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A FITTED NUMERICAL TECHNIQUE FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. In this paper, we present a fitted numerical scheme for singularly perturbed delay differential equations with integral boundary conditions. To develop the scheme, the exact and approximate rules of integration with finite difference approximations of the first derivative are used. In the developed scheme, a fitting factor is introduced whose value is evaluated from the theory of singular perturbation. The Runge–Kutta method of order four is used to solve the reduced problem, and for the integral boundary condition, Composite Simpson's rule of integration is applied. The proposed method is shown to be second-order convergent. Numerical illustrations for various values of perturbation parameters are presented to validate the proposed method. The numerical results clearly show the high accuracy and order of convergence of the proposed scheme as compared to some of the results available in the literature.

Keywords: Singular perturbation, delay differential equations, integral boundary, numerical integration

AMS Subject Classification: 65L10, 65L11.

1. Introduction

In real-world models, we frequently run into problems with small parameters multiplying the highest-order derivative terms, involving at least one shift term; we call them singularly perturbed delay differential equations. Such problems arise frequently in the mathematical modeling of diverse physical and biological phenomena. The delay terms in the models enable us to include some past behavior to get more realistic models for the field. The delay or lag represents the incubation period, gestation time, transport delays, etc.[1].

Because of the appearance of small perturbation parameter, the solution of these problems have a multi-scale character, such as there are thin transition layer(s) where the solution varies very rapidly, while the solution behaves regularly and varies slowly away from the layer(s). This leads to boundary and/or interior layer(s) in the solution of the

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problems. For more details on the analytical as well as the numerical solution of these problems, one can refer [2, 3, 4]. Because of the presence of boundary and/or interior layers in the solution of singularly perturbed differential equations, standard numerical methods for solving such problems can be difficult and fail to produce accurate results for small values of the perturbation parameter ε. As a result, appropriate numerical methods that are uniformly convergent with respect to ε must be developed.

Boundary value problems with integral boundary conditions in which a small parameter multiplies the highest order derivative are called singularly perturbed problems with integral boundary conditions. Such types of problems constitute a very interesting and important class of problems. These problems have been used to describe many phenomena in the area of science and engineering, such as heat conduction, underground water, oceanography, population dynamics, cellular systems, meteorology, plasma physics, and the blood flow models, etc.,for further applications, one can refer to [5, 6, 7] and the references therein. In addition to this, for a discussion of existence and uniqueness results and for further applications of problems with integral boundary conditions see [8, 9, 10, 11] and the references therein. From these problems, if at least one delay term is present, we call it a singularly perturbed delay differential equation(SPDDE) with integral boundary condition.

There is a lot of literature available on studies of singularly perturbed differential equations with non-integral initial or boundary conditions. A number of scholars have studied the numerical solutions to singularly perturbed differential equations with non-integral initial or boundary conditions. Among those are some recent studies, such as [12, 13, 14, 15, 16]. However, studies of singularly perturbed second order delay differential equations with integral boundary conditions have not been satisfactorily studied. These are some studies conducted on SPDDE with integral boundary condition. In [17], the authors presented a finite difference scheme with an appropriate piecewise Shishkin type mesh, which is a first order convergent scheme. For a similar problem, in [18], the authors have given an accelerated fitted finite difference method on uniform mesh and a Richardson extrapolation technique has been used to enhance the order of convergence from first order convergent to second order convergent. Also in [7], the authors suggest a scheme based on the basis of B-spline functions on a piecewise uniform to the problem stated in [17].

Motivated by the above work, we have designed an ε-uniform numerical scheme for SPDDE with integral boundary condition, using the exact and approximate rule of integration with finite difference approximations of the first derivative. In the developed scheme, a fitting factor is introduced whose value is evaluated from the theory of singular perturbation. The Runge–Kutta method of order four is used to solve the reduced problem, and for the integral boundary condition, Composite Simpson's rule of integration is applied.

Notation : Throughout the analysis, C is generic positive constant which is independent of the perturbation parameter ε and number of mesh points N. We assume that $\overline{\Omega} = [0,2], \Omega = (0,2), \Omega_1 = (0,1), \Omega_2 = (1,2)$, $\Omega^* = \Omega_1 \cup \Omega_2$, C^k is the set of k times continously differentiable function in $\Omega, ||u||_\Omega = \underset{x\in\Omega}{Sup}|u(x)|$, K is differential operator and, $\mathcal{L}, \mathcal{L}_1$ and $\mathcal{L}_2$ are the linear operator associated to the domain Ω, Ω_1 and Ω_2, respectively.

2. Statement of the problem

We consider the following SPDDE with integral boundary condition [17],

$$\begin{aligned} &-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), x \in \Omega, \\ &u(x) = \phi(x), x \in [-1, 0], \\ &Ku(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = l, \end{aligned} \tag{1}$$

where $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \overline{\Omega}$, it is assumed that sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy $a(x) \geq \alpha > 0, b(x) \geq \beta \geq 0, c(x) \leq \gamma \leq 0$ and $\beta + \gamma \geq 0$.

Furthermore, $g(x)$ is non negative and monotonic with $\int_0^2 g(x)dx < 1$. The above assumptions ensure that $u \in X = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$ [17].

When the delay term is $o(\varepsilon)$, most of the numerical algorithms use Taylor series approximation for a priori estimation of the retarded term but this fails when the delay term is $O(\varepsilon)$.

The problem (1) is equivalent to

$$\mathcal{L}u(x) = F(x), \tag{2}$$

where

$$\mathcal{L}u(x) = \begin{cases} \mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x), & x \in \Omega_1, \\ \mathcal{L}_2 u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) & \\ \qquad\qquad + c(x)u(x-1), & x \in \Omega_2. \end{cases} \tag{3}$$

$$F(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2, \end{cases} \tag{4}$$

with boundary conditions

$$u(x) = \phi(x), x \in [-1, 0],$$

$$u(1^-) = u(1^+), u'(1^-) = u'(1^+),$$

$$Ku(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = l.$$

In general, for ε near 0, the solution of the problem (1) has an interior layer and a boundary layer [7].

3. Analytical Results

Now, we establish some asymptotic estimates for $u(x)$ which are used for the analysis of the parameter-uniform convergence.

Lemma 3.1 (Maximum Principle). *Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, K\psi(2) \geq 0, \mathcal{L}_1\psi(x) \geq 0, \forall x \in \Omega_1, \mathcal{L}_2\psi(x) \geq 0, \forall x \in \Omega_2,$ and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0, \forall x \in \overline{\Omega}$.*

Proof. Define a test function

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, x \in [0, 1], \\ \frac{3}{8} + \frac{x}{4}, x \in [1, 2]. \end{cases} \tag{5}$$

Note that $s(x) > 0, \forall x \in \overline{\Omega}, \mathcal{L}s(x) > 0, \forall x \in (\Omega_1 \cup \Omega_2), s(0) > 0, Ks(2) > 0$ and $[s'](1) < 0$. Let

$$\mu = \max\left\{ \frac{-\psi(x)}{s(x)} : x \in \overline{\Omega} \right\}.$$

Then, there exists $x_0 \in \overline{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0, \forall x \in \overline{\Omega}$. Therefore, the function $\psi(x) + \mu s(x)$ attains its minimum at $x = x_0$. It is easy to observe that for each $x \in \overline{\Omega}, \psi(x) \geq 0$ if $\mu \leq 0$. Suppose the theorem does not hold true, then $\mu > 0$.

Case-1 $x_0 = 0$,

$$(\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0,$$

this contradict with $(\psi + \mu s)(0) > 0$, since $\psi(0) \geq 0, s(0) > 0$ and $\mu > 0$.

Case-2 $x_0 \in \Omega_1$,

$$\mathcal{L}_1(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \leq 0,$$

this contradict with $\mathcal{L}_1(\psi + \mu s)(x_0) = \mathcal{L}_1\psi(x_0) + \mu\mathcal{L}_1 s(x_0) > 0$, since $\mathcal{L}_1\psi(x_0) \geq 0$ and $\mathcal{L}_1 s(x_0) > 0$ and $\mu > 0$.

Case-3 $x_0 = 1$,

$$[(\psi + \mu s)'](1) \geq 0,$$

this contradict with $[(\psi + \mu s)'](1) < 0$,since $[\psi'](1) \leq 0, [s'](1) < 0$ and $\mu > 0$.

Case-4 $x_0 \in \Omega_2$,

$$\begin{aligned} \mathcal{L}(\psi + \mu s)(x_0) &= -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) \\ &+ b(x_0)(\psi + \mu s)(x_0) + c(x_0)(\psi + \mu s)(x_0 - 1) \leq 0, \end{aligned}$$

this contradict with $\mathcal{L}_1(\psi + \mu s)(x_0) = \mathcal{L}_1\psi(x_0) + \mu\mathcal{L}_1 s(x_0) > 0$, since $\mathcal{L}_1\psi(x_0) \geq 0$ and $\mathcal{L}_1 s(x_0) > 0$ and $\mu > 0$.

Case-5 $x_0 = 2$,

$$(\psi + \mu s)(2) - \varepsilon\int_0^2 g(x)(\psi + \mu s)(x)dx \leq 0,$$

this contradict with $K(\psi + \mu s)(2) > 0$, since $K\psi(2) \geq 0$, $Ks(2) \geq 0$ and $\mu > 0$. It is clear that we arrived at a contradiction in all cases. Therefore, $\mu > 0$ is not possible. □

Lemma 3.2 (Stability result)**.** *The solution $u(x)$ of the problem(1), satisfies the bound:*

$$||u(x)|| \leq C_1 \max\left\{ |u(0)|, |Ku(2)|, \sup_{x \in \Omega^*} |\mathcal{L}u(x)| \right\}, x \in \overline{\Omega}.$$

Proof. This theorem can be proved by using (Lemma 3.1) and the barrier function

$$\varphi^{\pm} = CMs(x) \pm u(x), x \in \Omega, \text{ where } M = \max\left\{ |u(0)|, |Ku(2)|, \sup_{x \in \Omega^*} |\mathcal{L}u(x)| \right\},$$

$s(x)$ is the test function as in (Lemma 3.1). □

Lemma 3.3. *The derivatives of the solution $u(x)$ of (1) satisfies the following bound:*

$$||u^{(k)}(x)|| \leq C\varepsilon^{-k}, \text{ for } k = 1, 2, 3.$$

Proof. Let $x \in (0, 1)$ and let $I = (c, c + \eta)$ be a neighborhood of x, where c be a positive constant chosen so that $x \in I$ and $I \subset (0, 1)$, and $\eta > \varepsilon$ is a constant then by mean value theorem there exist a point $\zeta \in I$, such that

$$u'(\zeta) = \frac{u(c + \eta) - u(c)}{\eta}.$$

Using the fact that the maximum norm of a function is always greater than the value of the function over the domain. From this we can get the relation

$$\eta||u'(\zeta)|| \leq 2||u||. \tag{6}$$

To bound $u'(x)$ on the interval Ω_1, we consider

$$\mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) - c(x)\phi(x-1). \tag{7}$$

Integrating equation(7) from ζ to x on both sides, we have

$$\begin{aligned} -\varepsilon(u'(x) - u'(\zeta)) = -[a(x)u(x) - a(\zeta)u(\zeta)] + \int_\zeta^x a'(x)u(x)d(x) \\ - \int_\zeta^x b(x)u(x)dx + \int_\zeta^x [f(x) - c(x)\phi(x-1)]dx. \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned} \varepsilon|u'(x)| \leq \varepsilon|u'(\zeta)| + (2||a|| + ||a'||\ |x-\zeta|)||u|| \\ + ||b||\ ||u||\ |x-\zeta| + ||f(x) - c(x)\phi(x-1)||\ |x-\zeta|. \end{aligned} \tag{9}$$

Taking the inequality (6) and using the fact $|x-\zeta| < \eta$, and Lemma 3.2, then from equation (9) and Lemma 3.2 we obtain

$$||u'(x)|| \leq C\varepsilon^{-1},$$

where C is a positive constant independent of ε,such that

$$C = (2+2||a||+||a'||+\eta||b||)(C_1 \max\left\{|u(0)|, |Ku(2)|, \sup_{x\in\Omega^*}|\mathcal{L}u(x)|\right\}) + \eta||f(x) - c(x)\phi(x-1)||.$$

By similar argument we can bound $u'(x)$ on Ω_2. The bound for $||u''||$ and $||u'''||$ can be obtained similarly. □

Theorem 3.1. *Let $u(x)$ be the solution of the problem (1) and $v_0(x)$ be the solution of the reduced problem. Then*

$$|u(x) - u_0(x)| \leq C\left(\varepsilon + exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right), x \in \overline{\Omega}.$$

Proof. For the proof, refer to [17]. □

4. Description of the method

Consider equation (2) on the interval **[0,1],**

$$\begin{aligned} -\varepsilon u'' + a(x)u'(x) + b(x)u(x) = F(x), \\ u(0) = \phi(0), u(1) = \gamma, \end{aligned} \tag{10}$$

where γ is evaluated from the reduced problem using Runge-Kutta method.
The solution of equation (10) is of the form [2]

$$u(x) = u_0(x) + \frac{a(1)}{a(x)}(\gamma - u_0(1))e^{\int_x^1(\frac{-a(x)}{\varepsilon} - \frac{b(x)}{a(x)})dx} + o(\varepsilon), \tag{11}$$

where $u_0(x)$ represents the solution of the reduced problem

$$a(x)u'(x) + b(x)u(x) = F(x).$$

Expanding $a(x)$ and $b(x)$ in (11) with the help of the Taylor's series about the point $x = 1$ and restricting to their first terms, we obtain:

$$u(x) = u_0(x) + (\gamma - u_0(1))e^{\int_x^1 (\frac{-a(1)}{\varepsilon} - \frac{b(1)}{a(1)})dx} + o(\varepsilon), \tag{12}$$

Now we divide the interval $[0,1]$ in to N equal subintervals of mesh size $h = 1/N$ so that $x = x_i = ih, i = 0,1,2,...,N$. Then equation (12) gives

$$u(x_i) = u_0(x_i) + (\gamma - u_0(1))e^{(\frac{-a^2(1)-\varepsilon b(1)}{\varepsilon a(1)})(1-x_i)} + o(\varepsilon), \tag{13}$$

And taking the limit as $h \to 0$, we obtain

$$\lim_{h\to 0} u(ih) = u_0(0) + (\gamma - u_0(1))e^{(\frac{-a^2(1)-\varepsilon b(1)}{a(1)})(\frac{1}{\varepsilon} - i\rho)} + o(\varepsilon), \tag{14}$$

where $\rho = h/\varepsilon$.
Rearranging equation(10) in the form

$$-\varepsilon u'' + (a(x)u(x))' - a(x)'u(x) + b(x)u(x) = F(x). \tag{15}$$

And using the usual rule of exact and trapezoidal rule of integration over $[x_{i-1}, x_i]$ for $(i = 1,2,...,N-1)$, then equation (15) becomes:

$$\varepsilon u'_{i-1} - \varepsilon u'_i + a(x_i)u_i - a(x_{i-1})u_{i-1} + \frac{h}{2}[R_i u_i + R_{i-1}u_{i-1}] = \frac{h}{2}[F_i + F_{i-1}], \tag{16}$$

where $R(x) = b(x) - a'(x)$.
Next approximating the first order derivative u'_i using central finite difference approximation

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2), \tag{17}$$

and u'_{i-1} using non-symmetric finite difference approximation [19]

$$u'_{i-1} = \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h} + O(h^2), \tag{18}$$

with notation $u(x_i) = u_i, R(x_i) = R_i, a(x_i) = a_i, etc.$
Substituting equation (17) and (18) in to equation (16), we obtain the difference scheme:

$$\begin{aligned} &\varepsilon(\frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h}) - \varepsilon(\frac{u_{i+1} - u_{i-1}}{2h}) + a_i u_i - a_{i-1}u_{i-1} \\ &+ \frac{h}{2}[R_i u_i + R_{i-1}u_{i-1}] = \frac{h}{2}[F_i + F_{i-1}], \text{ for } 1 \le i \le N-1. \end{aligned} \tag{19}$$

Introduce a fitting factor $\sigma_1(\rho)$ in the scheme (19), we get

$$\begin{aligned} &-\sigma_1(\rho)\varepsilon\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h}\right) + a_i u_i - a_{i-1}u_{i-1} + \frac{h}{2}[R_i u_i + R_{i-1}u_{i-1}] \\ &= \frac{h}{2}[F_i + F_{i-1}]. \end{aligned} \tag{20}$$

The fitting factor is to be determined in such a way that the solution of (20) converges uniformly to the solution of (10). In the evaluation of the limit as $h \to 0$ the equation (20) becomes:

$$\lim_{h\to 0}\left[-\varepsilon\sigma_1(\rho)\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h}\right) + a_i u_i - a_{i-1}u_{i-1}\right] = 0, \tag{21}$$

under the assumption that $\frac{h}{2}[R_iu_i + R_{i-1}u_{i-1}]$ and $\frac{h}{2}[V_i + V_{i-1}]$ are bounded. Substituting equation (14) in equation (21) and simplifying, we get a constant fitting factor,

$$\sigma_1(\rho) = \frac{a(0)\rho}{2}\left[\frac{1 - e^{\left(\frac{-a^2(1)-\varepsilon b(1)}{a(1)}\right)\rho}}{cosh\left(\left(\frac{-a^2(1)-\varepsilon b(1)}{a(1)}\right)\rho\right) - 1}\right]. \tag{22}$$

Applying the hyperbolic identity $sinh^2(\frac{x}{2}) = \frac{coshx-1}{2}$, with $e^{x/2}e^{-x/2} = 1, \quad e^x = e^{x/2}e^{x/2}$, in to equation (22) we get

$$\sigma_1(\rho) = \frac{-a(0)\rho}{2}\left[\frac{e^{\left(\frac{-a^2(1)-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}}}{sinh\left(\left(\frac{-a^2(1)-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)}\right]. \tag{23}$$

Remark 4.1. *Implementing a similar procedure on the interval* $[\mathbf{1}, \mathbf{2}]$ *we get,*

$$\sigma_2(\rho) = \frac{-a(1)\rho}{2}\left[\frac{e^{\left(\frac{-a^2(2)-\varepsilon b(2)}{a(2)}\right)\frac{\rho}{2}}}{sinh\left(\left(\frac{-a^2(2)-\varepsilon b(2)}{a(2)}\right)\frac{\rho}{2}\right)}\right]. \tag{24}$$

Now $\overline{\Omega}^{2N}$ is discretized as follows

$$\overline{\Omega}^{2N} = \{x_0 = 0 < x_1 < x_2, ..., x_N = 1 < x_2 < .. < x_{2N} = 2\},$$

and $\overline{\Omega}^k$ is the set of all mesh points from -1 to 0, with positive integer k where $k = N$,

$$\overline{\Omega}^k = \{x_k = -1 < x_{-k+1} <, ...x_{-1} < x_0 = 0\}.$$

Applying the fitting factors (23) and (24) on equation (20). Therefore, the required finite difference scheme becomes

$$-\sigma(\rho)\varepsilon\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h}\right) + a_iu_i - a_{i-1}u_{i-1} + \frac{h}{2}[R_iu_i + R_{i-1}u_{i-1}] = \frac{h}{2}[F_i + F_{i-1}], \tag{25}$$

where

$$F_i = \begin{cases} (f_i - c_i\phi(x_{i-k})), & \text{for } i = 1, 2, ..., N-1, \\ (f_i - c_iu_{i-k}), & \text{for } i = N+1, N+2, ..., 2N-1, \end{cases} \tag{26}$$

and

$$\sigma(\rho) = \begin{cases} \sigma_1(\rho), & x \in \Omega_1, \\ \sigma_2(\rho), & x \in \Omega_2. \end{cases}$$

Therefore, the developed scheme in each domain is expressed in the following algorithm.

4.1. **Numerical Algorithm. Step 1:** We obtain the reduced problem by setting $\varepsilon = 0$ in equation (1).

Let u_0 be the solution of the reduced problem of (1), i.e

$$\begin{aligned} &a(x)u_0' + b(x)u_0 + c(x)u_0(x-1) = f(x), x \in \Omega_1, \\ &u_0(x) = \phi(x), x \in [-1, 0], \end{aligned} \tag{27}$$

Approximate $u_0(1)$ using Runge-Kutta method and take $u_N = u_0(1) = \gamma$.

Step 2: To obtain the solution in the domain $\Omega_1 = (0, 1)$, the numerical scheme in (25) with (26) can be written in three term recurrence relations as follows

$$E_iu_{i-1} + D_iu_i + G_iu_{i+1} = H_i, \text{ for } i = 1, 2, 3, ..., N-1, \tag{28}$$

where

$$\begin{cases} E_i = -\frac{\sigma_1\varepsilon}{h} - a_{i-1} + \frac{h}{2}R_{i-1}, \\ D_i = \frac{2\sigma_1\varepsilon}{h} + a_i + \frac{h}{2}R_i, \\ G_i = -\frac{\sigma_1\varepsilon}{h}, \\ H_i = \frac{h}{2}\Big\{ f_i + f_{i-1} - \Big(c_i\phi(x_{i-N}) + c_{i-1}\phi(x_{i-1-N}) \Big) \Big\}, \\ \text{ for } i = 1, 2, ..., N-1. \end{cases}$$

Step 3: To obtain the solution in the domain $\Omega_2 = (1, 2)$, the numerical scheme in (25) with (26) can be written in three term recurrence relations as follows

$$E_i u_{i-1} + D_i u_i + G_{i+1} u_{i+1} = H_i, \text{ for } i = N+1, N+2, ..., 2N-1, \tag{29}$$

where

$$\begin{cases} E_i = -\frac{\sigma_2\varepsilon}{h} - a_{i-1} + \frac{h}{2}R_{i-1}, \\ D_i = \frac{2\sigma_2\varepsilon}{h} + a_i + \frac{h}{2}R_i, \\ G_i = -\frac{\sigma_2\varepsilon}{h}, \\ H_i = \frac{h}{2}[f_i + f_{i-1} - (c_i u_{i-N} + c_i u_{i-1-N})], \text{ for } i = N+1, N+2, ..., 2N-1. \end{cases}$$

Step 4: For i=2N (boundary condition), we apply composite simpson's rule approximation to the following integral

$$\int_0^2 g(x)u(x)dx \approx \frac{h}{3}\Big(g(0)u(0) + 2\sum_{i=1}^{k} g_{2i}u_{2i} + 4\sum_{i=1}^{k} g_{2i-1}u_{2i-1} + g(2)u(2) \Big). \tag{30}$$

Substituting equation (30) in to equation $u(2) - \varepsilon \int_0^2 g(x)udx = l$, we get

$$u(2) - \varepsilon\frac{h}{3}\Big(g(0)u(0) + 2\sum_{i=1}^{k} g_{2i}u_{2i} + 4\sum_{i=1}^{k} g_{2i-1}u_{2i-1} + g(2)u(2) \Big) \approx l,$$

after simplification, we obtain

$$u(2) \approx l + \varepsilon\frac{h}{3}\Big(g(0)u(0) + 2\sum_{i=1}^{k} g_{2i}u_{2i} + 4\sum_{i=1}^{k} g_{2i-1}u_{2i-1} + g(2)u(2) \Big). \tag{31}$$

The resulting tri-diagonal systems (28) and (29) can be solved using Thomas algorithm.

5. Convergence analysis

The matrix-Vector form of the tridiagonal system (28) is

$$AU = B, \tag{32}$$

where $A = m_{i,j}, 1 \le i, j \le N-1$ is a tri-diagonal matrix of order N-1, with

$$\begin{aligned} m_{i,i-1} &= -\tfrac{\sigma_1\varepsilon}{h} - a_{i-1} + \tfrac{h}{2}R_{i-1}, \\ m_{i,i} &= \tfrac{2\sigma_1\varepsilon}{h} + a_i + \tfrac{h}{2}R_i, \\ m_{i,i+1} &= -\tfrac{\sigma_1\varepsilon}{h}, \\ B &= \tfrac{h}{2}\{f_i + f_{i-1} - (c_i\phi(x_{i-N}) + c_{i-1}\phi(x_{i-1-N}))\}, \end{aligned}$$

with local truncation error

$$\tau_i(h) = C(h^2), \tag{33}$$

we also have

$$A\overline{U} - \tau(h) = B, \tag{34}$$

where $\overline{U} = (\overline{U}_0, \overline{U}_1, \overline{U}_2, ..., \overline{U}_N)^T$ denotes the approximate solution, and $\tau(h) = (\tau_1(h), \tau_2(h), ..., \tau_N(h))^T$ is the local truncation error. From the equations (34) and (32), we get $A(\overline{U} - U) = \tau(h),$. Thus, we obtain the error equation

$$AE = \tau(h), \tag{35}$$

where $E = \overline{U} - U = (e_0, e_1, e_2, ..., e_N)^T$.
Let S_i be the sum of elements of the i^{th} row of A. Then, we have

$$\begin{cases} S_1 = \sum_{j=1}^{N-1} m_{1,j} = \dfrac{\sigma_1 \varepsilon}{h} + a_1 + \dfrac{h}{2} R_1, & \text{for } i = 1, \\ S_i = \sum_{j=1}^{N-1} m_{i,j} = a_i - a_{i-1} + \dfrac{h}{2}[R_i + R_{i-1}], & \text{for } i = 2(1)N-2, \\ S_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j} = \dfrac{\sigma_1 \varepsilon}{h} + a_{N-1} - a_{N-2} + \dfrac{h}{2}[R_{N-1} + R_{N-2}], & \text{for } i = N-1. \end{cases}$$

Since $0 < \varepsilon << 1$, for a given h, the matrix A is irreducible and monotone. Hence A^{-1}exists and it's elements are non-negative [20]. From the error equation(35)

$$E = A^{-1}\tau(h), \tag{36}$$

and

$$||E|| \leq ||A^{-1}|| \; ||\tau(h)||. \tag{37}$$

Let $\overline{m}_{k,i}$ be the $(k,i)^{th}$ elements of A^{-1}.Since $\overline{m}_{k,i} \geq 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} S_i = 1, \text{ for } k = 1, 2, ..., N-1. \tag{38}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \leq \frac{1}{\min\limits_{1 \leq i \leq N-1} S_i} = \frac{1}{Q_j} \leq \frac{1}{|Q_j|}, \text{ for some } j \text{ between 1 and } N-1, \tag{39}$$

where

$$Q_j = \begin{cases} \dfrac{\sigma_1 \varepsilon}{h} + a_1 + \dfrac{h}{2} R_1, & \text{for } i = 1, \\ a_i - a_{i-1} + \dfrac{h}{2}[R_i + R_{i-1}], & \text{for } i = 2(1)N-2, \\ \dfrac{\sigma_1 \varepsilon}{h} + a_{N-1} - a_{N-2} + \dfrac{h}{2}[R_{N-1} + R_{N-2}], & \text{for } i = N-1. \end{cases}$$

From matrix norm

$$||A^{-1}|| = \max_{1 \leq k \leq N-1} \sum_{i=1}^{N-1} |\overline{m}_{k,i}| \text{ and } ||T(h)|| = \max_{1 \leq i \leq N-1} |T_i(h)|.$$

From equations(35), (36), (38) and (39), we get $e_j = \sum_{i=1}^{N-1} \overline{m}_{k,i} \tau_i(h), \quad j = 1(1)N-1,$ which implies

$$e_j \leq \frac{C(h^2)}{|Q_j|}, \quad j = 1(1)N-1. \tag{40}$$

Therefore, from Eq.(40) we have,

$$||E|| = O(h^2). \tag{41}$$

This establishes that the developed scheme is second order accurate.

Remark 5.1. *Similar analysis for convergence can be carried out for the scheme (29).*

6. Numerical illustrations

The efficiency of the proposed method is tested by considering three different test problems. The maximum absolute errors (point-wise error) and numerical rate of convergence are calculated and tabulated. The numerical results of the present method are compared with the results of some numerical methods that are available in the literature. For those problems where the exact solution is not known the maximum absolute error is calculated using the double mesh principle. The formula for maximum absolute error and its associated rate of convergence are expressed, respectively.

$$E_\varepsilon^N = \max_{0\le i\le N}\left|U^N(x_i) - U^{2N}(x_{2i})\right|, \qquad R_\varepsilon^N = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right),$$

where $U^N(x_i)$ denote the numerical solution. The $\varepsilon-$uniform error and rate of convergence are computed by the formula

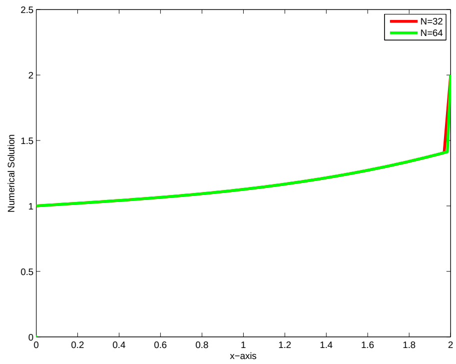
$$E^N = \max_\varepsilon E_\varepsilon^N \text{ and } R^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

Example 6.1. *Consider the following singularly perturbed delay differential equation with integral boundary condition* [17],

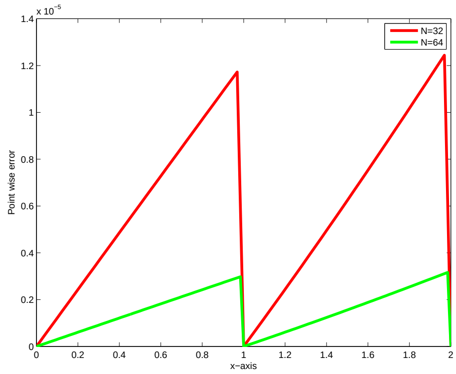
$$\begin{aligned}
&-\varepsilon u''(x) + (x+10)u'(x) - u(x-1) = x^2, \quad x\in(0,1)\cup(1,2),\\
&u(x) = 1,\ \textit{for } x\in[-1,0],\\
&u(2) - \varepsilon\int_0^2 \frac{x}{3}u(x)dx = 2.
\end{aligned}$$

Table 1. Computed E_ε^N, E^N and (R^N) for Example 6.1 at $2N$ number of mesh points

Method	$\varepsilon\downarrow$	$N\rightarrow 32$	64	128	256	512
present	2^{-10}	1.2442e-05	3.1636e-06	7.9754e-07	2.4655e-07	1.2325e-07
	2^{-11}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	6.1585e-08
	2^{-12}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-13}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-14}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-15}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-16}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-17}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-18}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-19}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	2^{-20}	1.2442e-05	3.1636e-06	7.9754e-07	2.0022e-07	5.0158e-08
	$\mathbf{E^N}$	**1.2442e-05**	**3.1636e-06**	**7.9754e-07**	**2.0022e-07**	**5.0158e-08**
	$\mathbf{R^N}$	**1.9756**	**1.9879**	**1.9940**	**1.9970**	
[17]	$\mathbf{E^N}$	**6.3216e-03**	**2.9562e-03**	**1.4271e-03**	**7.0074e-04**	**3.4709e-04**
	$\mathbf{R^N}$	**1.0965**	**1.050**	**1.0262**	**1.0135**	



(A) Numerical solution



(B) Point wise error

FIGURE 1. Numerical solution and Point wise error for Example 6.1 at $\varepsilon = 2^{-12}$.

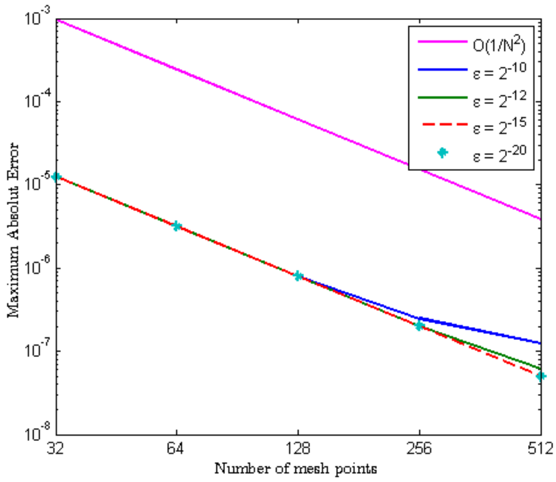


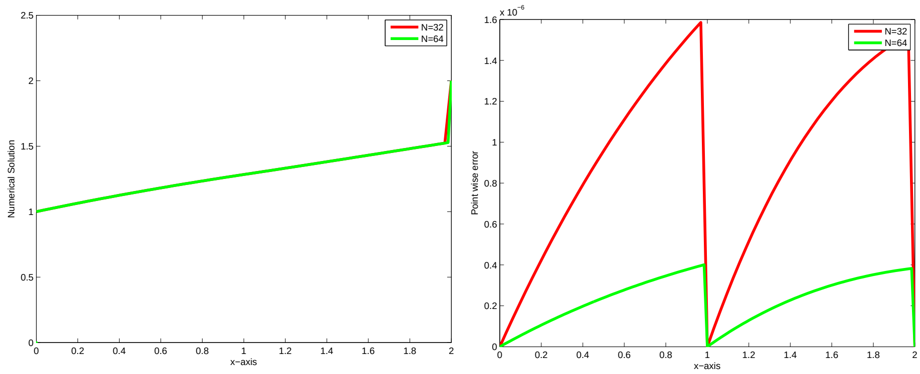
FIGURE 2. Loglog plot for Example 6.1

Example 6.2. *Consider the following singularly perturbed delay differential equation with integral boundary condition* [7, 18],

$$
\begin{aligned}
&-\varepsilon u''(x) + 3u'(x) + u(x) - u(x-1) = 1, x \in (0,1) \cup (1,2), \\
&u(x) = 1, x \in [-1, 0], \\
&u(2) - \varepsilon \int_0^2 \frac{x}{3} u(x) dx = 2.
\end{aligned}
$$

TABLE 2. Computed E_ε^N, E^N and (R^N) for Example 6.2 at $2N$ number of mesh points.

Method	$\varepsilon \downarrow$	**N →32**	**64**	**128**	**256**	**512**
present	10^{-4}	1.5856e-06	4.0070e-07	1.0071e-07	2.5243e-08	1.0175e-08
		1.9845	1.9924	1.9962	1.3109	
	10^{-8}	1.5856e-06	4.0070e-07	1.0071e-07	2.5243e-08	6.3190e-09
		1.9845	1.9924	1.9962	1.9981	
	10^{-12}	1.5856e-06	4.0070e-07	1.0071e-07	2.5243e-08	6.3190e-09
		1.9845	1.9924	1.9962	1.9981	
	10^{-16}	1.5856e-06	4.0070e-07	1.0071e-07	2.5243e-08	6.3190e-09
		1.9845	1.9924	1.9962	1.9981	
	10^{-20}	1.5856e-06	4.0070e-07	1.0071e-07	2.5243e-08	6.3190e-09
		1.9845	1.9924	1.9962	1.9981	
	$\mathbf{E^N}$	**1.5856e-06**	**4.0070e-07**	**1.0071e-07**	**2.5243e-08**	**6.3190e-09**
	$\mathbf{R^N}$	**1.9845**	**1.9924**	**1.9962**	**1.9981**	
[18]	$\mathbf{E^N}$	**6.8161e-06**	**1.7125e-07**	**4.2918e-07**	**1.0743e-07**	**2.6988e-08**
	$\mathbf{R^N}$	**1.9928**	**1.9964**	**1.9982**	**1.9930**	



(A) Numerical solution (B) Point wise error

FIGURE 3. Numerical solution and Point wise error for Example 6.2 at $\varepsilon = 2^{-12}$.

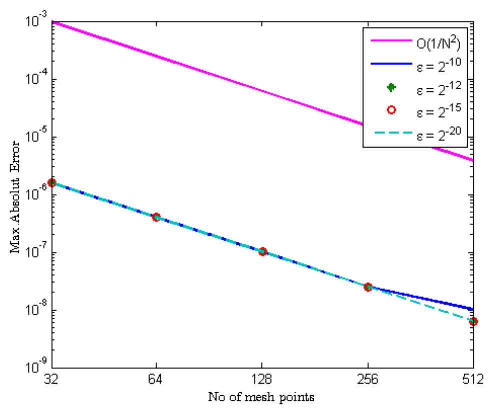


FIGURE 4. Loglog plot for Example 6.2

Example 6.3. *Consider the following singularly perturbed delay differential equation with integral boundary condition* [18, 17],

$$
\begin{aligned}
&-\varepsilon u''(x)+5u'(x)+(x+1)u(x)-u(x-1)=x^2, x\in(0,1)\cup(1,2),\\
&u(x)=1, x\in[-1,0],\\
&u(2)-\varepsilon\int_0^2 \frac{x}{3}u(x)dx=2.
\end{aligned}
$$

TABLE 3. Computed E_ε^N, E^N and R^N for Example 3 at $2N$ number of mesh points

Method	$\varepsilon\downarrow$	**N →32**	**64**	**128**	**256**	**512**
present	10^{-4}	2.0275e-05	5.1138e-06	1.2840e-06	3.2167e-07	8.0501e-08
		1.9872	1.9938	1.9970	1.9985	
	10^{-8}	2.0275e-05	5.1138e-06	1.2840e-06	3.2167e-07	8.0501e-08
		1.9872	1.9938	1.9970	1.9985	
	10^{-12}	2.0275e-05	5.1138e-06	1.2840e-06	3.2167e-07	8.0501e-08
		1.9872	1.9938	1.9970	1.9985	
	10^{-16}	2.0275e-05	5.1138e-06	1.2840e-06	3.2167e-07	8.0501e-08
		1.9872	1.9938	1.9970	1.9985	
	10^{-20}	2.0275e-05	5.1138e-06	1.2840e-06	3.2167e-07	8.0501e-08
		1.9872	1.9938	1.9970	1.9985	
	$\mathbf{E^N}$	**2.0275e-05**	**5.1138e-06**	**1.2840e-06**	**3.2167e-07**	**8.0501e-08**
	$\mathbf{R^N}$	**1.9872**	**1.9938**	**1.9970**	**1.9985**	
[18]	$\mathbf{E^N}$	**3.5556e-05**	**8.7577e-06**	**2.1733e-06**	**5.4134e-07**	**1.3509e-07**
	$\mathbf{R^N}$	**2.0215**	**2.0107**	**2.0053**	**2.0026**	

7. Discussion and Conclusion

We proposed and demonstrated an exponentially fitted numerical technique for SPDDE with an integral boundary condition. The error analysis of the scheme has been derived, and it is found that the present method is second order convergent, independent of the perturbation parameter. The numerical rate of convergence is also calculated. From the results clearly we see that the presented scheme is capable of producing highly accurate uniformly convergent solution for any fixed value of step size $h\geq\varepsilon$ when the perturbation parameter tends to zero. Furthermore, numerical experiments are carried out to demonstrate the performance of the proposed scheme. The numerical results clearly show the high accuracy and order of convergence of the proposed scheme as compared to some of the results available in the literature. The surface plot of Examples 6.1 and 6.2 in Figures 1a and 3a, respectively, clearly shows how the boundary layers behave as the perturbation parameters approach zero. Even though the interior layer is not visible in the surface plot of the numerical solution, it is possible to see this behavior in the point-wise error graphs 1b and 3b. In addition, Figures 2 and 4 show the log-log plot of the maximum point-wise error for Examples 6.1 and 6.2, respectively. From these, it is clear to see that the maximum absolute error decreases as the number of mesh points increases, demonstrating that the convergence is parameter-uniform. Lastly, the main feature of the proposed fitted scheme is that it does not depend on the very fine mesh size.

References

[1] Pramod Chakravarthy, P., Dinesh Kumar, S., Nageshwar Rao, R., Ghate, D. P., (2015), A fitted numerical scheme for second order singularly perturbed delay differential equations via cubic spline in compression, Advances in Difference Equations, 2015(1), pp.1–14.

[2] O'Malley, Jr., Robert, E., (1974), Introduction to singular perturbations, volume 14, Academic press.

[3] Miller, j. j., O'riordan, E., Shishkin, G. I., (1996), Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions, World scientific.

[4] Narasimhan, R., (2016), Singularly perturbed delay differential equations and numerical methods, In Differential Equations and Numerical Analysis, pp. 41–62, Springer.

[5] Nicoud, F., Schönfeld, T., (2002), Integral boundary conditions for unsteady biomedical cfd applications, International Journal for Numerical Methods in Fluids, 40(3-4), pp. 457–465.

[6] Ionkin, N.I. (1977), Solution of boundary value problem in heat conduction theory with nonclassical boundary conditions. Diff. Uravn, 13, pp. 1177–1182.

[7] Kumar, D., Kumari, P. (2020), A parameter-uniform collocation scheme for singularly perturbed delay problems with integral boundary condition. Journal of Applied Mathematics and Computing, 63(1), pp. 813–828.

[8] Jankowski, T. (2002), Differential equations with integral boundary conditions. Journal of Computational and Applied Mathematics, 147(1), pp. 1–8.

[9] Jankowski, T. (2003), Existence of solutions of boundary value problems for differential equations with delayed arguments, Journal of computational and applied mathematics, 156(1), 239–252.

[10] Samoilenko, A. M., Martynyuk, S. V., (1991), Justification of a numerical-analytic method of successive approximations for problems with integral boundary conditions, Ukrainian Mathematical Journal, 43(9), pp. 1150–1157.

[11] Samoilenko, A. M., Ronto, N. I., Martynyuk, S. V., (1991), On the numerical-analytic method for problems with integral boundary conditions, In Dokl. Akad. Nauk Ukrain. SSR, volume 4, pages 34–37.

[12] Ejere, A. H., Duressa, G. F., Woldaregay, M. M., Dinka, T.G., (2022), An exponentially fitted numerical scheme via domain decomposition for solving singularly perturbed differential equations with large negative shift. Journal of Mathematics, 2022, pp. 1–13.

[13] Gebeyaw, W., Andargie, A., Adamu, G., (2018), Solving second order singularly perturbed delay differential equations with layer behavior via initial value method. Journal of applied mathematics & informatics, 36(3_4), pp. 331–348.

[14] Melesse, W. G., Tiruneh, A. A., Derese, G. A., (2019), Solving linear second-order singularly perturbed differential difference equations via initial value method. International Journal of Differential Equations, 2019:1–10.

[15] Omkar, R. Phaneendra, K. (2022), Numerical simulation of singularly perturbed delay differential equations with large delay using an exponential spline. International Journal of Analysis and Applications, 20, pp. 63–63.

[16] Woldaregay, M. M., (2022), Solving singularly perturbed delay differential equations via fitted mesh and exact difference method, Research in Mathematics, 9(1):2109301.

[17] Sekar, E., Tamilselvan, A., (2019), Singularly perturbed delay differential equations of convection–diffusion type with integral boundary condition, Journal of Applied Mathematics and Computing, 59(1), pp. 701–722.

[18] Debela,H. G., Duressa, G. F., (2022), Accelerated fitted operator finite difference method for singularly perturbed delay differential equations with non-local boundary condition. Journal of the Egyptian Mathematical Society, 28, pp. 1–16.

[19] Soujanya, G. B. S. L., Phaneendra, K., Reddy, Y. N., (2013), An exponentially fitted non symmetric finite difference method for singular perturbation problems, WSEAS Transactions on Mathematics, 12(7).

[20] Mishra D., Sivakumar, K. C., (2012), Generalizations of matrix monotonicity and their relationships with certain subclasses of proper splittings, Linear algebra and its applications, 436(7), pp. 2604–2614.

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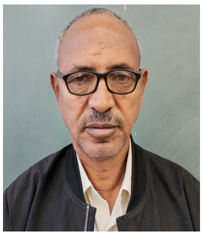

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