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HOMOMORPHIC PRODUCT OF SOFT DIRECTED GRAPHS

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ABSTRACT. A graph with directed edges is referred to be directed graph. It is possible to study and resolve problems with social connections, shortest paths, electrical circuits, etc. using directed graphs. D. Molodtsov proposed soft set theory as a mathematical framework for handling uncertain data. Nowadays, a lot of people employ soft set theory to solve decision-making problems. We present soft directed graphs by extending the notion of soft set to directed graphs. A parameterized perspective for directed graphs is provided by soft directed graphs. In this study, we look at various characteristics of soft directed graphs' homomorphic product and restricted homomorphic product.

Keywords: Soft Graph, Soft Directed Graph, Homomorphic Product.

AMS Subject Classification: 05C20, 05C76, 05C99

1. INTRODUCTION

Soft set theory was proposed by D. Molodtsov as a mathematical framework for dealing with uncertain data. Many academics are now applying soft set theory in decision-making problems. Authors like R. Biswas, P. K. Maji and A. R. Roy [10], [11] have delved deeper into the idea of soft sets and applied it to various decision-making situations. In 2014, R. K. Thumbakara and B. George [16] introduced the concept of soft graphs to provide a parameterized point of view for graphs. M. Akram and S. Nawas [1] updated R. K. Thumbakara and B. George's notion of the soft graph in 2015. They [2] also defined many varieties of soft graphs, such as regular soft graphs, soft trees, and soft bridges, as well as the notions of soft cut vertex, soft cycle and so on. More contributions to connected soft graphs came from J. D. Thenge, R. S. Jain and B. S. Reddy[13]. They [14] looked at the ideas of a soft graph's radius, diameter, and centre, as well as the concept of degree. They also addressed the notions of incidence and adjacency matrices of a soft graph in

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2020 [15]. B. George, R. K. Thumbakara and J. Jose [3],[5], [17] discussed some soft graph operations and introduced notions such as soft semigraphs and soft hypergraphs.

Directed graphs can be used to analyze and resolve problems with electrical circuits, project timelines, shortest routes, social links, and many other issues. J. Jose, B. George and R.K. Thumbakara [9] introduced the notion of the soft directed graph by applying the concepts of soft set in a directed graph. They also introduced the concepts of indegree, outdegree, degree, adjacency matrix and incidence matrix in soft directed graphs and investigated their properties. The directed graph product [7] is a binary operation on directed graphs. It is a process that takes two directed graphs, $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ and creates a directed graph D having the characteristics listed below: The vertex set of D is the Cartesian product $V_1 \times V_2$. Two vertices (v_1, v_2) and (v_1', v_2') of D are joined by an arc, if and only if some conditions about v_1, v_1' in D_1 and/or v_2, v_2' in D_2 are satisfied. Analogous to the definitions of directed graph products, we can define product operations in soft directed graphs. In [9], some product operations of soft directed graphs like the cartesian product, restricted cartesian product, lexicographic product and restricted lexicographic product are studied. B. George, J. Jose and R. K. Thumbakara [4] also introduced modular product and restricted modular product in soft directed graphs and investigated their properties. In this work, we introduce and study some of the features of homomorphic product and restricted homomorphic product of soft directed graphs.

2. Preliminaries

2.1. **Directed Graphs.** [6],[8] A *directed graph* or *digraph* D^* consists of a non-empty finite set V of elements called *vertices* and a finite set A of ordered pairs of distinct vertices called *arcs*. We often write $D^* = (V, A)$ to represent a directed graph. The number of vertices and arcs in a directed graph D^* are called *order* and *size* respectively. The first vertex u of an arc (u, v) is called its *tail* and the second vertex v is called its *head*. If (u, v) is an arc then v is *adjacent from* u and u is *adjacent to* v. A vertex u is *incident* to an arc a if u is the head or tail of a. A directed graph $D^{**} = (U, F)$ is called a *subdigraph* of $D^* = (V, A)$ if $U \subseteq V$ and $F \subseteq A$. The *in-degree* of a vertex v denoted by $ideg\ v$ is the number of vertices in D^* from which v is adjacent and *out-degree* of v denoted by $odeg\ v$ is the number of vertices in D^* to which v is adjacent. The sum $ideg\ v + odeg\ v$ is called the *degree* of the vertex v and is denoted by $deg\ v$. In a directed graph $D^* = (V, A)$, $\sum_{v\in V} ideg(v) = \sum_{v\in V} odeg(v) =$Number of arcs in D^* and $\sum_{v\in V} deg(v) = 2$(Number of arcs in D^*).

Some directed graph products can be defined in a manner that is similar to how the corresponding graph products are defined [7]. Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs. Their *homomorphic product* $D_1^* \ltimes D_2^*$ is a directed graph with vertex set $V(D_1^* \ltimes D_2^*) = V_1 \times V_2$ and arc set $A(D_1^* \ltimes D_2^*)$ where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D_1^* \ltimes D_2^*$ if and only if

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in D_1^* and (v_1', v_2') is not an arc in D_2^*.

2.2. **Soft Set.** [12] Let R be a set of parameters and U be an initial universe set. Then a pair (F, R) is called a *soft set* (over U) if and only F is a mapping of R into the power set of U. That is, $F : R \to \mathcal{P}(V)$.

2.3. **Soft Directed Graphs.** [9] Let $D^* = (V, A)$ be a directed graph having vertex set V and arc set A and let P be a non-empty set. Let a subset R of $P \times V$ be an arbitrary relation from P to V. Define a mapping $J : P \to \mathcal{P}(V)$ by $J(x) = \{u \in V | xRu\}$ where $\mathcal{P}(V)$ denotes the powerset of V. Define another mapping $L : P \to \mathcal{P}(A)$ by $L(x) =$

$\{(u,v) \in A | \{u,v\} \subseteq J(x)\}$ where $\mathcal{P}(A)$ denotes the powerset of E. Then $D = (D^*, J, L, P)$ is called a soft directed graph if it satisfies the following conditions:

(1) $D^* = (V, A)$ is a directed graph having vertex set V and arc set A,
(2) P is a nonempty set of parameters,
(3) (J, P) is a soft set over the vertex set V,
(4) (L, P) is a soft set over the arc set A,
(5) $(J(x), L(x))$ is a subdigraph of D^* for all $x \in P$.

If we represent $(J(x), L(x))$ by $M(x)$ then the soft directed graph D is also given by $\{M(x) : x \in P\}$. Then $M(x)$ corresponding to a parameter x in P is called a *directed part* or simply *dipart* of the soft directed graph D.

Let $D = (D^*, J, L, P)$ be a soft directed graph and let $M(x)$ be a dipart of D for some $x \in P$. Let v be a vertex of $M(x)$. Then dipart indegree of v in $M(x)$ denoted by *ideg* $v[M(x)]$ is defined as the number of vertices of $M(x)$ from which v is adjacent. That is, *ideg* $v[M(x)]$ is the number of arcs of $M(x)$ that have v as its head. Similarly, dipart outdegree of v in $M(x)$ denoted by *odeg* $v[M(x)]$ is defined as the number of vertices of $M(x)$ to which v is adjacent. That is, *odeg* $v[M(x)]$ is the number of arcs of $M(x)$ that have v as its tail. The dipart degree of v in $M(x)$ is defined as the sum, *ideg* $v[M(x)] +$ *odeg* $v[M(x)]$ and is denoted by *deg* $v[M(x)]$.

3. Homomorphic Product of Soft Directed Graphs

Definition 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D_2^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of the directed graphs D_1^* and D_2^* respectively. Then the homomorphic product of the soft directed graphs D_1 and D_2, which is represented by $D_1 \ltimes D_2$ is defined as $D_1 \ltimes D_2 = \{M_1(x_1) \ltimes M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. Here $M_1(x_1) \ltimes M_2(x_2)$ denotes the homomorphic product of the diparts $M_1(x)$ of D_1 and $M_2(y)$ of D_2 which is defined as follows: $M_1(x_1) \ltimes M_2(x_2)$ is a directed graph having set of vertices $V(M_1(x_1) \ltimes M_2(x_2)) = J_1(x_1) \times J_2(x_2)$ and set of arcs $A(M_1(x_1) \ltimes M_2(x_2))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x_1) \ltimes M_2(x_2)$ if and only if*

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in $M_1(x_1)$ and (v_1', v_2') is not an arc in $M_2(x_2)$.

Example 3.1. *Let $D_1^* = (V_1, A_1)$ be a directed graph which is shown in Fig. 1. Let $P_1 = \{v_6, v_3\} \subseteq V_1$ be a set of parameters. Define a mapping $J_1 : P_1 \to \mathcal{P}(V_1)$ by $J_1(x) = \{u \in V_1 \mid u = x$ or u is adjacent from $x\}, \forall x \in P_1$. That is, $J_1(v_6) = \{v_2, v_4, v_6\}$ and $J_1(v_3) = \{v_1, v_3, v_5\}$. Here (J_1, P_1) is a soft set over V_1. Define another mapping $L_1 : P_1 \to \mathcal{P}(A_1)$ by $L_1(x) = \{(u,v) \in A_1 \mid \{u,v\} \subseteq J_1(x)\}, \forall x \in P_1$. That is, $L_1(v_6) = \{(v_2, v_4), (v_6, v_2), (v_6, v_4)\}$ and $L_1(v_3) = \{(v_3, v_1), (v_3, v_5)\}$. Here, (L_1, P_1) is a soft set over A_1. Then $M_1(v_6) = (J_1(v_6), L_1(v_6))$ and $M_1(v_3) = (J_1(v_3), L_1(v_3))$ are subdigraphs of D_1^* as shown in Fig. 2. Therefore $D_1 = \{M_1(v_6), M_1(v_3)\}$ is a soft directed graph of D_1^*.*

Let $D_2^* = (V_2, A_2)$ be a directed graph which is shown in Fig. 3. Consider the parameter set $P_2 = \{u_4\} \subseteq V_2$. Define a mapping $J_2 : P_2 \to \mathcal{P}(V_2)$ by $J_2(x) = \{u \in V_2 \mid u = x$ or u is adjacent from $x\}, \forall x \in P_2$. That is, $J_2(u_4) = \{u_1, u_4\}$. Here, (J_2, P_2) is a soft set over V_2. Define another mapping $L_2 : P_2 \to \mathcal{P}(A_2)$ by $L_2(x) = \{(u,v) \in A_2 \mid \{u,v\} \subseteq J_2(x)\}, \forall x \in P_2$. That is, $L_2(u_4) = \{(u_4, u_1)\}$. Here, (L_2, P_2) is a soft set over A_2. Then, $M_2(u_4) = (J_2(u_4), L_2(u_4))$ is a subdigraph of D_2^* as shown in Fig. 4. Therefore, $D_2 = \{M_2(u_4)\}$ is a soft directed graph of D_2^*.

Then the homomorphic product of these two soft directed graphs D_1 and D_2 is given by

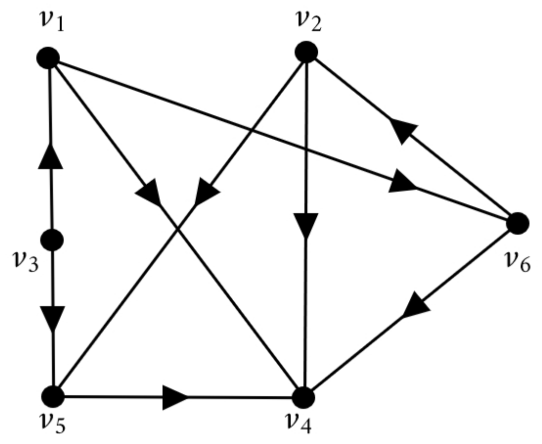

FIGURE 1. Directed Graph $D_1^* = (V_1, A_1)$

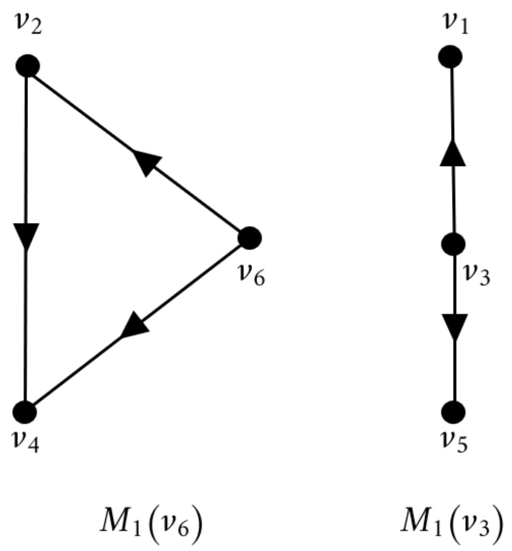

FIGURE 2. Soft Directed Graph $D_1 = \{M_1(v_6), M_1(v_3)\}$

$D = D_1 \ltimes D_2 = \{M_1(v_6) \ltimes M_2(u_4), M_1(v_3) \ltimes M_2(u_4)\}$ *and is shown in Fig.* 5.

Theorem 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and D_1 and D_2 be two soft directed graphs of D_1^* and D_2^* respectively. Then the homomorphic product of D_1 and D_2, which is represented by $D_1 \ltimes D_2$ is a soft directed graph of $D_1^* \ltimes D_2^*$.*

Proof. Let $D_1 = (D_1^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ be a soft directed graph of $D_1^* = (V_1, A_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be a soft directed graph of $D_2^* = (V_2, A_2)$. Then the homomorphic product $D_1 \ltimes D_2$ is defined as $D_1 \ltimes D_2 = \{M_1(x_1) \ltimes M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. Here $M_1(x_1) \ltimes M_2(x_2)$ denotes the homomorphic product of the diparts $M_1(x)$ of D_1 and $M_2(y)$ of D_2 which is defined as follows: $M_1(x_1) \ltimes M_2(x_2)$ is a directed graph having set of vertices $V(M_1(x_1) \ltimes M_2(x_2)) = J_1(x_1) \times J_2(x_2)$ and set

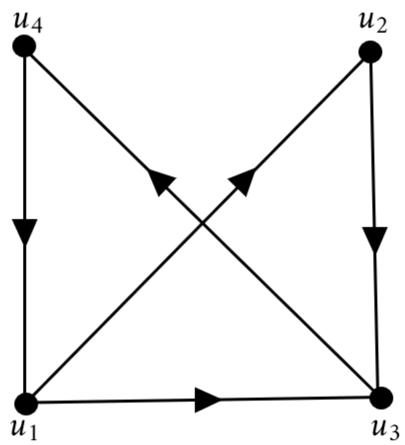

FIGURE 3. Directed Graph $D_2^* = (V_2, A_2)$

FIGURE 4. Soft Directed Graph $D_2 = \{M_2(u_4)\}$

of arcs $A(M_1(x_1) \ltimes M_2(x_2))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x_1) \ltimes M_2(x_2)$ if and only if

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in $M_1(x_1)$ and (v_1', v_2') is not an arc in $M_2(x_2)$.

The homomorphic product $D_1^* \ltimes D_2^*$ of the two directed graphs D_1 and D_2 is a directed graph having set of vertices $V(D_1^* \ltimes D_2^*) = V_1 \times V_2$ and set of arcs $A(D_1^* \ltimes D_2^*)$ where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D_1^* \ltimes D_2^*$ if and only if

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in D_1^* and (v_1', v_2') is not an arc in D_2^*.

Let the parameter set be $P_{D_1 \ltimes D_2} = P_1 \times P_2$. Define a mapping $J_{D_1 \ltimes D_2}$ from $P_{D_1 \ltimes D_2}$ to $\mathcal{P}[V(D_1^* \ltimes D_2^*)]$ by $J_{D_1 \ltimes D_2}(x_1, x_2) = J_1(x_1) \times J_2(x_2), \forall (x_1, x_2) \in P_1 \times P_2$ where $\mathcal{P}[V(D_1^* \ltimes D_2^*)]$ represents the power set of $V(D_1^* \ltimes D_2^*)$. Then $(J_{D_1 \ltimes D_2}, P_{D_1 \ltimes D_2})$ is a soft set over $V(D_1^* \ltimes D_2^*)$. Also, define another mapping $L_{D_1 \ltimes D_2}$ from $P_{D_1 \ltimes D_2}$ to

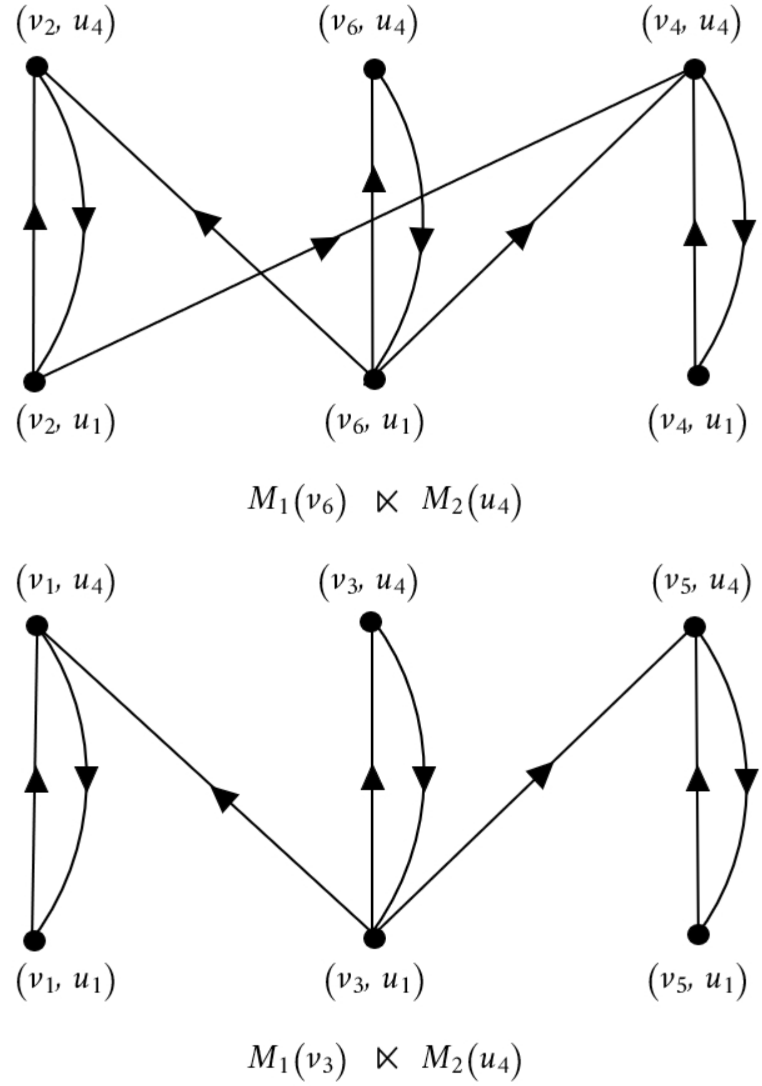

FIGURE 5. $D = D_1 \ltimes D_2 = \{M_1(v_6) \ltimes M_2(u_4), M_1(v_3) \ltimes M_2(u_4)\}$

$\mathcal{P}[A(D_1^* \ltimes D_2^*)]$ by $L_{D_1 \ltimes D_2}(x_1, x_2) = \{((u,v),(y,z)) \in A(D_1^* \ltimes D_2^*) \mid \{(u,v),(y,z)\} \in J_{D_1 \ltimes D_2}(x_1,x_2)\}, \forall (x_1,x_2) \in P_1 \times P_2$, where $\mathcal{P}[A(D_1^* \ltimes D_2^*)]$ represents the power set of $A(D_1^* \ltimes D_2^*)$. Then $(L_{D_1 \ltimes D_2}, P_{D_1 \ltimes D_2})$ is a soft set over $A(D_1^* \ltimes D_2^*)$. Also, if we denote $(J_{D_1 \ltimes D_2}(x_1,x_2), L_{D_1 \ltimes D_2}(x_1,x_2))$ by $M_{D_1 \ltimes D_2}(x_1,x_2)$, then $M_{D_1 \ltimes D_2}(x_1,x_2)$ is a subdigraph of $D_1^* \ltimes D_2^*, \forall (x_1,x_2) \in P_1 \times P_2$, since $J_1(x_1) \times J_2(x_2) \subseteq V_1 \times V_2$ and any arc in $L_{D_1 \ltimes D_2}(x_1,x_2)$ is also an arc in $A(D_1^* \ltimes D_2^*)$. Then $D_1 \ltimes D_2$ can be represented by the 4-tuple $(D_1^* \ltimes D_2^*, J_{D_1 \ltimes D_2}, L_{D_1 \ltimes D_2}, P_{D_1 \ltimes D_2})$ and also by $\{M_{D_1 \ltimes D_2}(x_1,x_2) : (x_1,x_2) \in P_1 \times P_2\}$ and $D_1 \ltimes D_2$ is a soft directed graph of $D_1^* \ltimes D_2^*$ since the conditions listed below are met:

(1) $D_1^* \ltimes D_2^* = (V(D_1^* \ltimes D_2^*), A(D_1^* \ltimes D_2^*))$ is a directed graph having set of vertices $V(D_1^* \ltimes D_2^*)$ and set of arcs $A(D_1^* \ltimes D_2^*)$,
(2) $P_{D_1 \ltimes D_2} = P_1 \times P_2 \neq \phi$ is the set of parameters,
(3) $(J_{D_1 \ltimes D_2}, P_{D_1 \ltimes D_2})$ is a soft set over $V(D_1^* \ltimes D_2^*)$,
(4) $(L_{D_1 \ltimes D_2}, P_{D_1 \ltimes D_2})$ is a soft set over $A(D_1^* \ltimes D_2^*)$,

(5) $M_{D_1 \ltimes D_2}(x_1, x_2) = (J_{D_1 \ltimes D_2}(x_1, x_2), L_{D_1 \ltimes D_2}(x_1, x_2))$ is a subdigraph of $D_1^* \ltimes D_2^*$, $\forall (x_1, x_2) \in P_{D_1 \ltimes D_2} = P_1 \times P_2$.

□

Remark 3.1. *In counting the number of vertices and arcs in various soft directed graph products, we count them as many times they appear in different diparts of the product.*

Theorem 3.2. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2)$ be two soft directed graphs of D_1^* and D_2^* respectively. Then the homomorphic product of D_1 and D_2, which is represented by $D_1 \ltimes D_2$ contains $\sum_{(x_i,x_j)\in P_1\times P_2} |J_1(x_i)||J_2(x_j)|$ vertices and $\sum_{(x_i,x_j)\in P_1\times P_2} (2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|])$ arcs, where $\binom{|J_2(x_j)|}{2}$ denotes the number of different combinations of vertices in $|J_2(x_j)|$ taking 2 at a time.*

Proof. By definition, $D_1 \ltimes D_2 = \{M_1(x_1) \ltimes M_2(x_2) : (x_1, x_2) \in P_1 \times P_2\}$. The parameter set of $D_1 \ltimes D_2$ is $P_1 \times P_2$. Consider the dipart $M_1(x_i) \ltimes M_2(x_j)$ of $D_1 \ltimes D_2$ corresponding to the parameter $(x_i, x_j) \in P_1 \times P_2$. The vertex set of $M_1(x_i) \ltimes M_2(x_j)$ is $J_1(x_i) \times J_2(x_j)$ which contains $|J_1(x_i)||J_2(x_j)|$ elements. This is a true statement for all diparts of $D_1 \ltimes D_2$. Therefore total count of vertices in $D_1 \ltimes D_2$ is $\sum_{(x_i,x_j)\in P_1\times P_2} |J_1(x_i)||J_2(x_j)|$. Also we know, $((v_q, v_r), (v_s, v_t))$ is an arc in $M_1(x_i) \ltimes M_2(x_j)$ if and only if

(1) $v_q = v_s$ or
(2) (v_q, v_s) is an arc in $M_1(x_i)$ and (v_r, v_t) is not an arc in $M_2(x_j)$.

Now, each arc in $M_1(x_i) \ltimes M_2(x_j)$ was made by just one of these two requirements and both of them can not be true at the same time. So to get the total count of arcs in $M_1(x_i) \ltimes M_2(x_j)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,$v_q = v_s$. Let v be any vertex in $M_1(x_i)$. The dipart $M_2(x_j)$ contains $|J_2(x_j)|$ vertices. We can choose 2 different vertices v' and v'' from $M_2(x_j)$ in $\binom{|J_2(x_j)|}{2}$ different ways. Corresponding to each choice we get two arcs $((v, v'), (v, v''))$ and $((v, v''), (v, v'))$ in $M_1(x_i) \ltimes M_2(x_j)$. Like v, there are totally $|J_1(x_i)|$ vertices in $M_1(x_i)$. Hence, the first condition of adjacency gives $2|J_1(x_i)|\binom{|J_2(x_j)|}{2}$ arcs in $M_1(x_i) \ltimes M_2(x_j)$. Now consider the second condition for adjacency, i.e., (v_q, v_s) is an arc in $M_1(x_i)$ and (v_r, v_t) is not an arc in $M_2(x_j)$. We can choose two different vertices v_q and v_s in $M_1(x_i)$ such that (v_q, v_s) is an arc in $M_1(x_i)$ in $|L_1(x_i)|$ different ways. Similarly we can choose two different vertices v_r and v_t in $M_2(x_j)$ such that (v_r, v_t) is not an arc in $M_2(x_j)$ in $(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|)$ different ways. Let v_q and v_s be two vertices in $M_1(x_i)$ such that (v_q, v_s) is an arc in $M_1(x_i)$ and let v_r and v_t be two vertices in $M_2(x_j)$ such that (v_r, v_t) is not an arc in $M_2(x_j)$. From this we get an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x_i) \ltimes M_2(x_j)$. Hence totally the second condition for adjacency gives $|L_1(x_i)|\,(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|)$ arcs in $M_1(x_i) \ltimes M_2(x_j)$. Hence, the total count of arcs in $M_1(x_i) \ltimes M_2(x_j)$ is $2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|(|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)|)$. This is a true statement for all diparts of $D_1 \ltimes D_2$. Therefore, total count of arcs in $D_1 \ltimes D_2$ is

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(2|J_1(x_i)| \binom{|J_2(x_j)|}{2} + |L_1(x_i)| \left[|J_2(x_j)|(|J_2(x_j)| - 1) - |L_2(x_j)| \right] \right)$$

. □

Corollary 3.1. *Let $D_1^* = (V_1, A_1)$ and $D_2^* = (V_2, A_2)$ be two directed graphs and $D_1 = (D_1^*, J_1, L_1, P_1)$ and $D_2 = (D_2^*, J_2, L_2, P_2)$ be two soft directed graphs of D_1^* and D_2^* respectively. Then*

$$(i) \sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]\right)$$

$$(ii) \sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} deg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(4|J_1(x_i)|\binom{|J_2(x_j)|}{2} + 2|L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]\right),$$

where $ideg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)], odeg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)]$ and $deg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)]$ denote the dipart in-degree, dipart out-degree and dipart degree respectively, of the vertex (u,v), in the dipart $M_{D_1\ltimes D_2}(x_i,x_j)$ of $D_1 \ltimes D_2$.

Proof. (i) Consider any dipart $M_{D_1\ltimes D_2}(x_i,x_j) = (J_{D_1\ltimes D_2}(x_i,x_j), L_{D_1\ltimes D_2}(x_i,x_j))$ of $D_1 \ltimes D_2$ which is given by $M_1(x_i) \times M_2(x_j)$. By *theorem* 3.2, we have number of arcs in $M_1(x_i) \times M_2(x_j)$ is $2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|]$.
Since the dipart $M_{D_1\ltimes D_2}(x_i,x_j)$ is a directed graph having $2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]$ arcs, we have

$$\sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\left(2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]\right),$$

since each arc in $M_{D_1\ltimes D_2}(x_i,x_j)$ contributes 1 each to the sums $\sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)]$ and $\sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)]$.
This is true for all the diparts $M_{D_1\ltimes D_2}(x_i,x_j)$ of $D_1 \ltimes D_2$. Hence,

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} ideg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} odeg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(2|J_1(x_i)|\binom{|J_2(x_j)|}{2} + |L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]\right).$$

(ii) Since $deg(u,v)[M_{D_1 \ltimes D_2}(x_i,x_j)] = ideg(u,v)[M_{D_1 \ltimes D_2}(x_i,x_j)] + odeg(u,v)[M_{D_1 \ltimes D_2}(x_i,x_j)]$ and by part (i) of this theorem we have,

$$\sum_{(x_i,x_j)\in P_1\times P_2} \sum_{(u,v)\in J_{D_1\ltimes D_2}(x_i,x_j)} deg(u,v)[M_{D_1\ltimes D_2}(x_i,x_j)] =$$

$$\sum_{(x_i,x_j)\in P_1\times P_2} \left(4|J_1(x_i)|\binom{|J_2(x_j)|}{2} + 2|L_1(x_i)|\left[|J_2(x_j)|(|J_2(x_j)|-1) - |L_2(x_j)|\right]\right).$$

□

4. Restricted Homomorphic Product of Soft Directed Graphs

Definition 4.1. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of D^* such that $P_1 \cap P_2 \neq \phi$. Then the restricted homomorphic product of D_1 and D_2, which is denoted by $D_1 \circledast D_2$ is defined as $D_1 \circledast D_2 = \{M_1(x) \ltimes M_2(x) : x \in P_1 \cap P_2\}$. Here $M_1(x) \ltimes M_2(x)$ denotes the homomorphic product of the diparts $M_1(x)$ of D_1 and $M_2(x)$ of D_2 which is defined as follows: $M_1(x) \ltimes M_2(x)$ is a directed graph having set of vertices $V(M_1(x) \ltimes M_2(x)) = J_1(x) \times J_2(x)$ and set of arcs $A(M_1(x) \ltimes M_2(x))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x) \ltimes M_2(x)$ if and only if*

(1) $v_1 = v_2$ or

(2) (v_1, v_2) is an arc in $M_1(x)$ and (v_1', v_2') is not an arc in $M_2(x)$.

Example 4.1. *Let $D^* = (V, A)$ be a directed graph which is shown in Fig. 6. Let*

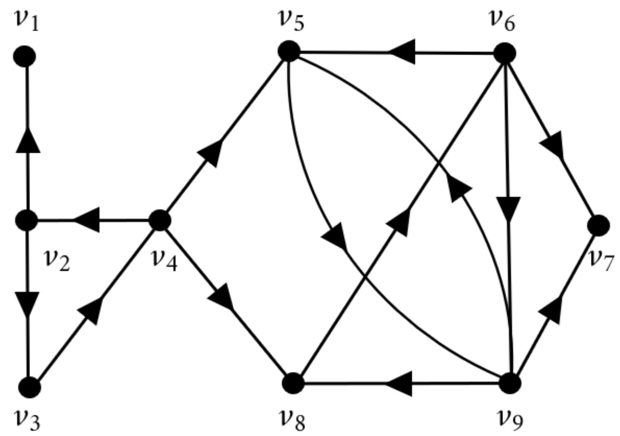

Figure 6. Directed Graph $D^* = (V, A)$

$P_1 = \{v_2, v_6\} \subseteq V$ be a set of parameters. Define a mapping $J_1 : P_1 \to \mathcal{P}(V)$ by $J_1(x) = \{u \in V \mid u = x$ or u is adjacent from x or u is adjacent to $x\}, \forall x \in P_1$. That is, $J_1(v_2) = \{v_1, v_2, v_3, v_4\}$ and $J_1(v_6) = \{v_5, v_6, v_7, v_8, v_9\}$. Here (J_1, P_1) is a soft set over V. Define another mapping $L_1 : P_1 \to \mathcal{P}(A)$ by $L_1(x) = \{(u,v) \in A \mid \{u,v\} \subseteq J_1(x)\}, \forall x \in P_1$. That is, $L_1(v_2) = \{(v_2,v_1),(v_2,v_3),(v_3,v_4),(v_4,v_2)\}$ and $L_1(v_6) = \{(v_6,v_5),(v_6,v_7),(v_6,v_9),(v_5,v_9),(v_9,v_5),(v_9,v_8),(v_8,v_6),(v_9,v_7)\}$. Here, (L_1, P_1) is a soft set over A. Then $M_1(v_2) = (J_1(v_2), L_1(v_2))$ and $M_1(v_6) = (J_1(v_6), L_1(v_6))$ are subdigraphs of D^* as shown in Fig. 7. Therefore $D_1 = \{M_1(v_2), M_1(v_6)\}$ is a soft directed graph of D^*.

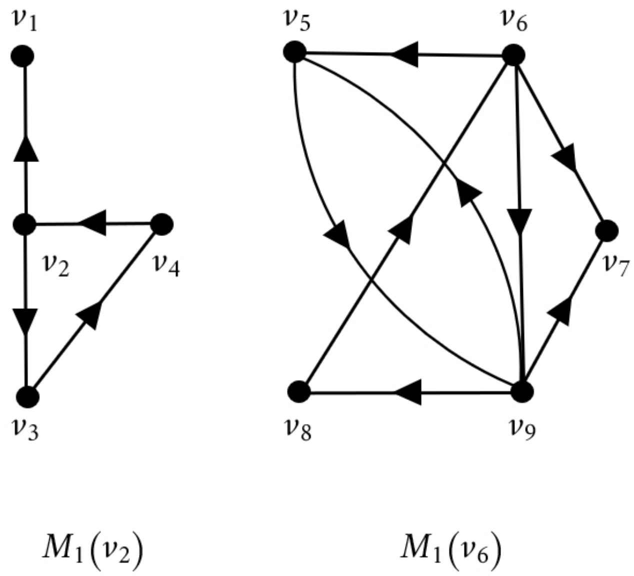

FIGURE 7. Soft Directed Graph $D_1 = \{M_1(v_2), M_1(v_6)\}$

Consider another parameter set $P_2 = \{v_2, v_9\} \subseteq V$. *Define a mapping* $J_2 : P_2 \to \mathcal{P}(V)$ *by* $J_2(x) = \{u \in V \mid u = x$ *or* u *is adjacent from* $x\}, \forall x \in P_2$. *That is,* $J_2(v_2) = \{v_1, v_2, v_3\}$ *and* $J_2(v_9) = \{v_5, v_7, v_8, v_9\}$. *Here,* (J_2, P_2) *is a soft set over* V. *Define another mapping* $L_2 : P_2 \to \mathcal{P}(A)$ *by* $L_2(x) = \{(u, v) \in A \mid \{u, v\} \subseteq J_2(x)\}, \forall x \in P_2$. *That is,* $L_2(v_2) = \{(v_2, v_1), (v_2, v_3)\}$ *and* $L_2(v_9) = \{(v_9, v_5), (v_5, v_9), (v_9, v_8), (v_9, v_7)\}$. *Here,* (L_2, P_2) *is a soft set over* A. *Then,* $M_2(v_2) = (J_2(v_2), L_2(v_2))$ *and* $M_2(v_9) = (J_2(v_9), L_2(v_9))$ *are subdigraphs of* D^* *as shown in Fig. 8. Therefore,* $D_2 = \{M_2(v_2), M_2(v_9)\}$ *is a soft directed graph of* D^*. *Then the restricted homomorphic product of these two soft directed graphs*

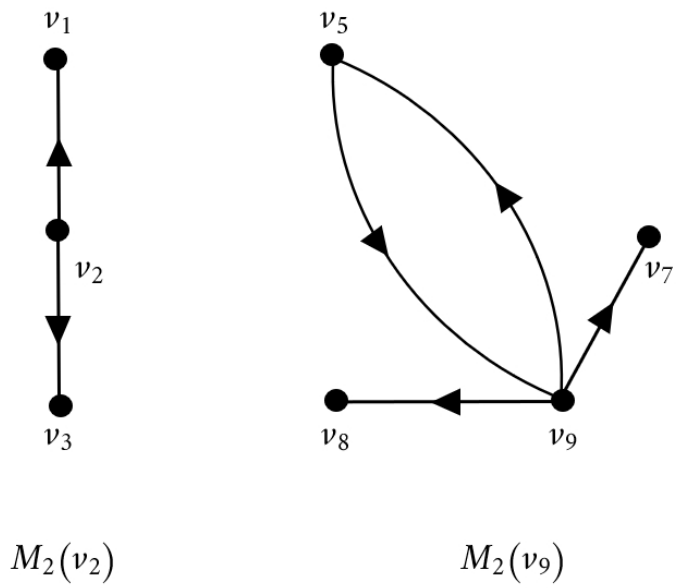

FIGURE 8. Soft Directed Graph $D_2 = \{M_2(v_2), M_2(v_9)\}$

D_1 and D_2 is given by $D = D_1 \circledast D_2 = \{M_1(v_2) \ltimes M_2(v_2)\}$ and is shown in Fig. 9.

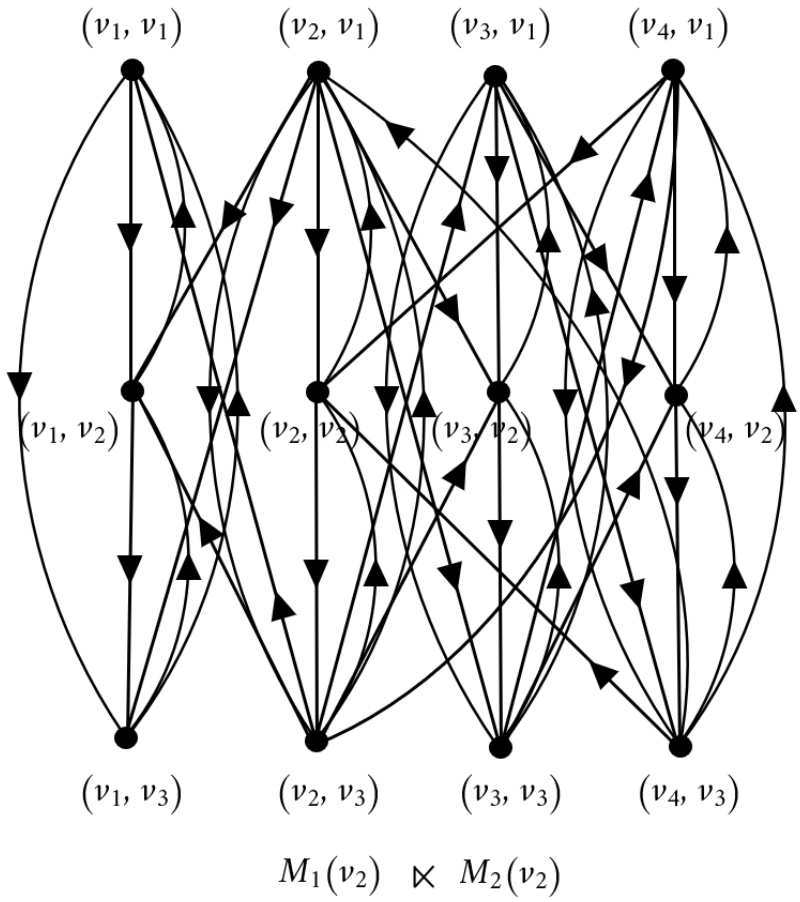

FIGURE 9. $D = D_1 \circledast D_2 = \{M_1(v_2) \ltimes M_2(v_2)\}$

Theorem 4.1. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be two soft directed graphs of D^* such that $P_1 \cap P_2 \neq \phi$. Then the restricted homomorphic product of D_1 and D_2, which is represented by $D_1 \circledast D_2$ is a soft directed graph of $D^* \ltimes D^*$.*

Proof. Let $D_1 = (D^*, J_1, L_1, P_1) = \{M_1(x) : x \in P_1\}$ and $D_2 = (D^*, J_2, L_2, P_2) = \{M_2(x) : x \in P_2\}$ be soft directed graphs of $D^* = (V, A)$ such that $P_1 \cap P_2 \neq \phi$. Then the restricted homomorphic product $D_1 \circledast D_2$ is defined as $D_1 \circledast D_2 = \{M_1(x) \ltimes M_2(x) : x \in P_1 \cap P_2\}$. Here $M_1(x) \ltimes M_2(x)$ denotes the homomorphic product of the diparts $M_1(x)$ of D_1 and $M_2(x)$ of D_2 which is defined as follows: $M_1(x) \ltimes M_2(x)$ is a directed graph having set of vertices $V(M_1(x) \ltimes M_2(x)) = J_1(x) \times J_2(x)$ and set of arcs $A(M_1(x) \ltimes M_2(x))$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $M_1(x) \ltimes M_2(x)$ if and only if

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in $M_1(x)$ and (v_1', v_2') is not an arc in $M_2(x)$.

The homomorphic product $D^* \ltimes D^*$ is a directed graph having set of vertices $V(D^* \ltimes D^*) = V \times V$ and set of arcs $A(D^* \ltimes D^*)$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $D^* \ltimes D^*$ if and only if

(1) $v_1 = v_2$ or
(2) (v_1, v_2) is an arc in D^* and (v_1', v_2') is not an arc in D^*.

Let the parameter set be $P_{D_1 \circledast D_2} = P_1 \cap P_2$. Define a mapping $J_{D_1 \circledast D_2}$ from $P_{D_1 \circledast D_2}$ to $\mathcal{P}[V(D^* \ltimes D^*)]$ by $J_{D_1 \circledast D_2}(x) = J_1(x) \times J_2(x), \forall x \in P_1 \cap P_2$ where $\mathcal{P}[V(D^* \ltimes D^*)]$ represents the power set of $V(D^* \ltimes D^*)$. Then $(J_{D_1 \circledast D_2}, P_{D_1 \circledast D_2})$ is a soft set over $V(D^* \ltimes D^*)$. Also define another mapping $L_{D_1 \circledast D_2}$ from $P_{D_1 \circledast D_2}$ to $\mathcal{P}[A(D^* \ltimes D^*)]$ by $L_{D_1 \circledast D_2}(x) = \{((u, v), (y, z)) \in A(D^* \ltimes D^*) \mid \{(u, v), (y, z)\} \in J_{D_1 \circledast D_2}(x)\}, \forall x \in P_1 \cap P_2$, where $\mathcal{P}[A(D^* \ltimes D^*)]$ represents the power set of $A(D^* \ltimes D^*)$. Then $(L_{D_1 \circledast D_2}, P_{D_1 \circledast D_2})$ is a soft set over $A(D^* \ltimes D^*)$. Also if we denote $(J_{D_1 \circledast D_2}(x), L_{D_1 \circledast D_2}(x))$ by $M_{D_1 \circledast D_2}(x)$, then $M_{D_1 \circledast D_2}(x)$ is a subdigraph of $D^* \ltimes D^*, \forall x \in P_1 \cap P_2$, since $J_1(x) \times J_2(x) \subseteq V \times V$ and any arc in $L_{D_1 \circledast D_2}(x)$ is also an arc in $A(D^* \ltimes D^*)$. Then $D_1 \circledast D_2$ can be represented by the 4-tuple $(D^* \ltimes D^*, J_{D_1 \circledast D_2}, L_{D_1 \circledast D_2}, P_{D_1 \circledast D_2})$ and also by $\{M_{D_1 \circledast D_2}(x) : x \in P_1 \cap P_2\}$ and $D_1 \circledast D_2$ is a soft directed graph of $D^* \ltimes D^*$ since the conditions listed below are met:

(1) $D^* \ltimes D^* = (V(D^* \ltimes D^*), A(D^* \ltimes D^*))$ is a directed graph having set of vertices $V(D^* \ltimes D^*)$ and set of arcs $A(D^* \ltimes D^*)$,
(2) $P_{D_1 \circledast D_2} = P_1 \cap P_2 \neq \phi$ is the set of parameters,
(3) $(J_{D_1 \circledast D_2}, P_{D_1 \circledast D_2})$ is a soft set over $V(D^* \ltimes D^*)$,
(4) $(L_{D_1 \circledast D_2}, P_{D_1 \circledast D_2})$ is a soft set over $A(D^* \ltimes D^*)$,
(5) $M_{D_1 \circledast D_2}(x) = (J_{D_1 \circledast D_2}(x), L_{D_1 \circledast D_2}(x))$ is a subdigraph of $D^* \ltimes D^*, \forall x \in P_{D_1 \circledast D_2} = P_1 \cap P_2$.

□

Theorem 4.2. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1)$ and $D_2 = (D^*, J_2, L_2, P_2)$ be two soft directed graphs of D^* such that $P_1 \cap P_2 \neq \phi$. Then their restricted homomorphic product $D_1 \circledast D_2$ contains $\sum_{x \in P_1 \cap P_2} |J_1(x)||J_2(x)|$ vertices and $\sum_{x \in P_1 \cap P_2} (2|J_1(x)|\binom{|J_2(x)|}{2} + |L_1(x)|[|J_2(x)|(|J_2(x)| - 1) - |L_2(x)|])$ arcs, where $\binom{|J_2(x)|}{2}$ denotes the number of different combinations of vertices in $|J_2(x)|$ taking 2 at a time.*

Proof. By definition, $D_1 \circledast D_2 = \{M_1(x) \ltimes M_2(x) : x \in P_1 \cap P_2\}$. The parameter set of $D_1 \circledast D_2$ is $P_1 \cap P_2$. Consider the dipart $M_1(x) \ltimes M_2(x)$ of $D_1 \circledast D_2$ corresponding to the parameter $x \in P_1 \cap P_2$. The vertex set of $M_1(x) \ltimes M_2(x)$ is $J_1(x) \times J_2(x)$ which contains $|J_1(x)||J_2(x)|$ elements. This is a true statement for all diparts of $D_1 \circledast D_2$. Therefore total count of vertices in $D_1 \circledast D_2$ is $\sum_{x \in P_1 \cap P_2} |J_1(x)||J_2(x)|$. Also we know, $((v_q, v_r), (v_s, v_t))$ is an arc in $M_1(x) \ltimes M_2(x)$ if and only if

(1) $v_q = v_s$ or
(2) (v_q, v_s) is an arc in $M_1(x)$ and (v_r, v_t) is not an arc in $M_2(x)$.

Now, each arc in $M_1(x) \ltimes M_2(x)$ was made by just one of these two requirements and both of them can not be true at the same time. So to get the total count of arcs in $M_1(x) \ltimes M_2(x)$, we add the number of arcs generated by each condition. Consider the first condition for adjacency, i.e.,$v_q = v_s$. Let v be any vertex in $M_1(x)$. The dipart $M_2(x)$ contains $|J_2(x)|$ vertices. We can choose 2 different vertices v' and v'' from $M_2(x)$ in $\binom{|J_2(x)|}{2}$ different ways. Corresponding to each choice we get 2 arcs $((v, v'), (v, v''))$ and $((v, v''), (v, v'))$ in $M_1(x) \ltimes M_2(x)$. Like v, there are totally $|J_1(x)|$ vertices in $M_1(x)$. Hence the first condition of adjacency gives $2|J_1(x)|\binom{|J_2(x)|}{2}$ arcs in $M_1(x) \ltimes M_2(x)$. Now consider the second condition for adjacency, i.e., (v_q, v_s) is an arc in $M_1(x)$ and (v_r, v_t) is not an arc in $M_2(x)$. We can choose two different vertices v_q and v_s in $M_1(x)$ such that there is an arc (v_q, v_s) in $M_1(x)$, in $|L_1(x)|$ different ways. Similarly we can choose two different vertices v_r and v_t in $M_2(x)$ such that (v_r, v_t) is not an arc in $M_2(x)$ in $(|J_2(x)|(|J_2(x)| - 1) - |L_2(x)|)$ different ways. Let v_q and v_s be two vertices in $M_1(x)$ such that (v_q, v_s) is an arc in $M_1(x)$ and let v_r and

v_t be two vertices in $M_2(x)$ such that (v_r, v_t) is not an arc in $M_2(x)$. From this we get an arc $((v_q, v_r), (v_s, v_t))$ in $M_1(x) \ltimes M_2(x)$. Hence totally the second condition for adjacency gives $|L_1(x)|\,(|J_2(x)|(|J_2(x)|-1) - |L_2(x)|)$ arcs in $M_1(x) \ltimes M_2(x)$. Hence, the total count of arcs in $M_1(x) \ltimes M_2(x)$ is $2|J_1(x)|\binom{|J_2(x)|}{2} + |L_1(x)|\,(|J_2(x)|(|J_2(x)|-1) - |L_2(x)|)$. This is a true statement for all diparts of $D_1 \circledast D_2$. Therefore total count of arcs in $D_1 \circledast D_2$ is

$$\sum_{x \in P_1 \cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right),$$

. □

Corollary 4.1. *Let $D^* = (V, A)$ be a directed graph and $D_1 = (D^*, J_1, L_1, P_1)$ and $D_2 = (D^*, J_2, L_2, P_2)$ be two soft directed graphs of D^*. Then*

$$(i) \sum_{x \in P_1 \cap P_2} \sum_{(u,v) \in J_{D_1 \circledast D_2}(x)} ideg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x \in P_1 \cap P_2} \sum_{(u,v) \in J_{D_1 \circledast D_2}(x)} odeg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x \in P_1 \cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right)$$

$$(ii) \sum_{x \in P_1 \cap P_2} \sum_{(u,v) \in J_{D_1 \circledast D_2}(x)} deg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x \in P_1 \cap P_2} \left(4|J_1(x)| \binom{|J_2(x)|}{2} + 2|L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right),$$

where $ideg(u,v)[M_{D_1 \circledast D_2}(x)], odeg(u,v)[M_{D_1 \circledast D_2}(x)]$ and $deg(u,v)[M_{D_1 \circledast D_2}(x)]$ denote the dipart in-degree, dipart out-degree and dipart degree respectively, of the vertex (u,v), in the dipart $M_{D_1 \circledast D_2}(x)$ of $D_1 \circledast D_2$.

Proof. (i) Consider any dipart $M_{D_1 \circledast D_2}(x) = (J_{D_1 \circledast D_2}(x), L_{D_1 \circledast D_2}(x))$ of $D_1 \circledast D_2$ which is given by $M_1(x) \times M_2(x)$. By *theorem* 4.2, we have number of arcs in $M_1(x) \times M_2(x)$ is

$$\sum_{x \in P_1 \cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right).$$

Since the dipart $M_{D_1 \circledast D_2}(x)$ is a directed graph having

$$\sum_{x \in P_1 \cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right)$$

arcs, we have

$$\sum_{(u,v) \in J_{D_1 \circledast D_2}(x)} ideg(u,v)[M_{D_1 \circledast D_2}(x)] = \sum_{(u,v) \in J_{D_1 \circledast D_2}(x)} odeg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x \in P_1 \cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right),$$

since each arc in $M_{D_1 \circledast D_2}(x)$ contributes 1 each to the sums $\sum_{(u,v)\in J_{D_1 \circledast D_2}(x)} ideg(u,v)[M_{D_1 \circledast D_2}(x)]$ and $\sum_{(u,v)\in J_{D_1 \circledast D_2}(x)} odeg(u,v)[M_{D_1 \circledast D_2}(x)]$. This is true for all the diparts $M_{D_1 \circledast D_2}(x)$ of $D_1 \circledast D_2$. Hence,

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \circledast D_2}(x)} ideg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \circledast D_2}(x)} odeg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} \left(2|J_1(x)| \binom{|J_2(x)|}{2} + |L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right).$$

(ii) Since $deg(u,v)[M_{D_1 \circledast D_2}(x)] = ideg(u,v)[M_{D_1 \circledast D_2}(x)] + odeg(u,v)[M_{D_1 \circledast D_2}(x)]$ and by part (i) of this theorem we have,

$$\sum_{x\in P_1\cap P_2} \sum_{(u,v)\in J_{D_1 \circledast D_2}(x)} deg(u,v)[M_{D_1 \circledast D_2}(x)] =$$

$$\sum_{x\in P_1\cap P_2} \left(4|J_1(x)| \binom{|J_2(x)|}{2} + 2|L_1(x)| \left[|J_2(x)|(|J_2(x)|-1) - |L_2(x)| \right] \right).$$

□

5. Conclusion

Soft directed graph generates a series of representations of a relationship given by a directed graph, through parameterization. We introduced and explored the features of homomorphic product and restricted homomorphic product of soft directed graphs, in this study.

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