TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1420-1433

ON NEW FORMS OF BI-IDEAL NANO OPEN SETS

K. KAUR[1], A. GUPTA[1*], §

Abstract. The idea of nano topology was proposed by M.L. Thivagar. Since then, various researchers have worked on generalizing this theory. Bi-ideal nano topology was generated by approximation using two ideals. Motivated by the fact that this technique of approximation is better than the existing ones due to higher accuracy and more versatility, there is a need to investigate some new forms of bi-ideal nano topology. The aim of this paper is to generate novel forms of bi-ideal nano open sets, namely bi-ideal nano semi-open, bi-ideal nano pre-open, bi-ideal nano regular open, and bi-ideal nano α open sets. The interrelation of all these weak forms is studied. A practical application of nano topology is also discussed towards the end of this paper to find the main factors responsible for covid-19 disease.

Keywords: Nano topology, Ideal topology, approximation, bi-ideal nano pre-open, bi-ideal nano semi-open, and bi-ideal nano α open.

AMS Subject Classification: 54A05, 54H99, 54F65

1. Introduction

Nano topology and rough set theory play a significant role as an interdisciplinary forum, which emphasizes the progress in sciences and has numerous applications in the engineering and biomedical world. The rough set [18] was originally instituted by Pawlak. It was a generalization of normal set theory where upper and lower approximations were introduced to split a set. A set was conceived in terms of two approximated sets, lower and upper approximated versions, which have successfully been applied in dealing with intelligent systems, specified by insufficient information or data. Since the past decade, many researchers proposed different theories to generate nano topology through various mathematical tools such as neutrosophic sets [20], which assimilates the notion of intuitionistic sets and fuzzy sets together, Pythagorean sets [1], ideals [11, 12, 13], soft sets [19], graphs [7, 14], nano operations [9], neighborhoods [8, 13], etc.

[1] Department of Mathematics, Punjab Engineering College (Deemed to be University), 160012, India.
e-mail: kamalpreetkaur.phd21appsc@pec.edu.in; https://orcid.org/0000-0003-3822-9516.
e-mail: ashagoel1968@gmail.com; https://orcid.org/0000-0002-2519-9517.
[*] Corresponding author.
§Manuscript received: March 27, 2023; accepted: July 28, 2023.

The first author is supported by the University Grants Commission, New Delhi - Junior Research fellowship (NTA ref no. 191620108838).

In topology, ideals are powerful elements for the analysis and representation of uncertainty and indiscernibility within data sets. They help in approximating and classifying the objects, based on the identification of relevant characteristics, and the extraction of crucial or significant information. The novel method of approximating using two ideals was suggested by A. Kandil [10], which is a more remarkable technique than the existing ones as it reduces the boundary and hence increases the accuracy degree in the theory of rough sets. Various researchers worked on ideal topology and their significant topological properties [4, 2]. Many significant topological concepts such as α local functions and weak separation axioms via $e-I$ sets were discussed thoroughly in ideal topological spaces. Also, $e-I$ continuous functions were investigated [3]. Further, in the paper [12], the bi-ideal approach of approximation, introduced by A. Kandil [10], where the approximation of a set is carried out by the maximum possibility of accuracy and precision, is directed to generate a new nano topology. The comparison of this approach with the previous ones has proven that it is a more efficient technique [12]. This is a modified version of the original definition of nano topology, which uses the technique of approximating spaces via two ideals. It must be noted that this method can also be further generalized to 'n' ideals, which may lead to better precision and accuracy in decision-making, etc. Furthermore, a new multi-ideal nano topological model was given via neighborhoods for diagnosis and cure of dengue disease [13].

Interestingly, the nano topology, when associated with the graph theory has achieved success in the easy interpretation of various biological functions such as blood circulation and the working of the heart [7, 16]. Nano topological graphs have been used to study a model of the respiratory system[14] and draw some imperative medical conclusions about the functioning of different organs. Nano topology has been applied to the physical world as it has also served in the reduction of electric transmission lines [15]. Also, the relation between a digraph theory and nano topology has been applied to detect urinary system diseases [5]. In addition, the approximations of a rough set have been used to study the physical properties of the fractals through their nano topological graphs [6]. Many topologists have studied the nearly open forms of nano topology which are also referred to as weak forms of nano open sets [1, 17, 21].

Open sets' generalizations and weak forms provide a strong toolset in topology, which enables the study of convergence, separation properties, and the characterization of topological properties. In this paper, some new forms of bi-ideal nano topology, such as bi-ideal nano pre-open, bi-ideal nano semi-open, bi-ideal nano α open, and bi-ideal nano regular open sets are introduced. The interrelation of these forms along with some results and theorems are discussed. Furthermore, a practical life application of nano topology is illustrated to find the main attributes that decide whether the region has high or low covid-19 susceptibility. Also, in this application, a comparison of the accuracy measures of the bi-ideal approach with the previous approaches is shown in a tabular form.

2. Preliminaries

In this section, we define the existing definitions and notations which are significant for this paper.

Definition 2.1. [21] *If $\tilde{U}$ is the universe and $\hat{\mathcal{R}}$ is an equivalence relation. Here, $(\tilde{U}, \hat{\mathcal{R}})$ is an approximation space. Also, let $\mathcal{W} \subseteq \tilde{U}$. Then*

(1) $\tilde{L}_{\hat{\mathcal{R}}}(\mathcal{W}) = \bigcup_{w \in U} \left\{ \hat{\mathcal{R}}(w) : \hat{\mathcal{R}}(w) \subseteq \mathcal{W} \right\}$ *is the lower approximation of $\mathcal{W}$ w.r.t $\hat{\mathcal{R}}$.*

(2) $\tilde{U}_{\hat{\mathcal{R}}}(\mathcal{W}) = \bigcup_{w \in U} \left\{ \hat{\mathcal{R}}(w) : \hat{\mathcal{R}}(w) \cap \mathcal{W} \neq \emptyset \right\}$ *is referred as the upper approximation space of* $\mathcal{W}$ *w.r.t* $\hat{\mathcal{R}}$.

(3) $\tilde{B}_{\hat{\mathcal{R}}}(\mathcal{W}) = U_{\hat{\mathcal{R}}}(\mathcal{W}) - L_{\hat{\mathcal{R}}}(\mathcal{W})$ *is referred as the boundary of* $\mathcal{W}$ *w.r.t* $\hat{\mathcal{R}}$.
Let $\tau_{\hat{\mathcal{R}}}(\mathcal{W}) = \left\{ \tilde{U},\ \emptyset,\ L_{\hat{\mathcal{R}}}(\mathcal{W}),\ U_{\hat{\mathcal{R}}}(\mathcal{W}),\ B_{\hat{\mathcal{R}}}(\mathcal{W}) \right\}$.

It is a nano topology w.r.t $\mathcal{W}$. *Then* $(\tilde{U}, \tau_{\hat{\mathcal{R}}}(\mathcal{W}))$ *is a nano topological space. Elements of* $(\tilde{U}, \tau_{\hat{\mathcal{R}}}(\mathcal{W}))$ *are known as nano open sets and their complements are nano closed sets.*

Remark 2.1. [21] *If* $K \subseteq \tilde{U}$, *then the union of all nano open subsets of* K *is known as nano-interior of* K, *written as* $\eta int(K)$ *and the intersection of all nano closed sets containing* K *is known as nano-closure of* K, *written as* $\eta cl(K)$.

Definition 2.2. [12] *An ideal* $\tilde{I}_d$ *on a set* $\tilde{U}$ *is a family of subsets of* $\tilde{U}$ *which satisfies :*
(i) $\tilde{H} \in \tilde{I}_d, \tilde{K} \subseteq \tilde{H} \implies \tilde{K} \in \tilde{I}_d$.
(ii) $\tilde{H} \in \tilde{I}_d,\ \tilde{K} \in \tilde{I}_d \implies \tilde{H} \cup \tilde{K} \in \tilde{I}_d$.

Definition 2.3. [12] *Given* $(\tilde{U}, \hat{\mathcal{R}}, \tilde{I}_{d1}, \tilde{I}_{d2})$ *a bi-ideal approximation space where* $\tilde{U}$ *is the universe and* $\mathcal{W} \subseteq \tilde{U}$. *Let* $\hat{\mathcal{R}}$ *be a binary relation on* $\tilde{U}$. *Also, let* $\tilde{I}_{d1}$ *and* $\tilde{I}_{d2}$ *be two ideals on* $\tilde{U}$ *and a pair of lower and upper approximations be* $\underline{\underline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ *and* $\overline{\overline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ *which are defined as :*

$$\underline{\underline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\ \tilde{I}_{d2}>}(\mathcal{W}) = \left\{ w \in \mathcal{W} : \hat{\mathcal{R}} < w > \hat{\mathcal{R}} \cap \mathcal{W}^c \in < \tilde{I}_{d1},\ \tilde{I}_{d2} > \right\}.$$
$$\overline{\overline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\ \tilde{I}_{d2}>}(\mathcal{W}) = \mathcal{W} \cup \left\{ w \in \tilde{U} : \hat{\mathcal{R}} < w > \hat{\mathcal{R}} \cap \mathcal{W} \notin < \tilde{I}_{d1},\ \tilde{I}_{d2} > \right\}.$$

Also, $\tilde{B}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \overline{\overline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) - \underline{\underline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ *is the boundary.*
Then $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\emptyset,\ \tilde{U},\ \overline{\overline{R}}_{<\tilde{I}_{d1},\ \tilde{I}_{d2}>}(\mathcal{W}),\ \underline{\underline{\hat{\mathcal{R}}}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}), B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$.

This topology is defined as the bi-ideal nano topology w.r.t. $\mathcal{W}$, *abbreviated as bi-ideal* $\mathfrak{N}.\mathfrak{T}$. *Then,* $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ *is bi-ideal nano topological space, abbreviated as bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$.

Definition 2.4. [12] *All elements of bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ *are defined as bi-ideal nano open* ($BINO$) *sets and complements of bi-ideal nano open sets are defined as bi-ideal nano closed* ($BINC$) *sets.*

Remark 2.2. [12] *If* $K \subseteq \tilde{U}$, *then the union of all bi-ideal nano open subsets of* K *is defined as bi-ideal nano interior of* K, *written as* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(K)$ *and the intersection of all bi-ideal nano closed sets containing* K *is defined as bi-ideal nano closure of* K, *written as* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(K)$.

3. New forms of bi-ideal nano open sets

In this section, we will study the new forms of bi-ideal nano open sets.

Definition 3.1. *Let* $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ *be a bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$. *Let* $\tilde{K} \subseteq \tilde{U}$.

(1) *If* $\tilde{K} \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\tilde{K}))$, *then* $\tilde{K}$ *is defined as bi-ideal nano semi open set* ($BINSO$).

(2) *If* $\tilde{K} \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\tilde{K}))$, *then* $\tilde{K}$ *is defined as bi-ideal nano pre open set* ($BINPO$).

(3) *If* $\tilde{K} \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\tilde{K})))$*, then* $\tilde{K}$ *is defined as bi-ideal nano* α *open set* $(BIN\alpha O)$.

(4) *If* $\tilde{K} = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\tilde{K}))$*, then* $\tilde{K}$ *is defined as bi-ideal nano regular open set* $(BINRO)$.

Let $BINSO(\tilde{U},\mathcal{W})$ *represent the family of all bi-ideal nano semi-open sets,*
$BINPO(\tilde{U},\mathcal{W})$ *represent the family of all bi-ideal nano pre-open sets,*
$BIN\alpha O(\tilde{U},\mathcal{W})$ *or* $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ *denote the family of all bi-ideal nano alpha open sets and*
$BINRO(\tilde{U},\mathcal{W})$ *denote the family of all bi-ideal nano regular open sets.*

Example 3.1. *Let* $\tilde{U} = \{\aleph, \aleph_2, \aleph_3, \aleph_4\}$ *and* $\mathcal{W}$ *be* $\{\aleph_4, \aleph_2\}$*. Also, let* $\tilde{I}_{d1} = \{\emptyset, \{\aleph_4\}\}$ *and* $\tilde{I}_{d2} = \{\emptyset, \{\aleph_3\}\}$*. Then,* $<\tilde{I}_{d1},\tilde{I}_{d2}> = \{\emptyset, \{\aleph_4\}, \{\aleph_3\}, \{\aleph_4, \aleph_3\}\}$.
If $\tilde{U}/R = \{\{\aleph_4\}, \{\aleph_2, \aleph_1\}, \{\aleph_3\}\}$ *, then* $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_4\}\}$.
$(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ *is a bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$.

$BINSO(\tilde{U},\mathcal{W}) = \{\emptyset, \tilde{U}, \{\aleph_1, \aleph_2\}, \{\aleph_3, \aleph_4\}, \{\aleph_4\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_1, \aleph_2, \aleph_3\}\}$.
$BINPO(\tilde{U},\mathcal{W}) = \{\emptyset, \tilde{U}, \{\aleph_1\}, \{\aleph_2\}, \{\aleph_1, \aleph_4\}, \{\aleph_4\}, \{\aleph_4, \aleph_2\}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_1, \aleph_3, \aleph_4\}, \{\aleph_2, \aleph_3, \aleph_4\}\}$.
$BIN\alpha O(\tilde{U},\mathcal{W})$ *or* $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_4\}\}$.
$BINRO(\tilde{U},\mathcal{W}) = \{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_4\}\}$.

Remark 3.1. *Note that* $BINSO(\tilde{U},\mathcal{W})$*,* $BINPO(\tilde{U},\mathcal{W})$ *and* $BINRO(\tilde{U},\mathcal{W})$ *doesn't form a topology but* $BIN\alpha O(\tilde{U},\mathcal{W})$ *or* $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ *always forms a topology.*

Theorem 3.1. *If* A *is* $BINO$*, then it is* $BIN\alpha O$ *in* $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$.

Proof. Since A is $BINO$ in $\tilde{U}$, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = A$.
So, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A) \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)$.
$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$.
$\implies A$ is $BIN\alpha O$ or $N_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\alpha$ open.
$\implies A \in \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$. □

Theorem 3.2. $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINSO(\tilde{U},\mathcal{W})$ *in bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$.

Proof. If $A \in \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
$\implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$.
For any set K, we know that $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(K) \subseteq K$.
$\therefore \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$
$\subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$.
$\implies A \in BINSO(\tilde{U},\mathcal{W})$.
So, $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINSO(\tilde{U},\mathcal{W})$. □

Theorem 3.3. $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINPO(\tilde{U},\mathcal{W})$ *in bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$.

Proof. Let $A \in \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Then, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$.
$\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A)$ $(\because \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) \subseteq A)$.

$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$
$\subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$.
$\implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$.
Thus, $A \in BINPO(\tilde{U}, \mathcal{W})$. □

Remark 3.2. *$\tilde{U}$ and $\emptyset$ are $BIN\alpha O$ since $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(U)))$ $= U$ and $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\emptyset))) = \emptyset$.*

Theorem 3.4. *$\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W})$ in $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$.*

Proof. Let $A \in \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Then, $A \in BINPO(\tilde{U}, \mathcal{W})$ and also, $A \in BINSO(\tilde{U}, \mathcal{W})$. (By Theorem 3.2 and 3.3).
Thus, $A \in BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W})$.
Therefore, $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W})$.
Conversely, if $A \in BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W})$, then
$A \in BINSO(\tilde{U}, \mathcal{W}) \implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(N_{<\tilde{I}_{d1},\tilde{I}_{d2}>)}int(A))$. ....**[i]**
Also, if $A \in BINPO(\tilde{U}, \mathcal{W}) \implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$. ....**[ii]**
From **[i]**, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A) \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$.
$\implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)$.
Thus, $BINSO(\tilde{U}, \mathcal{W}) \subseteq \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ . ...**[iii]**
From **[ii]** , $A \subseteq (\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$.
$\implies A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$.
Thus, $BINPO(\tilde{U}, \mathcal{W}) \subseteq \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$. ....**[iv]**
Equations **[iii]** and **[iv]** implies that $BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W}) \subseteq \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Hence, $\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = BINPO(\tilde{U}, \mathcal{W}) \cap BINSO(\tilde{U}, \mathcal{W})$. □

Theorem 3.5. *If in a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$, $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) =$ $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \mathcal{W}$, then $\tilde{U}, \emptyset, \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \mathcal{W}$ and any set A satisfying $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$ is the single $BIN\alpha O$ set in $\tilde{U}$.*

Proof. Since $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \mathcal{W}$, then the induced bi-ideal $\mathfrak{N}.\mathfrak{T}$ is $\{\tilde{U}, \emptyset, \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$. By theorem 3.1, the members of bi-ideal $\mathfrak{N}.\mathfrak{T}$ are all $BIN\alpha O$ sets. If $A \subseteq \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then A is not $BIN\alpha O$ set.
If $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ is the largest $BIN\alpha O$ subset of A.
$\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
So, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$
$= \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})))$
$= \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(U) = U$.
$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = U$.
Thus, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$.
$\implies A$ is $BIN\alpha O$ in $\tilde{U}$. □

Theorem 3.6. *$\tilde{U}$, $\emptyset$, $\overline{\overline{R}}_{<\tilde{I}_{d1},\ \tilde{I}_{d2}>}(\mathcal{W})$ and any set containing $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BIN\alpha O$ sets of a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ if $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$.*

Proof. Since $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})= \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Thus, bi-ideal $\mathfrak{N}.\mathfrak{T}$ is $\{\tilde{U},\ \emptyset,\ \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$ and by theorem 3.1,
the members of bi-ideal $\mathfrak{N}.\mathfrak{T}$ are all $BIN\alpha O$ sets.
Let $A \subseteq \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then A is not $BIN\alpha O$ set.
If $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ is the largest $BINO$ subset of A.
So, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))) = U$.
Thus, A is $BIN\alpha O$ in $\tilde{U}$. Hence, proved. □

Theorem 3.7. *If $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tilde{U}$ and $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq \emptyset$ in a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$, then $\tilde{U}$, $\emptyset$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BIN\alpha O$ sets in $\tilde{U}$.*

Proof. As $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = U$ and $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$,
the $BINO$ sets in $\tilde{U}$ are $\tilde{U}$, $\emptyset$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Hence, these are $BIN\alpha O$. If $A = \emptyset$, then it is $BIN\alpha O$.
Now, let $A \neq \emptyset$ and let $A \subseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$. $\implies$ $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \emptyset$.
As $\emptyset$ is the largest $BINO$ set in A, so $A \nsubseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$.
$\implies$ A is not $BIN\alpha O$ set.
Suppose $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Hence, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$
$=\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$.
$\implies$ $A \nsubseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$.
Thus, A can not be $BIN\alpha O$ set.
In a parallel manner, it can easily be verified that A can not be $BIN\alpha O$ because $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$ and $A \subseteq B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
If A has one element in $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ or $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then A is not $BIN\alpha O$ set.
Hence, proved. □

Corollary 3.1. *$\tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ if $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = U$.*

Theorem 3.8. *Let $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ where $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq \emptyset$ and $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq U$ in $(\tilde{U},\ \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$. Then, $\tilde{U}$, $\emptyset$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ and any set*
A containing $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BIN\alpha O$ sets in $\tilde{U}$.

Proof. The bi-ideal $\mathfrak{N}.\mathfrak{T}$ on $\tilde{U}$ is written as $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U},\ \emptyset,\ \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}),$ $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}), \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$ and hence $\tilde{U}$, $\emptyset$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$ and any set A containing $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BIN\alpha O$ sets in $\tilde{U}$.
Let A be a $BINO$ in $\tilde{U}$ where $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Thus, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)))$

$\implies A$ is $BIN\alpha O$ in $\tilde{U}$ if $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$.
If $A \subseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \emptyset$
$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))) = \emptyset$.
Thus, A is not $BIN\alpha O$ set.
When $A \subseteq \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ but it is neither a subset of $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ nor of $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \emptyset \implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))) = \emptyset$.
Thus, A is not $BIN\alpha O$ set.
Hence, proved that $\tilde{U}$, $\emptyset$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ and any set A containing $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BIN\alpha O$ sets in $\tilde{U}$. □

Remark 3.3. *$\tilde{U}$ and $\emptyset$ are always bi-ideal nano semi open ($BINSO$) since* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(U)) = U$ *and* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\emptyset)) = \emptyset$.

Remark 3.4. *$\tilde{U}$ and $\emptyset$ are always bi-ideal nano pre open ($BINPO$) since* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(U)) = U$ *and* $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\emptyset)) = \emptyset$.

Theorem 3.9. *Let $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ be a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ with $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then $\emptyset$ and the sets A where $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$ are the only $BINSO$ subsets of $\tilde{U}$.*

Proof. Since $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U},\ \emptyset,\ \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$.
$\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\emptyset) = \emptyset$ and $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\emptyset)) = \emptyset$. $\implies \emptyset$ is $BINSO$.
Let $A \subseteq \tilde{U}$ and $A \subseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then
$\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\emptyset) = \emptyset$.
If $A \subseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then A is not $BINSO$.
Consider $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, thus $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)$
$= \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})) = U$. $(\because \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$
Thus, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$ and A is $BINSO$.
Therefore, $\emptyset$ and the sets A such that $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$ are the only $BINSO$ subsets. □

Theorem 3.10. *If $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$ and $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq U$, then only those sets containing $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the $BINSO$ sets in $\tilde{U}$.*

Proof. Let $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U},\ \emptyset,\ \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$ and $A \subseteq \tilde{U}$.
If $A \subseteq \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \emptyset$.
$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = \emptyset$, then A is not $BINSO$.
Consider $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
$\implies \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = U$. $\implies A$ is $BINSO$ set.
Thus, the only type is set A, a superset of $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the $BINSO$ sets in $\tilde{U}$ whenever $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$ and $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq U$. □

Theorem 3.11. *Suppose $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = U$ in bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$, then $\tilde{U}$, $\emptyset$, $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are the only $BINSO$ sets in $\tilde{U}$.*

Proof. Let $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U},\ \emptyset,\ \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}),\ B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})\}$ and $A \neq \emptyset$ in $\tilde{U}$. Clearly, A is not $BINSO$ set when $A \subseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

If $A = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Hence, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$ which further implies that A is $BINSO$ set when $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$ since $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ but $A \nsubseteq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$. Similarly, if $A \subseteq B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ and $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq A$, then
$\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = \emptyset$ and $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) = B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ respectively. Hence, A isn't a $BINSO$ set.

If A has atleast one element in $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ and atleast one element in $B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then A is not a $BINSO$. Thus, $\tilde{U},\ \emptyset,\ \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}),\ B_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are those only $BINSO$ sets in $\tilde{U}$ where $\overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = U$ and $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \neq \emptyset$. □

Corollary 3.2. *If $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \emptyset$ in theorem 3.11, then $\emptyset$ and $\tilde{U}$ are the only $BINSO$ sets in $\tilde{U}$.*

Theorem 3.12. *If A and B are $BINSO$ in $\tilde{U}$, then $A \cup B$ is also $BINSO$ in $\tilde{U}$.*

Proof. As A and B are $BINSO$ in $\tilde{U}$,
$A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A))$ and $B \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(B))$.
If we take $A \cup B$ under consideration,
then, $A \cup B \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(B))$.
$\subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A)) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(B))$
$\subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A \cup B))$,
since, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(B) \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A \cup B)$.
Hence, proved. □

Remark 3.5. *The intersection of two $BINSO$ sets in $\tilde{U}$ need not be $BINSO$ in $\tilde{U}$.*

(Counterexample): *$\tilde{U} = \{x, y, z, r\}$. Let $\mathcal{W}$ be $\{r, y\}$. Also, let $\tilde{I}_{d1} = \{\emptyset, \{r\}\}$ and $\tilde{I}_{d2} = \{\emptyset, \{z\}\}$. Then, $< \tilde{I}_{d1}, \tilde{I}_{d2} >= \{\emptyset, \{r\}, \{z\}, \{r, z\}\}$.*
If $\tilde{U}/R = \{\{r\}, \{y, x\}, \{z\}\}$ and $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\emptyset, \tilde{U}, \{y, x\}, \{x, y, r\}, \{r\}\}$.
Then, $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ be a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$.
$BINSO(\tilde{U}, \mathcal{W})$ $=\{\emptyset, \tilde{U}, \{x, y\}, \{x, y, r\}, \{z, r\}, \{r\}, \{x, y, z\}\}$.
Here, $\{r\}$ and $\{z, r\}$ are both bi-ideal nano semi open but their intersection $\{z\}$ is not bi-ideal nano semi open.

Theorem 3.13. *If A as well as B are $BINPO$ in $\tilde{U}$, then $A \cup B$ is also $BINPO$ in $\tilde{U}$.*

Proof. As A and B are $BINPO$ in $\tilde{U}$, $A \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$ and $B \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(B)$. If we consider $A \cup B$, then
$A \cup B \subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(B))$.
$\subseteq \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A)) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(B)) = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A \cup B))$.
$(\because \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A) \cup \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(B) = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A \cup B).)$
So, the union of two $BINPO$ sets is a $BINPO$ set in $\tilde{U}$. □

Remark 3.6. *The intersection of two $BINPO$ sets in $\tilde{U}$ need not be $BINPO$ in $\tilde{U}$.*

(Counterexample): *$\tilde{U} = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4\}$ Let $\mathcal{W}$ be $\{\aleph_4, \aleph_2\}$. Let $\tilde{I}_{d1} = \{\emptyset, \{\aleph_4\}\}$ and $\tilde{I}_{d2} = \{\emptyset, \{\aleph_3\}\}$. Then, $< \tilde{I}_{d1}, \tilde{I}_{d2} >= \{\emptyset, \{\aleph_4\}, \{\aleph_3\}, \{\aleph_4, \aleph_3\}\}$.*
If $\tilde{U}/R = \{\{\aleph_4\}, \{\aleph_2, \aleph_1\}, \{\aleph_3\}\}$, then $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \big\{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_4\}\big\}$.
Here, $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ is a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$.
If $BINPO(\tilde{U}, \mathcal{W})$ $=\{\emptyset, \tilde{U}, \{\aleph_1\}, \{\aleph_2\}, \{\aleph_4\}, \{\aleph_4, \aleph_2\}, \{\aleph_1, \aleph_4\}, \{\aleph_2, \aleph_1\}, \{\aleph_4, \aleph_2, \aleph_3\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_1, \aleph_3, \aleph_4\}\}$.
Here, $\{\aleph_4, \aleph_2, \aleph_3\}$ and $\{\aleph_4\}$ both are bi-ideal nano pre open but their intersection $\{\aleph_2, \aleph_3\}$ is not bi-ideal nano pre open.

Theorem 3.14. *Any bi-ideal nano regular open $(BINRO)$ set is bi-ideal nano open $(BINO)$ set.*

Proof. If A is bi-ideal nano regular open in $\tilde{U}$, then $A = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))$.
Then, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(A) = \hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(A))) = A$.
Hence, A is $BINO$ in $\tilde{U}$. □

Remark 3.7. *The converse of the above statement is not true in general.*

(Counterexample): *Let $\tilde{U} = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4\}$ and $\mathcal{W}$ be $\{\aleph_4, \aleph_2\}$. Also, let $\tilde{I}_{d1} = \{\emptyset, \{\aleph_4\}\}$ and $\tilde{I}_{d2} = \{\emptyset, \{\aleph_3\}\}$. Then, $< \tilde{I}_{d1}, \tilde{I}_{d2} >= \{\emptyset, \{\aleph_4\}, \{\aleph_3\}, \{\aleph_4, \aleph_3\}\}$.*
If $\tilde{U}/R = \{\{\aleph_4\}, \{\aleph_2, \aleph_1\}, \{\aleph_3\}\}$ and $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \big\{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_4\}\big\}$.
Then, $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ be a bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$.
$BINRO(\tilde{U}, \mathcal{W})$ $=\big\{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_4\}\big\}$.
Here, $\{\aleph_1, \aleph_2, \aleph_4\}$ is bi-ideal nano open but not bi-ideal nano regular open.

Theorem 3.15. *In $(\tilde{U},\ \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$, if $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \overline{\overline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$, then the only bi-ideal nano regular open sets are $\tilde{U}$ and $\emptyset$.*

Proof. The bi-ideal nano open sets in $\tilde{U}$, $\emptyset$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ are $\tilde{U}$, $\emptyset$, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Also, $\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{int}(\hat{N}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}\tilde{cl}(\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})) = U \neq \underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
Hence, $\underline{\underline{R}}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$ is not bi-ideal nano regular open $(BINRO)$.
Thus, the only bi-ideal nano regular open sets are $\tilde{U}$ and $\emptyset$. □

Corollary 3.3. *If A and B are bi-ideal nano regular open sets in bi-ideal $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$, then $A \cap B$ is also bi-ideal nano regular open.*

Remark 3.8. *In view of the above theorems 3.1, 3.4, and 3.14, the following implications hold:*

- $BINRO \implies BINO \implies BIN\alpha O \implies BINSO.$
- $BINRO \implies BINO \implies BIN\alpha O \implies BINPO.$

In other words,

- $BINRO(\tilde{U}, \mathcal{W}) \subseteq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINSO(\tilde{U}, \mathcal{W}).$
- $BINRO(\tilde{U}, \mathcal{W}) \subseteq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq \tau^{\alpha}_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) \subseteq BINPO(\tilde{U}, \mathcal{W}).$

However, the reverse implications may not hold in general as illustrated by the example below:
(Counter example): *Let $\tilde{U} = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4\}$ and $\mathcal{W}$ be $\{\aleph_2, \aleph_4\}$.*
If $\tilde{I}_{d1} = \{\emptyset, \{\aleph_4\}\}$ and $\tilde{I}_{d2} = \{\emptyset, \{\aleph_3\}\}$, then $< \tilde{I}_{d1}, \tilde{I}_{d2} >= \{\emptyset, \{\aleph_4\}, \{\aleph_3\}, \{\aleph_4, \aleph_3\}\}$.
If $\tilde{U}/R = \{\{\aleph_4\}, \{\aleph_2, \aleph_1\}, \{\aleph_3\}$ then $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \big\{\emptyset, \tilde{U}, \{\aleph_2, \aleph_1\}, \{\aleph_1, \aleph_2, \aleph_4\}, \{\aleph_4\}\big\}$.

So, $(\tilde{U}, \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}))$ *is a bi-ideal* $\mathfrak{N}.\mathfrak{T}.\mathfrak{S}$ *w.r.t.* $\mathcal{W}$.
Here, $\{\aleph_1, \aleph_2, \aleph_4\}$ *is bi-ideal nano open but not bi-ideal nano regular open.*
$\{\aleph_4, \aleph_2, \aleph_3\}$ *is bi-ideal nano pre open but not bi-ideal nano* α *open.*
$\{\aleph_3, \aleph_4\}$ *is bi-ideal nano semi open but not bi-ideal nano* α *open.*

4. An application

In this section, we give a real-life example of Covid, where nano topology is applied to determine the key factors responsible for its high susceptibility in some cities. Consider Table 1:

Table1 demonstrates the various cities $\{\Pi_{o1}, \Pi_{o2}, \Pi_{o3}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o7}, \Pi_{o8}\}$ with re-

Cities	P.C.I	P.D	P.M.T	V.H	S.C	Decision
Π_{o1}	Good	High	Public	low	good	high
Π_{o2}	Bad	High	Private	high	good	high
Π_{o3}	Good	high	public	high	poor	high
Π_{o4}	Bad	Low	Private	high	good	low
Π_{o5}	Good	High	Public	low	good	low
Π_{o6}	Good	high	public	low	average	low
Π_{o7}	Good	high	public	high	poor	high
Π_{o8}	Good	low	public	low	poor	low

Table 1. Illustration of cities w.r.t various conditional attributes and decision attributes.

spect to different conditional attributes (factors) such as per capita income (P.C.I), population density (P.D), primary mode of transportation (P.M.T), vaccination hesitancy (V.H), Sanitary condition (S.C). Let H be the set of condition attributes.
The decision (covid susceptibility) is the decision attribute. The domains of attributes are given as follows:
Per capita income={Good, Bad}, Population density={high, low}, Primary mode of transportation ={public, private}, Vaccination hesitancy = {low, high)}, sanitary condition = {good, average, poor} and decision(covid susceptibility) = {high, low}.

So, the universe $\tilde{U}$ is $\{\Pi_{o1}, \Pi_{o2}, \Pi_{o3}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o7}, \Pi_{o8}\}$. Let R be an indiscernibility relation on $\tilde{U}$ if all attributes are taken together.
So, $\tilde{U}/R = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}\}, \{\Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$.
Also, $\tilde{I}_{d1} = \{\emptyset, \{\Pi_{o3}\}, \{\Pi_{o8}, \Pi_{o3}\}, \{\Pi_{o8}\}\}$ refers to the report of any external expert 1 and $\tilde{I}_{d2} = \{\emptyset, \{\Pi_{o2}\}, \{\Pi_{o8}, \Pi_{o2}\}, \{\Pi_{o8}\}\}$ refers to the report of any external report 2.
$<\tilde{I}_{d1}, \tilde{I}_{d2}>= \{\emptyset, \{\Pi_{o3}\}, \{\Pi_{o2}\}, \{\Pi_{o8}\}\}, \{\Pi_{o8}, \Pi_{o3}\}, \{\Pi_{o8}, \Pi_{o2}\}, \{\Pi_{o2}, \Pi_{o3}, \Pi_{o8}\}\}$ refers to the combined report of both external experts.

CASE I: Let $\mathcal{W} = \{\Pi_{o1}, \Pi_{o2}, \Pi_{o3}, \Pi_{o7}\}$, the cities with the high corona susceptibility. Now, $\tilde{U}/R = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}\}, \{\Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. By the definition,
$\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset, \{\Pi_{o2}, \Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}, \Pi_{o3}, \Pi_{o7}, \Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o1}, \Pi_{o5}\}\}$.

Now, if the per capita income is neglected from the H, then $\tilde{U}/R' = \tilde{U}/R - (P.C.I) =$

$\{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}\}, \{\Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. Hence, topology $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset, \{\Pi_{o2}, \Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}, \Pi_{o3}, \Pi_{o7}, \Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o1}, \Pi_{o5}\}\} = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Secondly, if population density is removed from the H, then $\tilde{U}/R' = \tilde{U}/R - (P.D)$ $= \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}, \Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. Hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) =$ $\{\tilde{U}, \emptyset, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o1}, \Pi_{o2}, \Pi_{o3}, \Pi_{o5}, \Pi_{o7}\}, \{\Pi_{o1}, \Pi_{o2}, \Pi_{o5}\}\} \neq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Thirdly, if the primary mode of transportation is excluded from the H, then $\tilde{U}/R' = \tilde{U}/R - (P.M.T) = \tilde{U}/R$ and hence,$\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Fourthly, if vaccination hesitancy is neglected from the H, then $\tilde{U}/R' = \tilde{U}/R - (V.H) = \tilde{U}/R$ and hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Finally, if the sanitary condition is neglected from the factors, then $\tilde{U}/R' = \tilde{U}/R-(S.C) = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o2}\}, \{\Pi_{o3}, \Pi_{o6}, \Pi_{o7}\}, \{\Pi_{o4}\}, \{\Pi_{o8}\}\}$.
Hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset, \{\Pi_{o2}\}, \{\Pi_{o1}, \Pi_{o2}, \Pi_{o3}, \Pi_{o6}, \Pi_{o7}\}, \{\Pi_{o1}, \Pi_{o3}, \Pi_{o6}, \Pi_{o7}\},$ $\neq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.
So, from Case I, we get Core $(R) = \{$ **population density, sanitary condition**$\}$.

CASE II: Let $\mathcal{W} = \{\Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o8}\}$, the cities with the low corona susceptibility. $\tilde{U}/R = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}, \Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. By the definition , the nano topology on $\tilde{U}$ w.r.t $\mathcal{W}$ is $\tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset, \{\Pi_{o4}, \Pi_{o8}\}, \{\Pi_{o1}, \Pi_{o2}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o8}\},$ $\{\Pi_{o1}, \Pi_{o2}, \Pi_{o5}, \Pi_{o6}\}\}$.

Now, if the per capita income is neglected from the H, then $\tilde{U}/R' = \tilde{U}/R - (P.C.I) = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o2}\}, \{\Pi_{o4}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. Hence, topology $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset, \{\Pi_{o4}, \Pi_{o6}, \Pi_{o8}\}, \{\Pi_{o4}, \Pi_{o6}, \Pi_{o8}, \Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o1}, \Pi_{o5}\}\} = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Secondly, if population density is removed from the H, then $\tilde{U}/R' = U/R - (P.D) = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o2}, \Pi_{o4}\}, \{\Pi_{o3}, \Pi_{o7}\}, \{\Pi_{o6}\}, \{\Pi_{o8}\}\}$. Hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) =$
$\{\tilde{U}, \emptyset, \{\Pi_{o6}, \Pi_{o8}\},$
$\{\Pi_{o1}, \Pi_{o2}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o8}\}, \{\Pi_{o1}, \Pi_{o2}, \Pi_{o4}, \Pi_{o5}\}\} \neq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Thirdly, if the primary mode of transportation is excluded from the H, then $\tilde{U}/R' = \tilde{U}/R - (P.M.T) = \tilde{U}/R$ and hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Fourthly, if vaccination hesitancy is neglected from the H, then $\tilde{U}/R' = \tilde{U}/R - (V.H) = \tilde{U}/R$ and hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

Finally, if the sanitary condition is neglected from the factors, then $\tilde{U}/R' = \tilde{U}/R-(S.C) = \{\{\Pi_{o1}, \Pi_{o5}\}, \{\Pi_{o2}\}, \{\Pi_{o3}, \Pi_{o6}, \Pi_{o7}\}, \{\Pi_{o4}\}, \{\Pi_{o8}\}\}$. Hence, $\tau'_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W}) = \{\tilde{U}, \emptyset,$ $\{\Pi_{o2}, \Pi_{o8}\}, \{\Pi_{o1}, \Pi_{o3}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o7}, \Pi_{o8}\}, \{\Pi_{o1}, \Pi_{o3}, \Pi_{o4}, \Pi_{o5}, \Pi_{o6}, \Pi_{o7}\}\}$ $\neq \tau_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(\mathcal{W})$.

So, from Case II, we get Core (R)=$\{$ **population density, sanitary condition**$\}$.

Observation: It can be concluded that “population density ” and “sanitary condition” are the most crucial attributes to judge whether the region has high susceptibility to Covid-19 disease or not.

Comparison with the existing approaches: The bi-ideal approach is more efficient than the single ideal approach [11] and basic approach [21] as accuracy in the case of bi-ideal nano topology is higher than the previous approaches as illustrated by the tables 2 and 3. Clearly, the accuracy degree $\lambda_R(Z) \leq \lambda_{\tilde{I}_{di}}(Z) \leq \lambda_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(Z)$ for i=1,2.
Here, λ stands for accuracy degree which is mathematically defined as the ratio of the cardinality of the lower approximation to the cardinality of upper approximation.

Relation R	$\lambda_{R'}(Z)$	$\lambda_{\tilde{I}_{d1}}(Z)$	$\lambda_{\tilde{I}_{d2}}(Z)$	$\lambda_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(Z)$
$R' = R$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$R' = R$- P.C.I.	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$R' = R$- P.D.	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{2}$
$R' = R$- P.M.T.	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$R' = R$- V.H.	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$R' = R$- S.C.	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

TABLE 2. A comparison of the accuracy degrees of $\mathcal{W}$ w.r.t R of the bi-ideal $\mathfrak{N}.\mathfrak{T}$ with the previous notions in the above problem (CASE-I).

Relation R	$\lambda_{R'}(Z)$	$\lambda_{\tilde{I}_{d1}}(Z)$	$\lambda_{\tilde{I}_{d2}}(Z)$	$\lambda_{<\tilde{I}_{d1},\tilde{I}_{d2}>}(Z)$
$R' = R$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$R' = R$- P.C.I.	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$R' = R$- P.D.	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$R' = R$- P.M.T.	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$R' = R$- V.H.	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$R' = R$- S.C.	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{7}$

TABLE 3. A comparison of the accuracy degrees of $\mathcal{W}$ w.r.t R of the bi-ideal $\mathfrak{N}.\mathfrak{T}$ with the previous notions in the above problem (CASE-II).

5. Future scope and conclusion

First introduced by M.L. Thivagar in 2013, nano topology holds a huge potential to serve in various domains such as medicine, information sciences, research, and technologies. In the last ten years, various researchers have stated and thoroughly investigated numerous ideologies wherein nano topology was induced via approximation using different mathematical tools. The notion of generation of nano topology by approximations via two ideals has been proved better than others, which implies that generated bi-ideal nano topology is far better than Thivagar's approach. This paper further investigates its new nearly open forms along with their properties and characterizations. Further, this theory holds scope in the future as the bi-ideal notion can also be generalized to multiple ideals, which can serve in forming an algorithm to study the inter-dependency of conditional and decision attributes through a collective ideology of multi perspectives. This theory has the potential to be merged with other artificial intelligence techniques to enhance their capabilities. Hybrid approaches integrate the rough set theory with machine learning algorithms, fuzzy logic, and genetic algorithms to handle complex problems that involve uncertainty, imprecision, and insufficient data. This amalgamation can lead to more robust and accurate intelligent systems. Further work on its applications is presently in progress.

Acknowledgement. "Authors are extremely grateful to the esteemed reviewers for their valuable feedback which really helped in improving the quality of the manuscript."

References

[1] Ajay D., Charisma J., (2020), Pythagorean nano topological space, International Journal of Recent Technology and Engineering, 8(5), pp. 3415-3419.

[2] Al-Omeri, W., Noorani, M. S. M., Al-omeri, A., Noiri, T., (2015), Weak Separation Axioms via e-I-sets in Ideal Topological Spaces, European Journal of Pure and Applied Mathematics, 8(4), pp. 502-513.

[3] Al-Omeri, W., Noiri, T., (2021), On almost $e-I$-continuous functions, Demonstratio Mathematica, 54(1), pp. 168-177.

[4] Al-Omeri, W., Noorani, M. S. M., Al-Omari, A., (2014), a-local function and its properties in ideal topological spaces, Fasciculi Mathematici.

[5] Arafa N. A., Shokry M., Hassan M., (2020), From graph theory to nano topology, Filomat, 34(1), pp. 1-17.

[6] El-Atik. A. E. F. A., Nasef A. A., (2020), Some topological structures of fractals and their related graphs, Filomat, 34(1), pp. 153-165.

[7] El-Atik A. E. F. A., Hassan H. Z., (2020), Some nano topological structures via ideals and graphs, Journal of the Egyptian Mathematical Society, 28(1), pp. 1-21.

[8] El Sayed M., El Safty M. A., El-Bably M. K., (2021), Topological approach for decision-making of COVID-19 infection via a nano topological model, AIMS Mathematics, 6(7), pp. 7872-7894.

[9] Ibrahim H. Z., (2021), Applications of nano topology via nano operations, The Pure and Applied Mathematics, 28(3), pp. 199-215.

[10] Kandil A., El-Sheikh. S. A, Hosny M., Raafat M., (2020), Bi-ideal approximation spaces and their applications, Soft Computing, 24(17), pp. 12989-13001.

[11] Kandil A., El-Sheikh S. A., Hosny M., Raafat M., (2021), Generalization of nano topological spaces induced by different neighborhoods based on ideals and its applications, Tbilisi Mathematical Journal, 14(1), pp. 135-148.

[12] Kaur K., Gupta A., (2023), A topological approach for improving accuracy in decision-making via bi-ideal approximation, Journal of Intelligent and Fuzzy systems, 44(3), pp. 4557 – 4567.

[13] Kaur K., Gupta A., (2023), A new multi-ideal nano topological model via neighborhoods for diagnosis and cure of dengue, (Communicated).

[14] Kaur K., Gupta A., (2023), A new nano topological model of the human respiratory system via graphs, (Communicated).

[15] Nasef A., Shokry M., Mukhtar S., (2020), Some methods to reduction on electrical transmission lines by using rough concepts, Filomat, 34(1), pp. 111-128.

[16] Nawar, Ashraf S., Abd El Fattah A., El Atik A. E. F. A., (2019), A model of a human heart via graph nano topological spaces, International Journal of Biomathematics, 12(01), pp. 1950006.

[17] Parimala M., Indirani C., Jafari S., (2016), On nano b-open sets in nano topological spaces, Jordan Journal of Mathematics and Statistics, 9(3), pp. 173-184.

[18] Pawlak Z., (1982), Rough sets, International journal of computer and information sciences, 11(5), pp. 341-356.

[19] Priyalatha, S., Al-Omeri, W. F., (2020), Multi granulation on nano soft topological space, Advances in Mathematics: Scientific Journal, 9(10), pp. 7711–7717

[20] Thivagar M. L., Jafari S., Devi V. S., Antonysamy V., (2018), A novel approach to nano topology via neutrosophic sets, Infinite Study

[21] Thivagar M. L., Richard C., (2013), On nano forms of weakly open sets, International journal of mathematics and statistics invention, 1(1), pp. 31-37.

Kamalpreet Kaur is a research scholar in the Department of Mathematics, Punjab Engineering College, Chandigarh(Deemed to be University). She completed her MSc in mathematics at Punjabi University. Her research interests include general topology, nano topology, rough set theory, soft computing, and decision-making.

Asha Gupta is a professor in the Department of Mathematics, Punjab Engineering College, Chandigarh(Deemed to be University). Her research interests include General Topology, Fuzzy Topology, Nano Topology, Rough set theory, Soft computing, and Decision-making.