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CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE (p,q)–DERIVATIVE OPERATOR FOR GENERALIZED DISTRIBUTION SATISFYING SUBORDINATE CONDITION

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Abstract. The main object of this paper is to study classes of analytic function associated with generalized Struve functions and using (p,q)–Jackson derivative. Furthermore, the bounds for generalized distribution using subordination principle involving modified sigmoid functions wear we estimate.

Keywords: Hadamard (convolution) product, subordination and quasi-subordination, Jackson (p,q)-derivative, generalized Struve functions, modified sigmoid functions.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ represent the class of all functions analytic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, and also denote by $\mathcal{A}$ the subclass of $\mathcal{H}(\mathbb{D})$ comprising of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ z\in\mathbb{D}, \tag{1}$$

which are normalized by the condition $f(0)=f'(0)-1=0$.

For $f_r(z)=\sum\limits_{n=0}^{\infty}a_{n,r}z^n$, $r=1,2$, which are two analytic functions in $\mathbb{D}$, the *Hadamard (or convolution) product* of f_1 and f_2 is defined by

$$(f_1*f_2)(z):=z+\sum_{n=0}^{\infty}a_{n,1}a_{n,2}\,z^n,\ z\in\mathbb{D}. \tag{2}$$

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For two functions $h,\ g \in \mathcal{H}(\mathbb{D})$ we say h is *subordinate* to g, carved as $h(z) \prec g(z)$, if there exists a *Schwarz function* $\psi(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ analytic in $\mathbb{D}$ with $\psi(0) = 0$ and $|\psi(z)| < 1,\ z \in \mathbb{D}$, such that $h(z) = g(\psi(z))$ for all $z \in \mathbb{D}$. Further, if the function g is univalent in $\mathbb{D}$, then the next equivalence holds:

$$h(z) \prec g(z) \Leftrightarrow h(0) = g(0) \text{ and } h(\mathbb{D}) \subset g(\mathbb{D}).$$

The perception of *quasi-subordination* for two analytic functions is due to Robertson [28]. Thus, for the functions $h,\ g \in \mathcal{H}(\mathbb{D})$, the function h is said to be *quasi-subordinate* to g, written as $h(z) \prec_{\varrho} g(z)$, if there exist the analytic functions ϱ and w, with $w(0) = 0$ such that $|\varrho(z)| \leq 1$, $|w(z)| < 1$, and $h(z) = \varrho(z)g(w(z))$ for all $z \in \mathbb{D}$.

If we fix $\varrho(z) \equiv 1$, in the above definition then $h(z) = g(w(z))$, that is $h(z) \prec g(z)$. Also, if $w(z) \equiv z$, then $h(z) = \varrho(z)g(z)$, and in this case we say that h is *majorized* by g, written as $h(z) \ll g(z)$. Thus, the quasi-subordination is a generalization of the usual subordination as well as of the majorization (for details, see [2, 13, 29]).

Recall that if $f \in \mathcal{A}$, with $zf'(z)/f(z) \prec \phi(z)$ and $(1 + zf''(z))/f'(z) \prec \phi(z)$, where $\phi(z) = (1+z)/(1-z)$, we acquire the two standard subclasses of (1) which are starlike and convex functions, respectively, and the classes comprising of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, correspondingly.

Ma and Minda [21] unified various subclasses $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ of starlike and convex functions sustaining the subordination $zf'(z)/f(z) \prec \varphi(z)$ and $1 + zf''(z)/f'(z) \prec \varphi(z)$ respectively, where $\varphi(z) = 1 + L_1 z + L_2 z^2 + \ldots$, with $L_1 \in \mathbb{R}$, $L_1 > 0$, and note that many results connected with these classes were obtained by several authors. For example, if $\varphi(z) = (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ represent, respectively, to the classes $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$ of *Janowski starlike and convex functions* [16], and in fact, $\mathcal{S}^* := \mathcal{S}^*[1, -1]$ and $\mathcal{K} := \mathcal{K}[1, -1]$.

Let we recall some basic notations of (p, q)-calculus. Letting $0 < q < p \leq 1$, then the *Jackson* (p, q)*-derivative* of the function f is defined as

$$\mathfrak{D}_{p,q} f(z) := \begin{cases} \dfrac{f(pz) - f(qz)}{(p-q)z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0. \end{cases} \tag{3}$$

For functions f of the form (1), from (3) we have

$$\mathfrak{D}_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},\ z \in \mathbb{D}, \tag{4}$$

where the symbol $[n]_{p,q}$ denotes the so-called (p, q)*-bracket* or *twin-basic number*, that is

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}.$$

We note that $[n]_{p,q} = [n]_{q,p}$, and for $p = 1$ the Jackson (p, q)-derivative reduces to the *Jackson* q*-derivative*

$$\mathfrak{D}_q f(z) := \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0. \end{cases}$$

Also, the *twin-basic number* is a natural generalization of the q*-number*, that is

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q},\ q \neq 1.$$

One can easily show that $\mathfrak{D}_{p,q}f(z) \to f'(z)$ as $p \to 1^-$ and $q \to 1^-$, and $\mathfrak{D}_{p,q}f(0) = f'(0) = 1$. The operator $\mathfrak{D}_{p,q}$ provide is an important tool that has been used to investigate the various subclasses of analytic functions of the form given by (3) (see [1, 5] and [25]).

Let S denote the sum of the convergent series of the form

$$S = \sum_{n=0}^{\infty} a_n,$$

where $a_n \geq 0$, for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Recently, Porwal [27] introduced the *generalized discrete probability distribution*, whose probability mass function is given by

$$p(n) = \frac{a_n}{S}, \ n \in \mathbb{N}_0,$$

where $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum\limits_{n\in\mathbb{N}_0} p_n = 1$. Moreover, if we let

$$\wp(w) = \sum_{n=0}^{\infty} a_n w^n,$$

then the above series is convergent for $|w| < 1$ and $w = 1$. Further, the sum of the power series whose coefficients are probabilities of the generalized distribution is given by

$$K(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{S} z^n, \ z \in \mathbb{D}, \tag{5}$$

where $S = \sum\limits_{n=0}^{\infty} a_n$.

Recently, Srivastava et al. [31] discussed the function H_σ associated with the *generalized Struve function* as follows:

$$H_\sigma(z) := z + \sum_{n=2}^{\infty} \frac{(1 + n\sigma - \sigma)(-c/4)^{n-1}}{(3/2)_{n-1}(k)_{n-1}} z^n, \ z \in \mathbb{D}, \ 0 \leq \sigma \leq 1, \ c \in \mathbb{C}, \tag{6}$$

where $(x)_n$ denotes the *Pochhammer symbol* given by $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\dots(x+n-1)$, $(x)_0 = 1$, and $(1)_n = n!$.

Using (5) and (6), we define the function

$$\mathcal{F}(z) := (K * H_\sigma)(z) = z + \sum_{n=2}^{\infty} \frac{(1 + n\sigma - \sigma)(-c/4)^{n-1} a_{n-1}}{(3/2)_{n-1}(k)_{n-1} S} z^n, \ z \in \mathbb{D}. \tag{7}$$

From (4) and (7) it easily follows that

$$\mathfrak{D}_{p,q}\mathcal{F}(z) = 1 + \sum_{n=2}^{\infty} \frac{(1 + n\sigma - \sigma)[n]_{p,q}(-c/4)^{n-1} a_{n-1}}{(3/2)_{n-1}(k)_{n-1} S} z^{n-1}, \ z \in \mathbb{D}.$$

In [15] Fadipe-Joseph et al. studied some geometrical properties of the modified sigmoid function (see Figure 1)

$$\begin{aligned} \varpi(z) = \frac{2}{1 + e^{-z}} &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} z^n \right)^m \\ &= 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17}{40320} z^7 + \dots, \ z \in \mathbb{D}. \end{aligned} \tag{8}$$

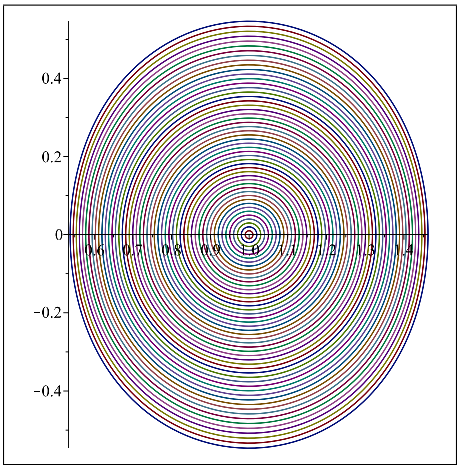

FIGURE 1. The image of $\varpi(\mathbb{D})$

Special functions and quantum calculus have important applications in almost every field of engineering and science (see, for details, [3, 6, 7, 8, 9, 10, 11, 14, 20, 26, 30, 31, 32, 34, 35]). Nowadays, specially applications of *Struve functions* occur in water-wave and surface-wave problems, unsteady aerodynamics, resistive MHD instability theory and optical diffraction. Further sigmoid function has recently taken the attention of many researchers these days because of its wide application in neural network, artificial intelligence, nonlinear approximation theory, statistics, and so on. By using the aforementioned concepts and recent studies on analytic functions, in this work the concept of subordinate principle involving modified sigmoid functions with (p, q)-derivative operator is used to define new classes of analytic functions as given below:

Definition 1.1. *Let $0 < q < p \leq 1$, $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and suppose that $\vartheta(z) = 1 + B_1 z + B_2 z^2 + \ldots$, with $B_1 \in \mathbb{R}$, $B_1 > 0$, and $\varrho(z) = \varrho_0 + \varrho_1 z + \varrho_2 z^2 + \ldots$, with $|\varrho(z)| \leq 1$ for all $z \in \mathbb{D}$, are two analytic functions in the unit disc $\mathbb{D}$.*

(i) We say that the function $f \in \mathcal{A}$ is in the class $\psi_\varrho S^{p,q}_{\beta,\gamma}(\vartheta)$ if

$$\frac{1}{\gamma}\left((1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_{p,q}\mathcal{F}(z)}{\mathcal{F}(z)} - i\,\tan\beta - 1\right) \prec_\varrho \vartheta(z) - 1.$$

(ii) The function $f \in \mathcal{A}$ belongs to the class $\hat{\psi}_\varrho C^{p,q}_{\beta,\gamma}(\vartheta)$ if

$$\frac{1}{\gamma}\left((1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_{p,q}\left(\mathfrak{D}_{p,q}\mathcal{F}(z)\right)}{\mathfrak{D}_{p,q}\mathcal{F}(z)} - i\,\tan\beta - 1\right) \prec_\varrho \vartheta(z) - 1.$$

Note that, if we consider the functions

$$f_*(z) = z + \frac{\gamma B_1 \varrho_0 w_1}{(1 + i\tan\beta)\left([2]_{p,q} - 1\right)} z^2,\ z \in \mathbb{D},$$

and

$$\hat{f}_*(z) = z + \frac{\gamma B_1 \varrho_0 w_1}{(1 + i\tan\beta)[2]_{p,q}([2]_{p,q} - 1)} z^2,\ z \in \mathbb{D},$$

then it is easy to check that $f_* \in \psi_\varrho S^{p,q}_{\beta,\gamma}(\vartheta)$ and $\hat{f}_* \in \hat{\psi}_\varrho C^{p,q}_{\beta,\gamma}(\vartheta)$, that is $\psi_\varrho S^{p,q}_{\beta,\gamma}(\vartheta) \neq \emptyset$ and $\hat{\psi}_\varrho C^{p,q}_{\beta,\gamma}(\vartheta) \neq \emptyset$.

Remark 1.1. *(i) For $p = 1$, $\varrho \equiv 1$ and $\vartheta := \varpi$, a function $f \in \mathcal{A}$ belongs to the class $\psi S^q_{\beta,\gamma}(\varpi) := \psi_\varrho S^{1,q}_{\beta,\gamma}(\vartheta)$ if*

$$1 + \frac{1}{\gamma}\left((1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_q\mathcal{F}(z)}{\mathcal{F}(z)} - i\,\tan\beta - 1\right) \prec \varpi(z),$$

and $f \in \mathcal{A}$ is in the class $\hat{\psi} C^q_{\beta,\gamma}(\varpi) := \hat{\psi}_\varrho C^{1,q}_{\beta,\gamma}(\vartheta)$ if

$$1 + \frac{1}{\gamma}\left((1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_q\left(\mathfrak{D}_q\mathcal{F}(z)\right)}{\mathfrak{D}_q\mathcal{F}(z)} - i\,\tan\beta - 1\right) \prec \varpi(z),$$

where $0 < q < 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and ϖ is defined by (8).

(ii) For $\gamma = 1$, $\varrho \equiv 1$ and $\vartheta := \varpi$, a function $f \in \mathcal{A}$ belongs to the class $\psi S^{p,q}_\beta(\varpi) := \psi_\varrho S^{p,q}_{\beta,1}(\vartheta)$ if

$$(1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_q\mathcal{F}(z)}{\mathcal{F}(z)} - i\,\tan\beta \prec \varpi(z),$$

and $f \in \mathcal{A}$ is in the class $\hat{\psi} C^q_\beta(\varpi) := \hat{\psi}_\varrho C^{p,q}_{\beta,1}(\vartheta)$ if

$$(1 + i\,\tan\beta)\,\frac{z\mathfrak{D}_q\left(\mathfrak{D}_q\mathcal{F}(z)\right)}{\mathfrak{D}_q\mathcal{F}(z)} - i\,\tan\beta \prec \varpi(z),$$

where $0 < q < p \le 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and ϖ as defined by (8).

(iii) For $\beta \equiv 0$, $\varrho \equiv 1$ and $\vartheta := \varpi$, a function $f \in \mathcal{A}$ belongs to the class $\psi S^{p,q}_\gamma(\varpi) := \psi_\varrho S^{p,q}_{0,\gamma}(\vartheta)$ if

$$1 + \frac{1}{\gamma}\left(\frac{z\mathfrak{D}_{p,q}\mathcal{F}(z)}{\mathcal{F}(z)} - 1\right) \prec \varpi(z),$$

and $f \in \mathcal{A}$ is in the class $\hat{\psi} C^{p,q}_\gamma(\varpi) := \hat{\psi}_\varrho C^{p,q}_{0,\gamma}(\vartheta)$ if

$$1 + \frac{1}{\gamma}\left(\frac{z\mathfrak{D}_{p,q}\left(\mathfrak{D}_{p,q}\mathcal{F}(z)\right)}{\mathfrak{D}_{p,q}\mathcal{F}(z)} - 1\right) \prec \varpi(z),$$

where $0 < q < p \le 1$, $\gamma \in \mathbb{C}^*$, and ϖ as defined by (8).

Motivated by several earlier investigation in connections between various subclasses of analytic functions by using special functions [12, 17, 19, 33], and inspired by the recent work of Altınkaya and Olatunji [4] and Oladipo [22, 23, 24], now we obtain the bounds for the first three coefficients of the above defined subclasses of functions.

Lemma 1.1. [18] *Let $w(z) = w_1 z + w_2 z^2 + \dots$, $z \in \mathbb{D}$, be an analytic function in $\mathbb{D}$, with $|w(z)| < 1$ in $\mathbb{D}$. Then, for any complex number t we have*

$$\left|w_2 + t w_1^2\right| \le \max\{1; |t|\},\ t \in \mathbb{C}.$$

The inequality is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2. Coefficients bounds

We begin this section by finding the estimates on the first three coefficients bounds for the functions that belongs to the classes $\psi_\varrho S^{p,q}_{\beta,\gamma}(\varpi)$ and $\hat{\psi}_\varrho C^{p,q}_{\beta,\gamma}(\varpi)$.

Unless otherwise mentioned, we assume throughout the paper that

(i) $w(z) = w_1 z + w_2 z^2 + \dots$, with $|w(z)| < 1$, $z \in \mathbb{D}$,
(ii) $\varrho(z) = \varrho_0 + \varrho_1 z + \varrho_2 z^2 + \dots$, with $|\varrho(z)| \le 1$, $z \in \mathbb{D}$,
(iii) $\vartheta(z) = 1 + B_1 z + B_2 z^2 + \dots$, with $B_1 \in \mathbb{R}$, $B_1 > 0$.

Theorem 2.1. *Let $0 < q < p \le 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. Set*

$$\kappa_1 = \frac{2(1+\sigma)}{3k}\left(-\frac{c}{4}\right), \qquad \kappa_2 = \frac{4(1+2\sigma)}{15k(k+1)}\left(-\frac{c}{4}\right)^2,$$
$$\kappa_3 = \frac{4(1+\sigma)^2}{9k^2}\left(-\frac{c}{4}\right)^2, \qquad \kappa_4 = \frac{8(1+3\sigma)}{105k(k^2+3k+2)}\left(-\frac{c}{4}\right)^3,$$
$$\kappa_5 = \frac{8(1+\sigma)(1+2\sigma)}{45k^2(k+1)}\left(-\frac{c}{4}\right)^3, \quad \kappa_6 = \frac{8(1+\sigma)^3}{27k^3}\left(-\frac{c}{4}\right)^3.$$

If $f \in \psi_\varrho S^{p,q}_{\beta,\gamma}(\vartheta)$ is given by (1), then

$$\left|\frac{a_1}{S}\right| \le \frac{B_1\varrho_0|\gamma|}{|\zeta|\left([2]_{p,q}-1\right)\kappa_1},$$
$$\left|\frac{a_2}{S}\right| \le \frac{B_1\varrho_0|\gamma|}{|\zeta|\left([3]_{p,q}-1\right)\kappa_2}\max\left\{1; \left|\frac{B_1\varrho_0\gamma\kappa_3}{\zeta\left([2]_{p,q}-1\right)\kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \le \frac{B_1\varrho_0|\gamma|}{|\zeta|\left([3]_{p,q}-1\right)\kappa_2}$$
$$\times \max\left\{1; \left|\frac{B_2\zeta\left([2]_{p,q}-1\right)^2\kappa_1^2 + B_1^2\varrho_0\gamma\left[\left([2]_{p,q}-1\right)\kappa_3 - \mu\left([3]_{p,q}-1\right)\kappa_2\right]}{B_1\zeta\left([2]_{p,q}-1\right)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{B_1\varrho_0|\gamma|}{|\zeta|\left([4]_{p,q}-1\right)\kappa_4}$$
$$\times \max\left\{1; \left|\xi\left[\frac{B_2\zeta\left([2]_{p,q}-1\right)^2\kappa_1^2 + B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta\left([2]_{p,q}-1\right)\kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1\xi} + \frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right)\right.\right.\right.$$
$$\left.\left.\left. + \left(1+\frac{2B_2}{B_1\xi}\right) + \frac{1}{\varrho_0\xi}\left(\varrho_1+\varrho_2\right)\right]\right|\right\},$$

where $\zeta = 1 + i\tan\beta$, $\eta = \dfrac{B_0^2\varrho_0^2\gamma^2\kappa_6}{\zeta^2\left([2]_{p,q}-1\right)^2\kappa_1^3}$, and $\xi = \dfrac{B_1\varrho_0\gamma\left[\left([2]_{p,q}-1\right)+\left([3]_{p,q}-1\right)\right]\kappa_5}{\zeta\left([2]_{p,q}-1\right)\left([3]_{p,q}-1\right)\kappa_1\kappa_2}$.

Proof. If $f \in \psi_\varrho S^{p,q}_{\beta,\gamma}(\vartheta)$ then there exists analytic functions in $\mathbb{D}$ that are ϱ and w, with $|\varrho(z)| \le 1$, $w(0) = 0$ and $|w| < 1$, such that

$$\frac{1}{\gamma}\left((1+i\tan\beta)\frac{z\mathfrak{D}_{p,q}\mathcal{F}(z)}{\mathcal{F}(z)} - i\tan\beta - 1\right) = \varrho(z)\left(\vartheta(w(z)) - 1\right). \tag{9}$$

We note that

$$\frac{z\mathfrak{D}_{p,q}\mathcal{F}(z)}{\mathcal{F}(z)} = 1 + \frac{\left([2]_{p,q}-1\right)\kappa_1 a_1}{S}z + \left(\frac{\left([3]_{p,q}-1\right)\kappa_2 a_2}{S} - \frac{\left([2]_{p,q}-1\right)\kappa_3 a_1^2}{S^2}\right)z^2 \tag{10}$$
$$+\left(\frac{\left([4]_{p,q}-1\right)\kappa_4 a_3}{S} - \frac{\left[\left([2]_{p,q}-1\right)+\left([3]_{p,q}-1\right)\right]\kappa_5 a_1 a_2}{S^2} + \frac{\left([2]_{p,q}-1\right)\kappa_6 a_1^3}{S^3}\right)z^3 + \dots,$$

and

$$\varrho(z)\left(\vartheta(w(z)) - 1\right) = B_1\varrho_0 w_1 z + [B_1\varrho_1 w_1 + \varrho_0(B_1 w_2 + B_2 w_1^2)]z^2 \tag{11}$$
$$+ \left[B_1\varrho_2 w_1 + \varrho_1(B_1 w_2 + B_2 w_1^2) + \varrho_0(B_1 w_3 + 2B_1 w_1 w_2 + B_3 w_1^3)\right]z^3 + \dots.$$

From (9), (10) and (11) we obtain

$$\frac{a_1}{S} = \frac{B_1 \varrho_0 \gamma w_1}{\zeta \left([2]_{p,q} - 1\right) \kappa_1}, \tag{12}$$

$$\frac{a_2}{S} = \frac{B_1 \varrho_0 \gamma}{\zeta \left([3]_{p,q} - 1\right) \kappa_2} \left[w_2 + \left(\frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta \left([2]_{p,q} - 1\right) \kappa_1^2} + \frac{B_2}{B_1} \right) w_1^2 + \frac{\varrho_1}{\varrho_0} w_1 \right], \tag{13}$$

and using Lemma 1.1 we deduce

$$\left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| \left([3]_{p,q} - 1\right) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta \left([2]_{p,q} - 1\right) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\}.$$

According to (12) and (13) it follows that

$$\frac{a_2}{S} - \mu \frac{a_1^2}{S^2} = \frac{B_1 \varrho_0 \gamma}{\zeta \left([3]_{p,q} - 1\right) \kappa_2}$$
$$\times \left[w_2 + \frac{B_2 \zeta \left([2]_{p,q} - 1\right)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \left[\left([2]_{p,q} - 1\right) \kappa_3 - \mu \left([3]_{p,q} - 1\right) \kappa_2 \right]}{B_1 \zeta \left([2]_{p,q} - 1\right)^2 \kappa_1^2} w_1^2 + \frac{\varrho_1}{\varrho_0} w_1 \right],$$

and moreover, from (9), (10) and (11) we get

$$\frac{a_3}{S} = \frac{B_1 \varrho_0 \gamma}{\zeta \left([4]_{p,q} - 1\right) \kappa_4} \times \left\{ w_3 + \xi \left[\left(\frac{B_2 \zeta \left([2]_{p,q} - 1\right)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \kappa_3}{B_1 \zeta \left([2]_{p,q} - 1\right) \kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1 \xi} \right) w_1^3 \right. \right.$$
$$\left. \left. + \frac{\varrho_1}{\varrho_0} \left(1 + \frac{B_2}{B_1 \xi} \right) w_1^2 + \left(1 + \frac{2B_2}{B_1 \xi} \right) w_1 w_2 + \frac{1}{\varrho_0 \xi} \left(\varrho_1 w_2 + \varrho_2 w_1 \right) \right] \right\},$$

where
$\zeta = 1 + i \tan \beta$, $\eta = \dfrac{B_0^2 \varrho_0^2 \gamma^2 \kappa_6}{\zeta^2 \left([2]_{p,q} - 1\right)^2 \kappa_1^3}$, and $\xi = \dfrac{B_1 \varrho_0 \gamma \left[\left([2]_{p,q} - 1\right) + \left([3]_{p,q} - 1\right) \right] \kappa_5}{\zeta \left([2]_{p,q} - 1\right) \left([3]_{p,q} - 1\right) \kappa_1 \kappa_2}$. □

If we take $p = 1$ in Theorem 2.1 we obtain the following result:

Corollary 2.1. *Let* $0 < q < 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. *If* $f \in \psi_\varrho S^q_{\beta,\gamma}(\vartheta)$ *is given by* (1), *then*

$$\left| \frac{a_1}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| \left([2]_q - 1\right) \kappa_1},$$
$$\left| \frac{a_2}{S} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| \left([3]_q - 1\right) \kappa_2} \max \left\{ 1; \left| \frac{B_1 \varrho_0 \gamma \kappa_3}{\zeta \left([2]_q - 1\right) \kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0} \right| \right\},$$
$$\left| \frac{a_2}{S} - \mu \frac{a_1^2}{S^2} \right| \leq \frac{B_1 \varrho_0 |\gamma|}{|\zeta| \left([3]_q - 1\right) \kappa_2}$$
$$\times \max \left\{ 1; \left| \frac{B_2 \zeta \left([2]_q - 1\right)^2 \kappa_1^2 + B_1^2 \varrho_0 \gamma \left[\left([2]_q - 1\right) \kappa_3 - \mu \left([3]_q - 1\right) \kappa_2 \right]}{B_1 \zeta \left([2]_q - 1\right)^2 \kappa_1^2} \right| \right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|\left([4]_q-1\right)\kappa_4} \times \max\left\{1; \left|\xi\left[\frac{B_2\zeta\left([2]_q-1\right)^2\kappa_1^2+B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta\left([2]_q-1\right)\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right) + \left(1+\frac{2B_2}{B_1\xi}\right)+\frac{1}{\varrho_0\xi}\left(\varrho_1+\varrho_2\right)\right]\right|\right\},$$

where $\zeta = 1 + i\,\tan\beta$, $\eta = \dfrac{B_0^2\varrho_0^2\gamma^2\kappa_6}{\zeta^2\left([2]_q-1\right)^2\kappa_1^3}$, *and* $\xi = \dfrac{B_1\varrho_0\gamma\left[\left([2]_q-1\right)+\left([3]_q-1\right)\right]\kappa_5}{\zeta\left([2]_q-1\right)\left([3]_q-1\right)\kappa_1\kappa_2}$.

For $p = 1$ and $\gamma = 1$ Theorem 2.1 yields to the next result:

Corollary 2.2. *Let* $0 < q < 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. *If* $f \in \psi_\varrho S_\beta^q(\vartheta)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{B_1\varrho_0}{|\zeta|\left([2]_q-1\right)\kappa_1},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{B_1\varrho_0}{|\zeta|\left([3]_q-1\right)\kappa_2}\max\left\{1; \left|\frac{B_1\varrho_0\kappa_3}{\zeta\left([2]_q-1\right)\kappa_1^2}+\frac{B_2}{B_1}+\frac{\varrho_1}{\varrho_0}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \leq \frac{B_1\varrho_0}{|\zeta|\left([3]_q-1\right)\kappa_2} \times \max\left\{1; \left|\frac{B_2\zeta\left([2]_q-1\right)^2\kappa_1^2+B_1^2\varrho_0\left[\left([2]_q-1\right)\kappa_3-\mu\left([3]_q-1\right)\kappa_2\right]}{B_1\zeta\left([2]_q-1\right)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{B_1\varrho_0}{|\zeta|\left([4]_q-1\right)\kappa_4} \times \max\left\{1; \left|\xi\left[\frac{B_2\zeta\left([2]_q-1\right)^2\kappa_1^2+B_1^2\varrho_0\kappa_3}{B_1\zeta\left([2]_q-1\right)\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right) + \left(1+\frac{2B_2}{B_1\xi}\right)+\frac{1}{\varrho_0\xi}\left(\varrho_1+\varrho_2\right)\right]\right|\right\},$$

where $\zeta = 1 + i\,\tan\beta$, $\eta = \dfrac{B_0^2\varrho_0^2\kappa_6}{\zeta^2\left([2]_q-1\right)^2\kappa_1^3}$, *and* $\xi = \dfrac{B_1\varrho_0\left[\left([2]_q-1\right)+\left([3]_q-1\right)\right]\kappa_5}{\zeta\left([2]_q-1\right)\left([3]_q-1\right)\kappa_1\kappa_2}$.

Considering the special case $p = 1$ and $q \to 1^-$ in Theorem 2.1 we obtain:

Corollary 2.3. *Let* $\gamma \in \mathbb{C}^*$ *and* $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. *If* $f \in \psi_\varrho S_{\beta,\gamma}(\vartheta)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{2|\zeta|\kappa_2}\max\left\{1; \left|\frac{B_1\varrho_0\gamma\kappa_3}{\zeta\kappa_1^2}+\frac{B_2}{B_1}+\frac{\varrho_1}{\varrho_0}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \leq \frac{B_1\varrho_0|\gamma|}{2|\zeta|\kappa_2}\max\left\{1; \left|\frac{B_2\zeta\kappa_1^2+B_1^2\varrho_0\gamma\left[\kappa_3-2\mu\kappa_2\right]}{B_1\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{B_1\varrho_0|\gamma|}{3|\zeta|\kappa_4}\max\left\{1;\left|\xi\left[\frac{B_2\zeta\kappa_1^2+B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(1+\frac{2B_2}{B_1\xi}\right)+\frac{1}{\varrho_0\xi}(\varrho_1+\varrho_2)\right]\right|\right\},$$

where $\zeta = 1+i\tan\beta$, $\eta = \dfrac{B_0^2\varrho_0^2\gamma^2\kappa_6}{\zeta^2\kappa_1^3}$, *and* $\xi = \dfrac{3B_1\varrho_0\gamma\kappa_5}{2\zeta\kappa_1\kappa_2}$.

If we take $\gamma = 1$, $p = 1$ and $q \to 1^-$ in Theorem 2.1 we obtain the next result:

Corollary 2.4. *Let* $\gamma \in \mathbb{C}^*$ *and* $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. *If* $f \in \psi_\varrho S_\beta(\vartheta)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \le \frac{B_1\varrho_0}{|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \le \frac{B_1\varrho_0}{2|\zeta|\kappa_2}\max\left\{1;\left|\frac{B_1\varrho_0\kappa_3}{\zeta\kappa_1^2}+\frac{B_2}{B_1}+\frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{B_1\varrho_0}{2|\zeta|\kappa_2}\max\left\{1;\left|\frac{B_2\zeta\kappa_1^2+B_1^2\varrho_0\left[\kappa_3-2\mu\kappa_2\right]}{B_1\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{B_1\varrho_0}{3|\zeta|\kappa_4}\max\left\{1;\left|\xi\left[\frac{B_2\zeta\kappa_1^2+B_1^2\varrho_0\kappa_3}{B_1\zeta\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(1+\frac{2B_2}{B_1\xi}\right)+\frac{1}{\varrho_0\xi}(\varrho_1+\varrho_2)\right]\right|\right\},$$

where $\zeta = 1+i\tan\beta$, $\eta = \dfrac{B_0^2\varrho_0^2\kappa_6}{\zeta^2\kappa_1^3}$, *and* $\xi = \dfrac{3B_1\varrho_0\kappa_5}{2\zeta\kappa_1\kappa_2}$.

If we take $\varrho \equiv 1$ and $\vartheta := \varpi$, then Theorem 2.1 reduces to the next special case:

Corollary 2.5. *Let* $0 < q < p \le 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f \in \psi S_{\beta,\gamma}^{p,q}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([2]_{p,q}-1\right)\kappa_1}, \quad \left|\frac{a_2}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([3]_{p,q}-1\right)\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{\zeta\left([2]_{p,q}-1\right)\kappa_1^2}\right|\right\},$$
$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{|\gamma|}{|\zeta|\left([3]_{p,q}-1\right)\kappa_2}\max\left\{1;\left|\frac{\gamma\left[\left([2]_{p,q}-1\right)\kappa_3-\mu\left([3]_{p,q}-1\right)\kappa_2\right]}{\zeta\left([2]_{p,q}-1\right)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([4]_{p,q}-1\right)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta\left([2]_{p,q}-1\right)\kappa_1^2}-\frac{1}{\xi}\left(\eta+\frac{1}{12}\right)+1\right]\right|\right\},$$

where $\zeta = 2(1+i\tan\beta)$, $\eta = \dfrac{\gamma^2\kappa_6}{\zeta^2\left([2]_{p,q}-1\right)^2\kappa_1^3}$, *and* $\xi = \dfrac{\left[\left([2]_{p,q}-1\right)+\left([3]_{p,q}-1\right)\right]\gamma\kappa_5}{\zeta\left([2]_{p,q}-1\right)\left([3]_{p,q}-1\right)\kappa_1\kappa_2}$.

Remark 2.1. *If we consider the function*

$$f_*(z) = z + \frac{\gamma w_1}{2(1+i\tan\beta)\left([2]_{p,q}-1\right)}z^2,$$

then it is easy to check that $f_* \in \psi S_{\beta,\gamma}^{p,q}(\varpi)$, *that is* $\psi S_{\beta,\gamma}^{p,q}(\varpi) \neq \emptyset$.

Replacing $p=1$, $\varrho\equiv 1$ and $\vartheta := \varpi$ in Theorem 2.1 we get:

Corollary 2.6. *Let* $0<q<1$, $\gamma\in\mathbb{C}^*$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f\in\psi S^q_{\beta,\gamma}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([2]_q-1\right)\kappa_1},\quad \left|\frac{a_2}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([3]_q-1\right)\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{\zeta\left([2]_q-1\right)\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{|\gamma|}{|\zeta|\left([3]_q-1\right)\kappa_2}\max\left\{1;\left|\frac{\gamma\left[\left([2]_q-1\right)\kappa_3-\mu\left([3]_q-1\right)\kappa_2\right]}{\zeta\left([2]_q-1\right)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{|\gamma|}{|\zeta|\left([4]_q-1\right)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta\left([2]_q-1\right)\kappa_1^2}-\frac{1}{\xi}\left(\eta+\frac{1}{12}\right)+1\right]\right|\right\}.$$

where $\zeta=2(1+i\tan\beta)$, $\eta=\dfrac{\gamma^2\kappa_6}{\zeta^2\left([2]_q-1\right)^2\kappa_1^3}$, *and* $\xi=\dfrac{\left[\left([2]_q-1\right)+\left([3]_q-1\right)\right]\gamma\kappa_5}{\zeta\left([2]_q-1\right)\left([3]_q-1\right)\kappa_1\kappa_2}$.

Taking $p=1$, $\gamma=1$, $\varrho\equiv 1$ and $\vartheta := \varpi$ Theorem 2.1 yields to the next result:

Corollary 2.7. *Let* $0<q<1$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f\in\psi S^q_{\beta}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \le \frac{1}{|\zeta|\left([2]_q-1\right)\kappa_1},\quad \left|\frac{a_2}{S}\right| \le \frac{1}{|\zeta|\left([3]_q-1\right)\kappa_2}\max\left\{1;\left|\frac{\kappa_3}{\zeta\left([2]_q-1\right)\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{1}{|\zeta|\left([3]_q-1\right)\kappa_2}\max\left\{1;\left|\frac{\left([2]_q-1\right)\kappa_3-\mu\left([3]_q-1\right)\kappa_2}{\zeta\left([2]_q-1\right)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{1}{|\zeta|\left([4]_q-1\right)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\kappa_3}{\zeta\left([2]_q-1\right)\kappa_1^2}-\frac{1}{\xi}\left(\eta+\frac{1}{12}\right)+1\right]\right|\right\}$$

where $\zeta=2(1+i\tan\beta)$, $\eta=\dfrac{\kappa_6}{\zeta^2\left([2]_q-1\right)^2\kappa_1^3}$, *and* $\xi=\dfrac{\left[\left([2]_q-1\right)+\left([3]_q-1\right)\right]\kappa_5}{\zeta\left([2]_q-1\right)\left([3]_q-1\right)\kappa_1\kappa_2}$.

For $p=1$, $\varrho\equiv 1$ and $\vartheta := \varpi$ and $q\to 1^-$ Theorem 2.1 reduces to the next special case:

Corollary 2.8. *Let* $\gamma\in\mathbb{C}^*$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f\in\psi S_{\beta,\gamma}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \le \frac{|\gamma|}{|\zeta|\kappa_1},\quad \left|\frac{a_2}{S}\right| \le \frac{|\gamma|}{2|\zeta|\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{\zeta\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{|\gamma|}{2|\zeta|\kappa_2}\max\left\{1;\left|\frac{\gamma\left[\kappa_3-2\mu\kappa_2\right]}{\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{|\gamma|}{3|\zeta|\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta\kappa_1^2}-\frac{1}{\xi}\left(\eta+\frac{1}{12}\right)+1\right]\right|\right\}.$$

where $\zeta=2(1+i\tan\beta)$, $\eta=\dfrac{\gamma^2\kappa_6}{\zeta^2\kappa_1^3}$, *and* $\xi=\dfrac{3\gamma\kappa_5}{2\zeta\kappa_1\kappa_2}$.

If we take $\gamma=1$, $p=1$, $\varrho\equiv 1$, $\vartheta := \varpi$ and $q\to 1^-$ in Theorem 2.1 we obtain the next result:

Corollary 2.9. *Let* $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$*, and* ϖ *be defined by* (8)*. If* $f \in \psi S_\beta(\varpi)$ *is given by* (1)*, then*

$$\left|\frac{a_1}{S}\right| \leq \frac{1}{|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \leq \frac{1}{2|\zeta|\kappa_2} \max\left\{1; \left|\frac{\kappa_3}{\zeta\kappa_1^2}\right|\right\},$$
$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{1}{2|\zeta|\kappa_2} \max\left\{1; \left|\frac{\kappa_3 - 2\mu\kappa_2}{\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{1}{3|\zeta|\kappa_4} \max\left\{1; \left|\xi\left[\frac{\kappa_3}{\zeta\kappa_1^2} - \frac{1}{\xi}\left(\eta + \frac{1}{12}\right) + 1\right]\right|\right\}.$$

where $\zeta = 2(1 + i\tan\beta)$*,* $\eta = \frac{\kappa_6}{\zeta^2\kappa_1^3}$*, and* $\xi = \frac{3\kappa_5}{2\zeta\kappa_1\kappa_2}$.

Theorem 2.2. *Let* $0 < q < p \leq 1$*,* $\gamma \in \mathbb{C}^*$*,* $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$*. Set*

$$\kappa_1 = \frac{2(1+\sigma)}{3k}\left(-\frac{c}{4}\right), \qquad \kappa_2 = \frac{4(1+2\sigma)}{15k(k+1)}\left(-\frac{c}{4}\right)^2,$$
$$\kappa_3 = \frac{4(1+\sigma)^2}{9k^2}\left(-\frac{c}{4}\right)^2, \qquad \kappa_4 = \frac{8(1+3\sigma)}{105k(k^2+3k+2)}\left(-\frac{c}{4}\right)^3,$$
$$\kappa_5 = \frac{8(1+\sigma)(1+2\sigma)}{45k^2(k+1)}\left(-\frac{c}{4}\right)^3, \quad \kappa_6 = \frac{8(1+\sigma)^3}{27k^3}\left(-\frac{c}{4}\right)^3.$$

If $f \in \psi_\varrho C^{p,q}_{\beta,\gamma}(\vartheta)$ *is given by* (1)*, then*

$$\left|\frac{a_1}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[2]_{p,q}([2]_{p,q}-1)\kappa_1},$$
$$\left|\frac{a_2}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_{p,q}([3]_{p,q}-1)\kappa_2} \max\left\{1; \left|\frac{B_1\varrho_0\gamma\kappa_3}{\zeta([2]_{p,q}-1)\kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_{p,q}([3]_{p,q}-1)\kappa_2}$$
$$\times \max\left\{1; \left|\frac{B_2\zeta[2]_{p,q}^2([2]_{p,q}-1)^2\kappa_1^2 + B_1^2\varrho_0\gamma\left\{[2]_{p,q}^2([2]_{p,q}-1)\kappa_3 - \mu[3]_{p,q}([3]_{p,q}-1)\kappa_2\right\}}{B_1\zeta[2]_{p,q}^2([2]_{p,q}^2-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{|\zeta|[4]_{p,q}([4]_{p,q}-1)\kappa_4} \max\left\{1; \left|\xi\left[\frac{B_2\zeta([2]_{p,q}-1)\kappa_1^2 + B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta([2]_{p,q}-1)\kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1\xi}\right.\right.\right.$$
$$\left.\left.\left. + \frac{\varrho_1}{\varrho_0}\left(1 + \frac{B_2}{B_1\xi}\right) + \left(1 + \frac{2}{\xi}\right) + \frac{1}{\varrho_0\xi}(\varrho_1 + \varrho_2)\right]\right|\right\},$$

where $\zeta = 1 + i\tan\beta$*,* $\eta = \frac{B_1^2\varrho_0^2\gamma^2\kappa_6}{\zeta^3\left([2]_{p,q}-1\right)^2\kappa_1^3}$*, and* $\xi = \frac{B_1\varrho_0\gamma\left[([2]_{p,q}-1) + ([3]_{p,q}-1)\right]\kappa_5}{\zeta^2\left([2]_{p,q}-1\right)\left([3]_{p,q}-1\right)\kappa_1\kappa_2}$.

Proof. Proceeding as in the proof of Theorem 2.1 excepting that instead of using (9) we will use the subordination

$$\frac{1}{\gamma}\left((1 + i\tan\beta)\frac{z\mathfrak{D}_{p,q}\left(\mathfrak{D}_{p,q}\mathcal{F}(z)\right)}{\mathfrak{D}_{p,q}\mathcal{F}(z)} - i\tan\beta - 1\right) \prec_\rho \vartheta(z) - 1,$$

our result follows easily. □

If we take $p=1$ in Theorem 2.2 we obtain the following result:

Corollary 2.10. *Let $0<q<1$, $\gamma\in\mathbb{C}^*$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. If $f\in\psi_\varrho C^q_{\beta,\gamma}(\vartheta)$ is given by (1), then*

$$\left|\frac{a_1}{S}\right|\le\frac{B_1\varrho_0|\gamma|}{|\zeta|[2]_q([2]_q-1)\kappa_1},$$
$$\left|\frac{a_2}{S}\right|\le\frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{B_1\varrho_0\gamma\kappa_3}{\zeta([2]_q-1)\kappa_1^2}+\frac{B_2}{B_1}+\frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\le\frac{B_1\varrho_0|\gamma|}{|\zeta|[3]_q([3]_q-1)\kappa_2}$$
$$\times\max\left\{1;\left|\frac{B_2\zeta[2]_q^2([2]_q-1)^2\kappa_1^2+B_1^2\varrho_0\gamma\left\{[2]_q^2([2]_q-1)\kappa_3-\mu[3]_q([3]_q-1)\kappa_2\right\}}{B_1\zeta[2]_q^2([2]_q^2-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right|\le\frac{B_1\varrho_0|\gamma|}{|\zeta|[4]_q([4]_q-1)\kappa_4}\max\left\{1;\left|\xi\left[\frac{B_2\zeta([2]_q-1)\kappa_1^2+B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta([2]_q-1)\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right)+\left(1+\frac{2}{\xi}\right)+\frac{1}{\varrho_0\xi}(\varrho_1+\varrho_2)\right]\right|\right\},$$

where $\zeta=1+i\tan\beta$, $\eta=\dfrac{B_1^2\varrho_0^2\gamma^2\kappa_6}{\zeta^3([2]_q-1)^2\kappa_1^3}$, and $\xi=\dfrac{B_1\varrho_0\gamma\left[([2]_q-1)+([3]_q-1)\right]\kappa_5}{\zeta^2([2]_q-1)([3]_q-1)\kappa_1\kappa_2}$.

For $p=1$ and $\gamma=1$ in Theorem 2.2 yields to the next result:

Corollary 2.11. *Let $0<q<1$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. If $f\in\psi_\varrho C^q_{\beta}(\vartheta)$ is given by (1), then*

$$\left|\frac{a_1}{S}\right|\le\frac{B_1\varrho_0}{|\zeta|[2]_q([2]_q-1)\kappa_1},$$
$$\left|\frac{a_2}{S}\right|\le\frac{B_1\varrho_0}{|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{B_1\varrho_0\kappa_3}{\zeta([2]_q-1)\kappa_1^2}+\frac{B_2}{B_1}+\frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right|\le\frac{B_1\varrho_0}{|\zeta|[3]_q([3]_q-1)\kappa_2}$$
$$\times\max\left\{1;\left|\frac{B_2\zeta[2]_q^2([2]_q-1)^2\kappa_1^2+B_1^2\varrho_0\left\{[2]_q^2([2]_q-1)\kappa_3-\mu[3]_q([3]_q-1)\kappa_2\right\}}{B_1\zeta[2]_q^2([2]_q^2-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right|\le\frac{B_1\varrho_0}{|\zeta|[4]_q([4]_q-1)\kappa_4}\max\left\{1;\left|\xi\left[\frac{B_2\zeta([2]_q-1)\kappa_1^2+B_1^2\varrho_0\kappa_3}{B_1\zeta([2]_q-1)\kappa_1^2}-\frac{\eta}{\xi}+\frac{B_3}{B_1\xi}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\varrho_1}{\varrho_0}\left(1+\frac{B_2}{B_1\xi}\right)+\left(1+\frac{2}{\xi}\right)+\frac{1}{\varrho_0\xi}(\varrho_1+\varrho_2)\right]\right|\right\},$$

where $\zeta=1+i\tan\beta$, $\eta=\dfrac{B_1^2\varrho_0^2\kappa_6}{\zeta^3([2]_q-1)^2\kappa_1^3}$, and $\xi=\dfrac{B_1\varrho_0\left[([2]_q-1)+([3]_q-1)\right]\kappa_5}{\zeta^2([2]_q-1)([3]_q-1)\kappa_1\kappa_2}$.

Considering the special case $p=1$ and $q\to1^-$ in Theorem 2.2 we obtain:

Corollary 2.12. *Let* $\gamma \in \mathbb{C}^*$, $-\dfrac{\pi}{2} < \beta < \dfrac{\pi}{2}$. *If* $f \in \psi_\varrho C_{\beta,\gamma}(\vartheta)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{2|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{6|\zeta|\kappa_2}\max\left\{1; \left|\frac{B_1\varrho_0\gamma\kappa_3}{\zeta\kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{B_1\varrho_0|\gamma|}{6|\zeta|\kappa_2}\max\left\{1; \left|\frac{4B_2\zeta\kappa_1^2 + B_1^2\varrho_0\gamma\left(4\kappa_3 - 6\mu\kappa_2\right)}{36B_1\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{B_1\varrho_0|\gamma|}{12|\zeta|\kappa_4}\max\left\{1; \left|\xi\left[\frac{B_2\zeta\kappa_1^2 + B_1^2\varrho_0\gamma\kappa_3}{B_1\zeta\kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1\xi} + \frac{\varrho_1}{\varrho_0}\left(1 + \frac{B_2}{B_1\xi}\right)\right.\right.\right.$$
$$\left.\left.\left. + \left(1 + \frac{2}{\xi}\right) + \frac{1}{\varrho_0\xi}\left(\varrho_1 + \varrho_2\right)\right]\right|\right\},$$

where $\zeta = 1 + i\,\tan\beta$, $\eta = \dfrac{B_1^2\varrho_0^2\gamma^2\kappa_6}{\zeta^3\kappa_1^3}$, *and* $\xi = \dfrac{3B_1\varrho_0\gamma\kappa_5}{2\zeta^2\kappa_1\kappa_2}$.

If we take $\gamma = 1$, $p = 1$ and $q \to 1^-$ in Theorem 2.2 we obtain the next result:

Corollary 2.13. *Let* $\gamma \in \mathbb{C}^*$, $-\dfrac{\pi}{2} < \beta < \dfrac{\pi}{2}$. *If* $f \in \psi_\varrho C_{\beta}(\vartheta)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{B_1\varrho_0}{2|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \leq \frac{B_1\varrho_0}{6|\zeta|\kappa_2}\max\left\{1; \left|\frac{B_1\varrho_0\kappa_3}{\zeta\kappa_1^2} + \frac{B_2}{B_1} + \frac{\varrho_1}{\varrho_0}\right|\right\},$$
$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{B_1\varrho_0}{6|\zeta|\kappa_2}\max\left\{1; \left|\frac{4B_2\zeta\kappa_1^2 + B_1^2\varrho_0\left(4\kappa_3 - 6\mu\kappa_2\right)}{36B_1\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{B_1\varrho_0}{12|\zeta|\kappa_4}\max\left\{1; \left|\xi\left[\frac{B_2\zeta\kappa_1^2 + B_1^2\varrho_0\kappa_3}{B_1\zeta\kappa_1^2} - \frac{\eta}{\xi} + \frac{B_3}{B_1\xi} + \frac{\varrho_1}{\varrho_0}\left(1 + \frac{B_2}{B_1\xi}\right)\right.\right.\right.$$
$$\left.\left.\left. + \left(1 + \frac{2}{\xi}\right) + \frac{1}{\varrho_0\xi}\left(\varrho_1 + \varrho_2\right)\right]\right|\right\},$$

where $\zeta = 1 + i\,\tan\beta$, $\eta = \dfrac{B_1^2\varrho_0^2\kappa_6}{\zeta^3\kappa_1^3}$, *and* $\xi = \dfrac{3B_1\varrho_0\kappa_5}{2\zeta^2\kappa_1\kappa_2}$.

For $\varrho \equiv 1$ and $\vartheta := \varpi$ Theorem 2.2 yields to the next special case:

Corollary 2.14. *Let* $0 < q < p \leq 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f \in \hat{\psi}C^{p,q}_{\beta,\gamma}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[2]_{p,q}([2]_{p,q}-1)\kappa_1},$$

$$\left|\frac{a_2}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[3]_{p,q}([3]_{p,q}-1)\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{2\zeta([2]_{p,q}-1)\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \leq \frac{|\gamma|}{2|\zeta|[3]_{p,q}([3]_{p,q}-1)\kappa_2} \times\max\left\{1;\left|\frac{\gamma([2]_{p,q}^2([2]_{p,q}-1)\kappa_3-\mu[3]_{p,q}([3]_{p,q}-1)\kappa_2)}{2\zeta[2]_{p,q}^2([2]_{p,q}-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[4]_{p,q}([4]_{p,q}-1)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta([2]_{p,q}-1)\kappa_1^2}+\frac{2\eta}{\xi}+\frac{23}{12\xi}\right]\right|\right\},$$

where $\zeta = 1 + i\tan\beta$, $\eta = \frac{\gamma^3\kappa_6}{8\zeta^3([2]_{p,q}-1)^2\kappa_1^3}$, *and* $\xi = \frac{(([2]_{p,q}-1)+([3]_{p,q}-1))\gamma^2\kappa_5}{4\zeta^2([2]_{p,q}-1)([3]_{p,q}-1)\kappa_1\kappa_2}$.

Remark 2.2. *If we consider the function*

$$\hat{f}_*(z) = z + \frac{\gamma w_1}{(1+i\tan\beta)[2]_{p,q}([2]_{p,q}-1)}z^2, z \in \mathbb{D},$$

then it is easy to check that $\hat{f}_* \in \hat{\psi}C^{p,q}_{\beta,\gamma}(\varpi)$, *that is* $\hat{\psi}C^{p,q}_{\beta,\gamma}(\varpi) \neq \emptyset$.

For $p = 1$, $\varrho \equiv 1$ and $\vartheta := \varpi$ Theorem 2.2 yields to the next special case:

Corollary 2.15. *Let* $0 < q < 1$, $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, *and* ϖ *be defined by* (8). *If* $f \in \hat{\psi}C^{q}_{\beta,\gamma}(\varpi)$ *is given by* (1), *then*

$$\left|\frac{a_1}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[2]_q([2]_q-1)\kappa_1},\quad \left|\frac{a_2}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{2\zeta([2]_q-1)\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \leq \frac{|\gamma|}{2|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{\gamma([2]_q^2([2]_q-1)\kappa_3-\mu[3]_q([3]_q-1)\kappa_2)}{2\zeta[2]_q^2([2]_q-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \leq \frac{|\gamma|}{2|\zeta|[4]_q([4]_q-1)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta([2]_q-1)\kappa_1^2}+\frac{2\eta}{\xi}+\frac{23}{12\xi}\right]\right|\right\},$$

where $\zeta = 1 + i\tan\beta$, $\eta = \frac{\gamma^3\kappa_6}{8\zeta^3([2]_q-1)^2\kappa_1^3}$, *and* $\xi = \frac{(([2]_q-1)+([3]_q-1))\gamma^2\kappa_5}{4\zeta^2([2]_q-1)([3]_q-1)\kappa_1\kappa_2}$.

Putting $p = 1$, $\gamma = 1$, $\varrho \equiv 1$ and $\vartheta := \varpi$ in Theorem 2.2 we get:

Corollary 2.16. *Let $0 < q < 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and ϖ be defined by* (8)*. If $f \in \hat{\psi}C^q_\beta(\varpi)$ is given by* (1)*, then*

$$\left|\frac{a_1}{S}\right| \le \frac{1}{2|\zeta|[2]_q([2]_q-1)\kappa_1}, \quad \left|\frac{a_2}{S}\right| \le \frac{1}{2|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{\kappa_3}{2\zeta([2]_q-1)\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{1}{2|\zeta|[3]_q([3]_q-1)\kappa_2}\max\left\{1;\left|\frac{([2]_q^2([2]_q-1)\kappa_3-\mu[3]_q([3]_q-1)\kappa_2)}{2\zeta[2]_q^2([2]_q-1)^2\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{1}{2|\zeta|[4]_q([4]_q-1)\kappa_4}\max\left\{1;\left|\xi\left[\frac{\kappa_3}{\zeta([2]_q-1)\kappa_1^2}+\frac{2\eta}{\xi}+\frac{23}{12\xi}\right]\right|\right\},$$

where $\zeta = 1+i\tan\beta$, $\eta = \frac{\kappa_6}{8\zeta^3([2]_q-1)^2\kappa_1^3}$, and $\xi = \frac{(([2]_q-1)+([3]_q-1))\kappa_5}{4\zeta^2([2]_q-1)([3]_q-1)\kappa_1\kappa_2}$.

Considering the special case $p = 1$, $\varrho \equiv 1$, $\vartheta := \varpi$ and $q \to 1^-$ in Theorem 2.2 we obtain:

Corollary 2.17. *Let $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and ϖ be defined by* (8)*. If $f \in \hat{\psi}C_{\beta,\gamma}(\varpi)$ is given by* (1)*, then*

$$\left|\frac{a_1}{S}\right| \le \frac{|\gamma|}{4|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \le \frac{|\gamma|}{12|\zeta|\kappa_2}\max\left\{1;\left|\frac{\gamma\kappa_3}{2\zeta\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{|\gamma|}{12|\zeta|\kappa_2}\max\left\{1;\left|\frac{2\gamma\kappa_3-3\mu\kappa_2)}{4\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{|\gamma|}{24|\zeta|\kappa_4}\max\left\{1;\left|\xi\left[\frac{\gamma\kappa_3}{\zeta\kappa_1^2}+\frac{2\eta}{\xi}+\frac{23}{12\xi}\right]\right|\right\},$$

where $\zeta = 1+i\tan\beta$, $\eta = \frac{\gamma^3\kappa_6}{8\zeta^3\kappa_1^3}$, and $\xi = \frac{3\gamma^2\kappa_5}{8\zeta^2\kappa_1\kappa_2}$.

If we take $\gamma = 1$, $p = 1$, $\varrho \equiv 1$, $\vartheta := \varpi$ and $q \to 1^-$ in Theorem 2.2 we get the next special case:

Corollary 2.18. *Let $\gamma \in \mathbb{C}^*$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and ϖ be defined by* (8)*. If $f \in \hat{\psi}C_{\beta}(\varpi)$ is given by* (1)*, then*

$$\left|\frac{a_1}{S}\right| \le \frac{1}{4|\zeta|\kappa_1}, \quad \left|\frac{a_2}{S}\right| \le \frac{1}{12|\zeta|\kappa_2}\max\left\{1;\left|\frac{\kappa_3}{2\zeta\kappa_1^2}\right|\right\},$$

$$\left|\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\right| \le \frac{1}{12|\zeta|\kappa_2}\max\left\{1;\left|\frac{2\kappa_3-3\mu\kappa_2)}{4\zeta\kappa_1^2}\right|\right\},$$

and

$$\left|\frac{a_3}{S}\right| \le \frac{1}{24|\zeta|\kappa_4}\max\left\{1;\left|\xi\left[\frac{\kappa_3}{\zeta\kappa_1^2}+\frac{2\eta}{\xi}+\frac{23}{12\xi}\right]\right|\right\},$$

where $\zeta = 1+i\tan\beta$, $\eta = \frac{\kappa_6}{8\zeta^3\kappa_1^3}$, and $\xi = \frac{3\kappa_5}{8\zeta^2\kappa_1\kappa_2}$.

Concluding Remarks

In our present investigation we have made use of the (p, q)–Jackson derivative to introduce and investigate the new classes $\psi_\varrho S^{p,q}_{\beta,\gamma}(\varpi)$ and $\hat{\psi}_\varrho C^{p,q}_{\beta,\gamma}(\varpi)$ of analytic functions in the open unit disk $\mathbb{D}$. Using the subordination principle we have obtained the bounds of the three first coefficients for the functions belonging to these classes.

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