d(T\mu,T_n\mu) + \alpha_1 d(\mu,\mu_n) + \alpha_2 d(\mu,T_n\mu) + \alpha_3\left[\frac{d(\mu,\mu_n)+d(\mu_n,\mu)+d(\mu,T_n\mu)}{2}\right] \\ &= \left(1+\alpha_2+\frac{\alpha_3}{2}\right)d(T\mu,T_n\mu) + (\alpha_1+\alpha_3)d(\mu,\mu_n) \end{aligned}$$

$$d(\mu,\mu_n) \leq \frac{(1+\alpha_2+\frac{\alpha_3}{2})}{(1-(\alpha_1+\alpha_3))}d(T\mu,T_n\mu).$$

$\to 0\, as\, T_n\mu \to T\mu\, as\, n \to +\infty$.
Thus, $\{\mu_n\}$ converges to μ.
□

Theorem 4.3. *Let (K, d) be a complete dislocated metric space and $\{S_i\}$, $\{T_i\}$ be a sequences of continuous mappings of K into itself which are converging pointwise to continuous functions $S, T : K \to K$ respectively. Let*

$$d(S_il, T_im) \leq \alpha_1 d(l, m) + \alpha_2[d(l, S_il) + d(m, T_im)] \left[\frac{d(l, T_im)}{d(l, m) + d(m, T_im)}\right] + \alpha_3[d(l, T_im) + d(m, S_il)] \left[\frac{d(l, T_im)}{d(l, m) + d(m, T_im) + d(l, S_il)}\right], \tag{4.3}$$

for all $l, m \in K$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$, $\alpha_1 + 2\alpha_2 + 4\alpha_3 < 1$, $i = 1, 2, 3, \cdots$. If each S_i and T_i has a common fixed point μ_i and S, T has a fixed point μ, where $\mu_i \neq \mu$ for all i then the sequence $\{\mu_n\}$ converges to μ.

Proof. Since μ_i is a common fixed point of S_i, T_i, we have

$$d(\mu, \mu_n) = d(S\mu, T_n\mu_n) \leq d(S\mu, S_n\mu) + d(S_n\mu, T_n\mu_n)$$

Now using (4.3), we have

$$\begin{aligned} d(\mu, \mu_n) &\leq d(S\mu, S_n\mu) + \alpha_1 d(\mu, \mu_n) + \alpha_2[d(\mu, S_n\mu) + d(\mu_n, T_n\mu_n)] \\ &\quad \left[\frac{d(\mu, T_n\mu_n)}{d(\mu, \mu_n) + d(\mu_n, T_n\mu_n)}\right] + \alpha_3[d(\mu, T_n\mu_n) + d(\mu_n, S_n\mu)] \\ &\quad \left[\frac{d(\mu, T_n\mu_n)}{d(\mu, \mu_n) + d(\mu_n, T_n\mu_n) + d(\mu, S_n\mu)}\right] \\ &\leq d(S\mu, S_n\mu) + \alpha_1 d(\mu, \mu_n) + \alpha_2 d(\mu, S_n\mu) + \alpha_3[d(\mu, \mu_n) + d(\mu_n, \mu) + d(\mu, S_n\mu)] \end{aligned}$$

$$d(\mu, \mu_n) \leq \frac{(1 + \alpha_2 + \alpha_3)}{(1 - (\alpha_1 + 2\alpha_3))} d(S\mu, S_n\mu).$$

$$\to 0 \, as \, S_n\mu \to S\mu \, as \, n \to +\infty.$$

Thus, $\{\mu_n\}$ converges to μ. □

2. Conclusions

Ciric [5], Dass and Gupta [6] proved the existence and uniqueness of fixed point result for mapping satisfying generalized contraction conditions in context of complete metric space. In this research article, we have established and proved the existence and uniqueness of fixed point for maps satisfying rational type contraction condition in the context of complete dislocated metric space. Our established results generalizes and extends related results in the existing literature. Suitable examples are also provided in support of some results. Any interested researchers can search for the existence and uniqueness of fixed points for maps satisfying different contraction conditions in dislocated metric space or any other generalization of metric space to conduct their thesis work on this topic.

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