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MORPHISMS ON MIDDLE GRAPH OF SEMIRING VALUED GRAPHS

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Abstract. The middle graph $M(G)$ of a graph G is an intersection graph on the vertex set $V(G)$ of any graph G. Let $E(G)$ be an edge set of G and $F = V'(G) \cup E(G)$, where $V'(G)$ indicates the family of all one vertex subsets of the set $V(G)$. This concept was introduced by T. Hamada and I. Yoshimura [4]. M. Chandramouleeswaran et al., studied isomorphism and automorphism groups for semiring valued graph (S-valued graph). In this paper, we study the morphisms and its properties of middle graph of S-valued graphs.

Keywords: S-valued graph, homomorphism, exact sequence, regularity, automorphism.

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1. Introduction

Jonathan S. Golan, was the first person who introduced the notion of S-valued graphs where he defined a function $g : V \times V \to S$ such that $g(v_1, v_2) \neq \emptyset$. Here V is the vertex set of a graph G and S is a semiring. Golan consider the S-valued graph by assinging values to the edges only. Further, M.Rajkumar, S. Jeyalakshmi and M. Chandramouleeswaran precisely studied the graphs whose vertices and edges are assigned values from the semiring. However, they assign values to every vertex and the every edges of G in relation to the values of vertices incident with the edges. In [6], we study the middle graph of semiring valued graphs.

Homomorphisms are generalisation of graph colourings. In order to achieve a good correspondence between two graphs using such graph representation, the widely used concept is the one, that we are already familiar with, called graph isomorphism. The concept of homomorphisms, retract, core and exactness in graphs were studied by many authors [1, 3] and so on. We follow the algebraic concepts from [2].

In section 2, we give the basic definitions and results which are required. The novelty of this paper lights in connecting the algebraic concepts of faithful and full mappings between two structure of middle graph of S-valued graphs. In this way, we have included the concept of exact sequence, core and retract in the concept of middle graph of S-valued graphs. We study the homomorphism, isomorphism, retract, core, exact sequence and

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regularity properties of middle graph of S-valued graphs in section 3. In section 4, we determine the automorphism group on middle graph of S-valued graphs.

2. Preliminaries

In this section, we present some basic definitions which are required to our work.

Definition 2.1. [7] *A semiring $(S,+,\cdot)$ is an algebraic system with a non-empty set S together with two binary operation $+$ and . such that*

1. *$(S,+)$ is a commutative monoid and $(S,\cdot)$ is a semigroup.*
2. *For all $a,b,c \in S, a\cdot(b+c) = a\cdot b+a\cdot c$ and $(a+b)\cdot c = a\cdot c+b\cdot c$ and $0\cdot a = a\cdot 0 = 0$.*

Definition 2.2. [7] *Let $G = (V,E)$ be given graph with both $V,E \neq \emptyset$. For any semiring $(S,+,\cdot)$, a semiring-valued graph (or a S- valued graph), G^S, is defined to be the graph $G^S = (V,E,\sigma,\psi)$, where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be*

$$\psi(x,y) = \begin{cases} min\{\sigma(x),\sigma(y)\} & if\ \sigma(x) \preceq \sigma(y)\ or\ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

Definition 2.3. [7] *If $\sigma(x) = a$, $\forall x \in V$ and some $a \in S$, then the corresponding S-valued graph G^S is called vertex regular S-valued graph. If $\psi(x,y) = a$, $\forall(x,y) \in E$ and some $a \in S$, then the corresponding S- valued graph G^S is called edge regular S-valued graph. An S-valued graph G^S is said to be S-regular if it is both a vertex regular and an edge regular S-valued graph.*

Definition 2.4. [5] *Let $G^S = (V,E,\sigma,\psi)$ be a S-valued graph corresponding to a graph G, and $a \in S$. G^S is said to be a (a,k)- regular if the graph G is k-regular and $\sigma(v) = a$ for every $v \in V$.*

Definition 2.5. [8] *Let $G^S = (V,E,\sigma,\psi)$ be a S-valued graph with n vertices and m edges. The order of a S-valued graph G^S is defined as $p_S = (\sum\limits_{v\in V}\sigma(v), n)$ where $n = \#$ vertices in G. The size of a S-valued graph G^S is defined as $q_S = (\sum\limits_{(u,v)\in E}\psi(u,v), m)$ where $m = \#$ edges in G. The degree of the vertex v_i of the S-valued graph G^S is defined as $deg_S(v_i) = (\sum\limits_{(v_i,v_j)\in E}\psi(v_i,v_j), l)$ where $l = \#$ edges incident with v_i*

Definition 2.6. [6] *Let $G = (V,E)$ be a graph with $E \neq \emptyset$. The middle graph $M(G)$ of a graph G is the graph whose vertex set is V_M and edge set is E_M, where $V_M = V \cup \{e_i^j = [v_i,v_j] : (v_i,v_j) \in E\}$ and $E_M = \{(e,f) : e$ and f are adjacent$\}$.*
Note: Adjacent in the sense that the corresponding edges are adjacent in G (in case of both vertices are edges). Otherwise, one is a vertex and the other is an edge incident with it.

Definition 2.7. [6] *Let $G = (V,E)$ be a graph, $G^S = (V,E,\sigma,\psi)$ be a semiring valued graph and $M(G) = (V_M,E_M)$ be a middle graph of G. Define the middle graph $M(G^S) = (V_M,E_M,\sigma_M,\psi_M)$ of a S-valued graph is the S-valued graph, where $\sigma_M : V_M \to S$ and $\psi_M : E_M \to S$ are defined by*

$$\sigma_M(v) = \begin{cases} \sigma(v) & if\ v \in V \\ \psi(v_i,v_j) & if\ v = e_i^j \in V_M \setminus V \end{cases}$$

$$\psi_M(e,f) = \begin{cases} min\{\sigma_M(e),\sigma_M(f)\} & if\ \sigma_M(e) \preceq \sigma_M(f)\ or\ \sigma_M(f) \preceq \sigma_M(e) \\ 0 & otherwise \end{cases}$$

Remark 2.1. [6] *The vertices and edges of $M(G^S)$ are the vertices and edges as in its underlying middle graph $M(G)$. Since every semiring posses a canonical pre-order, σ_M, ψ_M are well defined. In general, both vertices and edges of a S-valued graph have values in the semiring S, called S-values. We call σ_M, a S-vertex set and ψ_M, a S-edge set of S-valued graph $M(G^S)$.*

Theorem 2.1. [6] *If $M(G^S)$ is vertex regular S-valued graph then $M(G^S)$ is edge regular S-valued graph.*

Theorem 2.2. [6] *$M(G^S)$ is S-regular graph if and only if G^S is S-regular.*

Definition 2.8. [9] *Let S_1 and S_2 be semirings. A function $\beta : S_1 \to S_2$ is a homomorphism of semirings if $\beta(0_{S_1}) = 0_{S_2}$, $\beta(a+b) = \beta(a) + \beta(b)$ and $\beta(a \cdot b) = \beta(a) \cdot \beta(b)$, $\forall a, b \in S_1$.*

Definition 2.9. [10] *An S-valued automorphism of G^S is a pair of isomorphisms (α, β) that satisfies the property that $\{(v_i, a), (v_j, b)\} \in E(G^S)$ if and only if*

$$\{(\alpha(v_i), \beta(a)), (\alpha(v_j), \beta(b))\} \in E(G^S).$$

The set of all S-valued automorphisms of a S-valued graph G^S is denoted by $Aut(G^S)$.

Definition 2.10. [10] *An Edge automorphism is a pair of mapping $\phi = (\alpha, \beta)$, where $\alpha : E(G^S) \to E(G^S)$ that satisfies the property that (v_i, v_j), (v_i, v_k) are adjacent if and only if $\alpha(v_i, v_j)$ and $\alpha(v_i, v_k)$ are adjacent and $\beta(\psi(v_i, v_j)) = \psi(\alpha(v_i, v_j))$ and $\beta(\psi(v_i, v_k)) = \psi(\alpha(v_i, v_k))$.*

The set of all edge automorphisms of a S-valued graph G^S is denoted by $Aut_E(G^S)$. The induced edge automorphism is a particular case of vertex automorphism that presetves the adjacency of edges of S-valued graphs and the set of all such automorphisms will be represented by $Aut_I(G^S)$.

It is clear that $Aut_I(G^S) \subset Aut_E(G^S)$.

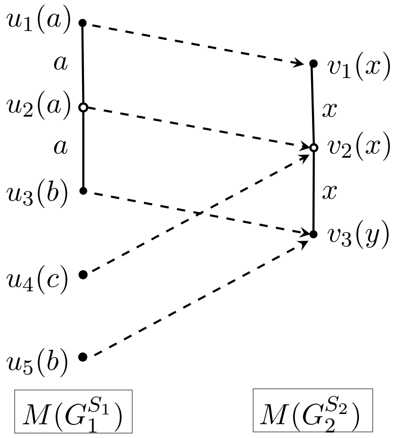
3. Homomorphism on $M(G^S)$

In this section, we introduce the notion of homomorphism of middle graph of S-valued graphs and some of its properties.

Definition 3.1. *Let $M(G_1^{S_1}) = (V'_M, E'_M, \sigma'_M, \psi'_M)$ and $M(G_2^{S_2}) = (V''_M, E''_M, \sigma''_M, \psi''_M)$ be the middle graph of S_1-valued and S_2-valued graphs respectively.*

(1) *A mapping $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is called a S-valued homomorphism if*
   (a) *$\alpha_M : V'_M \to V''_M$ is a graph homomorphism.*
   (b) *$\beta_M : S_1 \to S_2$ is a semiring homomorphism*
   such that
$$\beta_M(\sigma'_M(v)) \preceq \sigma''_M(\alpha_M(v)), \forall v \in V'_M \text{ and}$$
$$\beta_M(\psi'_M(u,v)) \preceq \psi''_M(\alpha_M(u), \alpha_M(v)), \forall (u,v) \in E'_M.$$
(2) *If f is a S-valued homomorphism, it induces a mapping $f_E : E'_M \to E''_M$ such that $f_E((u,v)) = (f(u), f(v))$ $\forall (u,v) \in E'_M$. The homomorphic image of $M(G_1^{S_1})$ in $M(G_2^{S_2})$ is defined as $f(M(G_1^{S_1})) = (f(V'_M), f_E(E'_M))$.*
(3) *A homomorphism $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is faithful if $f(M(G_1^{S_1}))$ is an induced subgraph of $M(G_2^{S_2})$. f is full $(u,v) \in E'_M$ if and only if $(f(u), f(v)) \in E''_M$.*

Example 3.1. *Let $M(G_1^{S_1})$ and $M(G_2^{S_2})$ be the middle graph of S-valued graphs.*



Then $V'_M = \{u_1, u_2, u_3, u_4, u_5\}$, $E'_M = \{(u_1, u_2), (u_2, u_3)\}$, $\sigma'_M(u_1) = a$, $\sigma'_M(u_2) = a$, $\sigma'_M(u_3) = b$, $\sigma'_M(u_4) = c$, $\sigma'_M(u_5) = b$, $\psi'_M((u_1, u_2)) = a$, $\psi'_M((u_2, u_3)) = a$, $V''_M = \{v_1, v_2, v_3\}$, $E''_M = \{(v_1, v_2), (v_2, v_3)\}$, $\sigma''_M(v_1) = x$, $\sigma''_M(u_2) = x$, $\sigma''_M(u_3) = y$, $\psi'_M((v_1, v_2)) = x$ *and* $\psi'_M((v_2, v_3)) = x$.

Define $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ *such that* $\alpha_M(u_1) = v_1$, $\alpha_M(u_2) = \alpha_M(u_4) = v_2$ *and* $\alpha_M(u_3) = \alpha_M(u_5) = v_3$. *Therefore,* α_M *is a graph homomorphism.* $\beta_M(a) = x$, $\beta_M(b) = x$ *and* $\beta_M(c) = x$. *Therefore,* β_M *is a semiring homomorphism.*

$$\begin{array}{ll}
\beta_M(\sigma'_M(u_1)) = \beta_M(a) = x & \sigma''_M(\alpha_M(u_1)) = \sigma''_M(v_1) = x \\
\beta_M(\sigma'_M(u_2)) = \beta_M(a) = x & \sigma''_M(\alpha_M(u_2)) = \sigma''_M(v_2) = x \\
\beta_M(\sigma'_M(u_3)) = \beta_M(b) = x & \sigma''_M(\alpha_M(u_3)) = \sigma''_M(v_3) = y \\
\beta_M(\sigma'_M(u_4)) = \beta_M(c) = x & \sigma''_M(\alpha_M(u_4)) = \sigma''_M(v_2) = x \\
\beta_M(\sigma'_M(u_5)) = \beta_M(b) = x & \sigma''_M(\alpha_M(u_5)) = \sigma''_M(v_3) = y
\end{array}$$

$$\begin{array}{ll}
\beta_M(\psi'_M((u_1, u_2))) = \beta_M(a) = x & \psi''_M((\alpha_M(u_1), \alpha_M(u_2))) = \psi''_M((v_1, v_2)) = x \\
\beta_M(\psi'_M((u_2, u_3))) = \beta_M(a) = x & \psi''_M((\alpha_M(u_2), \alpha_M(u_3))) = \psi''_M((v_2, v_3)) = x.
\end{array}$$

Therefore, $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ *is a* S*-valued homomorphism on middle graph of* S*-valued graphs.*

$$\begin{aligned}
f_E((u_1, u_2)) &= (f(u_1), f(u_2)) = (\alpha_M(u_1), \alpha_M(u_2)) = (v_1, v_2) \\
f_E((u_2, u_3)) &= (f(u_2), f(u_3)) = (\alpha_M(u_2), \alpha_M(u_3)) = (v_2, v_3).
\end{aligned}$$

Therefore, $f(M(G_1^{S_1})) = (f(V'_M), f_E(E'_M)) = (V''_M, E''_M)$. *Hence* f *is faithful but not full because there is an edge between* $(f(u_4), f(u_5))$ *but not between* (u_4, u_5).

Definition 3.2. *Let* $M(G_1^{S_1}) = (V'_M, E'_M, \sigma'_M, \psi'_M)$ *and* $M(G_2^{S_2}) = (V''_M, E''_M, \sigma''_M, \psi''_M)$ *be the middle graph of* S_1*-valued and* S_2*-valued graphs respectively.*

1. *A map* $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ *is* S*-valued injective if* α_M *is injective, i.e., no two vertices with a common neighbor in* V'_M *are mapped to a single vertex in* V''_M *and* $\beta_M : S_1 \to S_2$ *such that* $\beta_M(\sigma'_M(v)) = \sigma''_M(\alpha_M(v))$, $\forall v \in V'_M$ *and* $\beta_M(\psi'_M(u, v)) = \psi''_M(\alpha_M(u), \alpha_M(v))$, $\forall (u, v) \in E'_M$. *An* S*-valued injective homomorphism is called* S*-valued monomorphism.*
2. *A map* $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ *is* S*-valued surjective if* α_M *is surjective, i.e., every element of* V''_M *is the image of at least one element of* V'_M *and* $\beta_M : S_1 \to S_2$ *such that* $\beta_M(\sigma'_M(v)) = \sigma''_M(\alpha_M(v))$, $\forall v \in V'_M$ *and*

$\beta_M(\psi'_M(u,v)) = \psi''_M(\alpha_M(u),\alpha_M(v))$, $\forall(u,v)\in E'_M$. A S-valued surjective homomorphism is called S-valued epimorphism.

(3) *The middle graph of S-valued graphs $M(G_1^{S_1})$ and $M(G_2^{S_2})$ are said to be homomorphically equivalent, denoted by $M(G_1^{S_1}) \leftrightarrow M(G_2^{S_2})$, if there exists S-valued homomorphisms $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ and $g : M(G_2^{S_2}) \to M(G_1^{S_1})$.*

Definition 3.3. *Let $M(G_1^{S_1}) = (V'_M, E'_M, \sigma'_M, \psi'_M)$ and $M(G_2^{S_2}) = (V''_M, E''_M, \sigma''_M, \psi''_M)$ be the middle graph of S_1-valued and S_2-valued graphs respectively.*

(1) *A weak isomorphism $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is a pair of homomorphisms $\alpha_M : V'_M \to V''_M$, a graph isomorphism and $\beta_M : S_1 \to S_2$, a semiring homomorphism satisfying $\beta_M(\sigma'_M(v)) = \sigma''_M(\alpha_M(v))$, $\forall v \in V'_M$.*

(2) *A co-weak isomorphism $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is a pair of homomorphisms $\alpha_M : V'_M \to V''_M$, a graph isomorphism and $\beta_M : S_1 \to S_2$, a semiring homomorphism satisfying $\beta_M(\psi'_M(u,v)) = \psi''_M(\alpha_M(u),\alpha_M(v))$, $\forall(u,v)\in E'_M$.*

(3) *A mapping $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is called a S-valued isomorphism if $\alpha_M : V'_M \to V''_M$ is a graph isomorphism and $\beta_M : S_1 \to S_2$ is a semiring homomorphism such that $\beta_M(\sigma'_M(v)) = \sigma''_M(\alpha_M(v))$, $\forall v \in V'_M$ and $\beta_M(\psi'_M(u,v)) = \psi''_M(\alpha_M(u),\alpha_M(v))$, $\forall(u,v)\in E'_M$.*
If such an isomorphism from $M(G_1^{S_1})$ to $M(G_2^{S_2})$ exists and if both α_M and β_M are onto, then $M(G_1)^{S_1}$ is said to be S-isomorphic to $M(G_2)^{S_2}$ and we write it as $M(G_1)^{S_1} \cong M(G_2)^{S_2}$.

Theorem 3.1. *If $\phi = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a S-valued isomorphism, then there exist a S-valued isomorphism $\phi_M = (\alpha_M, \beta_M)$ from $M(G_1^{S_1})$ to $M(G_2^{S_2})$.*

Proof. The proof follows from the definition of middle graph and isomorphism of S-valued graphs. □

In particular, if G_1^S is isomorphic to G_2^S, then their corresponding middle graph $M(G_1^S)$ is isomorphic to $M(G_2^S)$.

Definition 3.4. *Let $M(G^S) = (V_M, E_M, \sigma_M, \psi_M)$ be the middle graph of S-valued graphs and let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be a partition of the vertex set of $M(G^S)$ into non-empty classes. The quotient $M(G^S)/\mathcal{P}$ of $M(G^S)$ by $\mathcal{P}$ is the graph $(\overline{V}_M, \overline{E}_M, \overline{\sigma}_M, \overline{\psi}_M)$ where $\overline{V}_M = \{V_1, \ldots, V_k\}$, $\overline{E}_M = \{(V_i, V_j) | (u_i, u_j) \in E_M$ for some $u_i \in V_i$ and $u_j \in V_j$, for $i \neq j\}$, $\overline{\sigma}_M(V_i) = \min\{\sigma(u_1), \ldots, \sigma(u_k)\}$, $u_1, \ldots, u_k \in V_i$ and*

$$\overline{\psi}_M((V_i, V_j)) = \begin{cases} min\{\overline{\sigma}_M(V_i), \overline{\sigma}_M(V_j)\} & \text{if } \overline{\sigma}_M(V_i) \preceq \overline{\sigma}_M(V_j) \text{ or } \overline{\sigma}_M(V_j) \preceq \overline{\sigma}_M(V_j) \\ 0 & \text{otherwise.} \end{cases}$$

The map $\pi_{\mathcal{P}} : V_M \to \overline{V}_M$ defined by $\pi_{\mathcal{P}}(u) = V_i$ such that $u \in V_i$, is the natural map for $\mathcal{P}$.

Proposition 3.1. *A map $f = (\alpha_M, \beta_M) : M(G_1^{S_1}) \to M(G_2^{S_2})$ is a S-valued homomorphism if and only if the preimage of every independent subset of V''_M is an independent set.*

Corollary 3.1. *Let $M(G^S)$ be a middle graph of S-valued graphs and $\mathcal{P}$ be a partition of vertex set of $M(G^S)$. Then $\pi_{\mathcal{P}}$ is a S-valued homomorphism if and only if V_i is an independent set for each i.*

Proposition 3.2. *For every S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ there is a partition $\mathcal{P}$ of V'_M into independent sets and a S-valued monomorphism $g : M(G_1^{S_1})/\mathcal{P} \to M(G_2^{S_2})$ such that $f = g \circ \pi_{\mathcal{P}}$.*

Definition 3.5. *A S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ gives rise to an equivalence relation (reflexive, symmetry and transitive) $\equiv_f$, the kernel of f, defined on V'_M by $u \equiv_f v$ if and only if $f(u) = f(v)$.*

Clearly, kernel is the null graph (null graph is a graph with isolated vertices) iff f is an S-valued injective homomorphism. This induces a partition $\mathcal{P}_f$ on V'_M.

Proposition 3.3. *A S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ is complete if and only if $g : M(G_1^{S_1})/\mathcal{P}_f \to M(G_2^{S_2})$ is an S-valued isomorphism.*

Definition 3.6.
(1) *A complete S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ is elementary if there is unique pair of nonadjacent vertices $u, v \in V'_M$ which are identified by f. We call $M(G_2^{S_2})$ an elementary quotient of $M(G_1^{S_1})$.*
(2) *An elementary S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ is a simple fold if the two vertices which are identified have a common neighbor.*
(3) *A folding is a S-valued homomorphism obtained as a sequence of simple folds. If $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ and f is a folding, we say that $M(G_1^{S_1})$ folds onto $M(G_2^{S_2})$.*

Definition 3.7. *For every middle graph of S-valued graph $M(G^S)$, there exists a S-valued homomorphisms $id_{M(G^S)} : M(G^S) \to M(G^S)$ called the identity S-valued homomorphism on $M(G^S)$, such that for every S-valued homomorphisms $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ we have $id_{M(G_2^{S_2})} \circ f = f = f \circ id_{M(G_1^{S_1})}$.*

3.1. **Retract and Core.**

Definition 3.8. *Let $M(G_1^{S_1})$ and $M(G_2^{S_2})$ be middle graph of S-valued graphs. Then $M(G_2^{S_2})$ is called a retract of $M(G_1^{S_1})$ if there are S-valued homomorphisms $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ and $g : M(G_2^{S_2}) \to M(G_1^{S_1})$ such that $f \circ g = id_{M(G_2^{S_2})}$. The S-valued homomorphism f is called a retraction (or split epimorphism) and g a co-retraction (split monomorphism or section).*

Every retraction is an S-valued epimorphism and every co-retraction is a S-valued monomorphism. The composition of two retractions is again a retraction and so a retract of a retract of $M(G^S)$ is a retract of $M(G^S)$. A co-retraction is always a faithful monomorphism and $g(M(G_2^{S_2}))$ an induced subgraph of $M(G_1^{S_1})$. Thus retracts of $M(G^S)$ are (isomorphic to) induced subgraphs of $M(G^S)$.

A retract of a graph $M(G^S)$ is always a quotient of $M(G^S)$ (i.e., a homomorphic image), and retractions are complete homomorphisms.

Proposition 3.4. *Any retraction of a connected graph of middle graph of S-valued graph is a folding.*

Definition 3.9. *A middle graph of S-valued graph $M(G^S)$ is a core if no proper subgraph of $M(G^S)$ is a retract of $M(G^S)$.*

Equivalently, a middle graph of S-valued graph $M(G^S)$ is a core if it has the minimum number of vertices of any graph in its S-valued homomorphism equivalence class.

Proposition 3.5. *Every finite middle graph of S-valued graph $M(G^S)$ has a core.*

Proposition 3.6. *If $M(G_1^{S_1})$ and $M(G_2^{S_2})$ are cores of a graph $M(G^S)$ then they are S-valued isomorphic.*

Proposition 3.7. *Let the middle graph of S-valued graphs $M(G_1^{S_1})$ and $M(G_2^{S_2})$ are homomorphically equivalent. Then the cores of $M(G_1^{S_1})$ and $M(G_2^{S_2})$ are isomorphic. Every finite middle graph of S-valued graph is homomorphically equivalent to a unique core.*

Clearly, $\leftrightarrow$ is an equivalence relation on the class of all middle graph of S-valued graphs. The equivalence class $\mathcal{H}(M(G^S))$ is the set of all middle graph of S-valued graphs which are homomorphically equivalent.

3.2. Exact sequence.

Definition 3.10.
(1) *The pair of S-valued homomorphisms $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ and $g : M(G_2^{S_2}) \to M(G_3^{S_3})$ is said to be exact at $M(G_2^{S_2})$ if image of f = kernel of g.*
(2) *A sequence $\cdots \to M(G_{i-1}^{S_{i-1}}) \to M(G_i^{S_i}) \to M(G_{i+1}^{S_{i+1}}) \to \cdots$ of S-valued homomorphisms is said to be an exact sequence if it is exact at every $M(G_i^{S_i})$ between a pair of S-valued homomorphisms.*

Proposition 3.8. *Let $M(G_1^{S_1})$, $M(G_2^{S_2})$ and $M(G_3^{S_3})$ be middle graph of S-valued graphs and 0 be the null graph of S-valued graph. Then*
(1) *The sequence $0 \to M(G_1^{S_1}) \xrightarrow{f} M(G_2^{S_2})$ of S-valued homomorphisms is exact at $M(G_1^{S_1})$ if f is S-valued injective.*
(2) *The sequence $M(G_2^{S_2}) \xrightarrow{g} M(G_3^{S_3}) \to 0$ of S-valued homomorphisms is exact at $M(G_3^{S_3})$ if g is S-valued surjective.*
(3) *The sequence $0 \to M(G_1^{S_1}) \xrightarrow{f} M(G_2^{S_2}) \xrightarrow{g} M(G_3^{S_3}) \to 0$ of S-valued homomorphisms is exact if f is S-valued injective, g is S-valued surjective and image of f = kernel of g. i.e., $M(G_2^{S_2})$ is an extension of $M(G_3^{S_3})$ by $M(G_1^{S_1})$.*

Definition 3.11. *The exact sequence $0 \to M(G_1^{S_1}) \xrightarrow{f} M(G_2^{S_2}) \xrightarrow{g} M(G_3^{S_3}) \to 0$ of S-valued homomorphisms is called a short exact sequence.*

Definition 3.12. *Let $0 \to M(G_1^{S_1}) \xrightarrow{f} M(G_2^{S_2}) \xrightarrow{g} M(G_3^{S_3}) \to 0$ and $0 \to M'(G_1^{S_1}) \xrightarrow{f} M'(G_2^{S_2}) \xrightarrow{g} M'(G_3^{S_3}) \to 0$ be two short exact sequences.*
(1) *A S-valued homomorphism of short exact sequences is a triple τ_1, τ_2, τ_3 of S-valued homomorphisms such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(G_1^{S_1}) & \xrightarrow{f} & M(G_2^{S_2}) & \xrightarrow{g} & M(G_3^{S_3}) & \longrightarrow & 0 \\ & & \downarrow{\tau_1} & & \downarrow{\tau_2} & & \downarrow{\tau_3} & & \\ 0 & \longrightarrow & M'(G_1^{S_1}) & \xrightarrow{f'} & M'(G_2^{S_2}) & \xrightarrow{g'} & M'(G_3^{S_3}) & \longrightarrow & 0 \end{array}$$

The S-valued homomorphism is an S-valued isomorphism of short exact sequences if τ_1, τ_2, τ_3 are all S-valued isomorphisms, in which case the extensions $M(G_2^{S_2})$ and $M'(G_2^{S_2})$ are said to be isomorphic extensions.
(2) *The two exact sequences are called S-valued equivalent if $M(G_1^{S_1}) = M'(G_1^{S_1})$, $M(G_3^{S_3}) = M'(G_3^{S_3})$ and τ_1, τ_3 are S-valued identity isomorphism. In this case the corresponding extensions $M(G_2^{S_2})$ and $M'(G_2^{S_2})$ are said to be S-valued equivalent extensions.*

Proposition 3.9. (1) *Composition of S-valued homomorphisms of short exact sequences is also a S-valued homomorphism.*
(2) *If the triple τ_1, τ_2, τ_3 is an S-valued isomorphism then $\tau_1^{-1}, \tau_2^{-1}, \tau_3^{-1}$ is an S-valued isomorphism in the reverse direction.*
(3) *S-valued isomorphism is an equivalence relation on any set of short exact sequences.*

Proposition 3.10. *Let τ_1, τ_2, τ_3 be a S-valued homomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(G_1^{S_1}) & \longrightarrow & M(G_2^{S_2}) & \longrightarrow & M(G_3^{S_3}) & \longrightarrow & 0 \\ & & \downarrow \tau_1 & & \downarrow \tau_2 & & \downarrow \tau_3 & & \\ 0 & \longrightarrow & M'(G_1^{S_1}) & \longrightarrow & M'(G_2^{S_2}) & \longrightarrow & M'(G_3^{S_3}) & \longrightarrow & 0 \end{array}$$

(1) *If τ_1 and τ_3 are S-valued injective then τ_2 is S-valued injective.*
(2) *If τ_1 and τ_3 are S-valued surjective then τ_2 is S-valued surjective.*
(3) *If τ_1 and τ_3 are S-valued isomorphisms then τ_2 is an S-valued isomorphism.*

3.3. **Regularity.**

Theorem 3.2. *If $\phi = (\alpha, \beta)$ is a S-valued isomorphism from a vertex regular graph $G_1^{S_1}$ with S_1-vertex set $\{a\}$ into a S_2-valued graph $G_2^{S_2}$, then $M(G_2^{S_2})$ is a S_2-vertex regular with S_2-vertex set $\{\beta_M(a)\}$.*

Proof. Let $\phi = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ be the S-valued isomorphism. Then by theorem 3.1, there exist S-valued isomorphism $\phi_M = (\alpha_M, \beta_M)$ from $M(G_1^{S_1})$ to $M(G_2^{S_2})$, where $M(G_1^{S_1}) = (V'_M, E'_M, \sigma'_M, \psi'_M), M(G_2^{S_2}) = (V''_M, E''_M, \sigma''_M, \psi''_M)$.

Since $G_1^{S_1}$ is a vertex regular graph with S_1-vertex set $\{a\}$, by theorem 2.2, $M(G_1^{S_1})$ is a vertex regular graph with $\sigma'_M(v) = a$, $\forall v \in V'_M$.

By theorem 3.1, since α_M is a graph isomorphism,

$$\sigma''_M(w) = \sigma''_M(\alpha_M(v)) = \beta_M(\sigma'_M(v)) = \beta_M(a),\ \forall w \in V''_M.$$

Hence $M(G_2^{S_2})$ is S_2-vertex regular graph with S_2-vertex set $\{\beta_M(a)\}$. □

Corollary 3.2. *If $\phi = (\alpha, \beta)$ is a S-valued isomorphism from a S_1-regular graph $G_1^{S_1}$ with S_1-vertex set $\{a\}$ into a S_2-valued graph $G_2^{S_2}$, then $M(G_2^{S_2})$ is a S_2- regular graph with S_2-vertex set $\{\beta(a)\}$.*

Proof. Since $G_1^{S_1}$ is S_1-regular, $G_1^{S_1}$ is S_1-vertex regular graph. By theorem 3.2, $M(G_2^{S_2})$ is a S_2-vertex regular graph and hence it is a S_2-edge regular graph with S_2-vertex set $\{\beta(a)\}$ by theorem 2.1. Therefore $M(G_2^{S_2}))$ is both vertex and edge regular which implies $M(G_2^{S_2})$ is a S_2-regular graph. □

Theorem 3.3. *If $\phi_M = (\alpha_M, \beta_M)$ is a S-valued isomorphism from a S_1-edge regular graph $M(G_1^{S_1})$ with S_1-edge set $\{a\}$ into a S_2-valued graph $M(G_2^{S_2})$ and $\beta_M(a) = \beta_M(\sigma'_M(v)), \forall v \in V'_M$ then $M(G_2^{S_2})$ is a S_2-edge regular graph.*

Proof. Let $\phi_M = (\alpha_M, \beta_M)$ be a S-valued isomorphism and $M(G_1^{S_1})$ be a S_1-edge regular graph. Therefore $\psi'_M(v, u) = a$ for some $a \in S_1$ and for all $(v, u) \in E'_M$. Since α_M is a graph isomorphism, for every $(x, w) \in E''_M$ there exist $(u, v) \in E'_M$ such that $(x, w) = (\alpha_M(u), \alpha_M(v)) \in E''_M$.

$$\begin{aligned} \psi''_M(x, w) = \psi''_M(\alpha_M(u), \alpha_M(v)) &= min\{\sigma''_M(\alpha_M(u)), \sigma''_M(\alpha_M(v))\} \\ &= min\{\beta_M(\sigma'_M(u)), \beta_M(\sigma'_M(v))\} = \beta_M(a). \end{aligned}$$

Therefore $\psi''_M(x,w) = \beta_M(a)$ for every $(x,w) \in E''_M$. This implies $M(G_2^{S_2})$ is a S_2-edge regular graph if $\beta_M(a) = \beta_M(\sigma'_M(v))$, $\forall v \in V'_M$. □

Remark 3.1. *From theorem 3.3, S-valued isomorphism does not preserve S-edge regularity.*

Theorem 3.4. *Let $\phi = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ be the S-valued isomorphism.*

(1) *If $G_1^{S_1}$ is a S_1-regular with S_1-vertex set $\{a\}, a \in S_1$ then*

a) Order of $M(G_2^{S_2})$ is $p_{S_2} = \left(\sum_{v\in V''_M} \beta(a), n+m\right)$, where n is the number of vertices in $G_2^{S_2}$ and m is the number of edges in $G_2^{S_2}$. Further if $\beta(a) \in S_2$ is additively idempotent then $p_{S_2} = (\beta(a), n+m)$.

b) Size of $M(G_2^{S_2})$ is $q_{S_2} = \left(\sum_{(v_i,v_j)\in E''_M} \beta(a), \frac{1}{2}\left(\sum_{u_i\in V''_M} (\rho_G(u_i))^2\right) + m\right)$, where m is the number of edges in $G_2^{S_2}$. Further if $\beta(a) \in S_2$ is additively idempotent then $q_{S_2} = \left(\beta(a), \frac{1}{2} \sum_{u_i\in V''_M} (\rho_G(u_i))^2) + m)\right)$.

(2) *If $u \in V'_M$ such that $deg_{S_1}(u) = (a,l)$, l is the number of edges incident with u then there exist vertex $v \in V''_M$ with degree $\left(\sum_{(v,v_i)\in E''_M} \beta(a), l\right)$. Further if $\beta(a)$ is additively idempotent then its degree is $(\beta(a), l)$.*

Proof. 1) Let $G_1^{S_1}$ be a S_1-regular graph with S_1-vertex set $\{a\}$. 3.2, $\phi_M(M(G_1^{S_1})$ is a S_2-regular graph with S_2-vertex set $\{\beta(a)\}$ by corollary. i.e., $\sigma''_M(v) = \beta(a), \forall v \in V''_M$. Since α_M is a graph isomorphism, n is the number of vertices in V'_M and V''_M, we have for every $v \in V''_M$ there exist $u \in V'_M$, $v \in V''_M$ such that $v = \alpha(u)$.

$$O(M(G_2^{S_2})) = p_{S_2} = \left(\sum_{v\in V''_M} \sigma''_M(v), n+m\right) = \left(\sum_{u\in V'_M} \sigma'_M(\alpha_M(u)), n+m\right)$$
$$= \left(\sum_{u\in V'_M} \beta_M(\sigma'_M(u)), n+m\right) = \left(\sum_{u\in V'_M} \beta(a), n+m\right).$$

If $\beta(a) \in S_2$ is additively idempotent, then $p_{S_2} = (\beta(a), n+m)$.

Since α_M is graph isomorphism and m is number of edges in E'_M and E''_M,

$$\text{Size of } M(G_2^{S_2}) = q_{S_2} = \left(\sum_{(v_i,v_j)\in E''_M} \psi''_M(v_i,v_j), \frac{1}{2}\left(\sum_{u_i\in V''_M} (\rho_G(u_i))^2\right) + m\right).$$

$$\psi_2(\alpha(u_i), \alpha(u_j)) = min\{\sigma''_M(\alpha_M(u_i)), \sigma''_M(\alpha_M(u_j))\}$$
$$= min\{\beta_M(\sigma'_M(u_i)), \beta_M(\alpha_M(u_j))\} = min\{\beta(a), \beta(a)\} = \beta(a)$$

If $\beta(a) \in S_2$ is additively idempotent, then

$$q_{S_2} = \left(\beta(a), \frac{1}{2}\left(\sum_{u_i\in V''_M} (\rho_G(u_i))^2\right) + m\right).$$

2) Given $\deg_{S_1}(u) = (a,l), l = \deg(u)$. Since α_M is a graph isomorphism, there exist $v \in V''_M$ such that $\alpha_M(u) = v$ having degree l.

$$deg_{S_2}(v) = \left(\sum_{(v,v_i)\in E''_M} \psi''_M(v,v_i), l \right) = \left(\sum_{(u,u_i)\in E'_M} \psi''_M(\alpha_M(u), \alpha_M(u_i)), l \right)$$
$$= \left(\sum_{(u,u_i)\in E'_M} \beta(\psi'_M(u,u_i), l \right) = (\beta(a), l).$$

If $\beta(a) \in S_2$ is additively idempotent, then $\deg_{S_2}(v) = (\beta(a), l)$. □

4. Automorphism Groups of $M(G^S)$

In this section, we determine the automorphism groups on $M(G^S)$.

A S-valued homomorphism from a middle graph of S-valued graph to itself is called a S-valued endomorphism of $M(G^S)$. An S-valued endomorphism which is also an S-valued isomorphism is called an S-valued automorphism. The set of all endomorphisms of $M(G^S)$ is denoted $\mathrm{End}(M(G^S))$, while the set of all S-valued automorphisms of $M(G^S)$ is denoted $Aut(M(G^S))$.

Theorem 4.1. *The set* $\mathrm{End}(M(G^S))$ *is a monoid under composition of S-valued homomorphisms. Also the set $Aut(M(G^S))$ of all S-valued graph automorphisms of a S-valued graph $M(G^S)$ forms a group under function composition.*

Theorem 4.2. $Aut(M(G^S)) = Aut(\overline{M(G^S)})$

Proof. First we prove that $Aut(M(G^S)) \subseteq Aut(\overline{M(G^S)})$. Let $(f_M, g_M) \in Aut(M(G^S))$ and an edge $(v_i, v_j) \in E(\overline{M(G^S)})$. Since by the definition of S-valued automorphism, we have $(f_M, g_M)(v_i, v_j) \notin E(M(G^S))$. Hence $(f_M, g_M)(v_i, v_j) \in E(\overline{M(G^S)})$. Thus we have shown that $Aut(M(G^S)) \subseteq Aut(\overline{M(G^S)})$. Similarly we can prove that $Aut(\overline{M(G^S)}) \subseteq Aut(M(G^S))$. Hence the two automorphism groups are equal. □

Theorem 4.3. *The set $Aut_E(M(G^S))$ of all S-edge automorphisms of a graph $M(G^S)$ forms a group under function composition.*

Definition 4.1. *A graph G^S is said to be self complementary if G^S and $\overline{G}^S$ are isomorphic.*

Theorem 4.4. *If $M(G^S)$ is self-complementary, then $Aut_E(M(G^S)) = Aut_E(\overline{M(G^S)})$.*

Proof. Let $(\alpha_M, \beta_M) \in Aut_E(M(G^S))$, $(v_i, v_j), (v_i, v_k)$ are adjacent in $\overline{M(G^S)}$ iff (v_i, v_j), (v_i, v_k) are not adjacent in $M(G^S)$ where $\alpha_M : E(M(G^S)) \rightarrow E(M(G^S))$. By the definition of an edge automorphism, $(v_i, v_j), (v_i, v_k)$ are adjacent in $\overline{M(G^S)}$ iff $\alpha_M(v_i, v_j), \alpha_M(v_i, v_k)$ are not adjacent in $M(G^S)$. i.e., $(v_i, v_j), (v_i, v_k)$ are adjacent in $\overline{M(G^S)}$ iff $\alpha_M(v_i, v_j)$ and $\alpha_M(v_i, v_k)$ are adjacent in $\overline{M(G^S)}$. This implies $(\alpha_M, \beta_M) \in Aut_E(\overline{M(G^S)})$. Hence $Aut_E(M(G^S)) \subseteq Aut_E(\overline{M(G^S)})$. Similarly, $Aut_E(\overline{M(G^S)}) \subseteq Aut_E(M(G^S))$. Hence the two automorphism groups are equal. □

Theorem 4.5. *Let G^S be any connected S-valued graph. Then $G^S \ncong K_2^S$ if and only if $Aut(M(G^S)) = Aut_I(M(G^S))$*

Proof. First we assume that G^S is a connected S-valued graph on atleast three vertices. That is G^S must have at least two edges. Hence $M(G^S)$ have at least two edges.

Define $\tau_M : Aut(M(G^S)) \to Aut_I(M(G^S))$ such that $\tau_M(\phi_M) = \tau_M(\alpha_M, \beta_M) = (\alpha^I_M, \beta^I_M)$,, where (α^I_M, β^I_M) is the edge automorphism induced by (α_M, β_M).
We need to prove that this τ_M is an isomorphism.

1. Let $\phi_{1_M} = (\alpha_{1_M}, \beta_{1_M}), \phi_{2_M} = (\alpha_{2_M}, \beta_{2_M}) \in Aut(M(G^S))$ such that $\phi_{1_M} \neq \phi_{2_M}$ and let $a, b, c \in S$ with $a \preceq b \preceq c$.

Therefore, there must be a vertex $(v_i, a) \in V(M(G^S))$ such that $\phi_{1_M}(v_i, a) \neq \phi_{2_M}(v_i, a)$ and let (v_j, b) be a vertex adjacent to (v_i, a) (by the connectedness of G^S).
Case(i) $\phi_{1_M}(v_i, a) \neq \phi_{2_M}(v_j, b)$ or $\phi_{1_M}(v_j, b) \neq \phi_{2_M}(v_i, a)$, then we found that for the edge (w_{ij}, a), the induced automorphism $\phi^I_{1_M} \neq \phi^I_{2_M}$.
Case(ii) $\phi_{1_M}(v_i, a) = \phi_{2_M}(v_j, b)$ and $\phi_{1_M}(v_j, b) = \phi_{2_M}(v_i, a)$.

Since $M(G^S)$ has atleast three vertices, there exists another vertex (v_k, c) such that (v_k, c) is adjacent to either (v_i, a) or (v_j, b) or both. Suppose that $(w_{jk}, b) = (v_j, b)(v_k, c) \in E(M(G^S))$, we arrive that $\phi^I_{1_M}(w_{jk}, b) \neq \phi^I_{2_M}(w_{jk}, b)$. Thus in all the cases, we have shown that τ_M is one-one.

2. By the construction of induced automorphism, for each $\phi^I_M \in Aut_I(M(G^S))$, there is $\phi_M \in Aut(M(G^S))$ such that $\tau_M(\phi_M) = \phi^I_M$. This proves τ_M is onto.

3. Let $(w_{ij}, a) = \{(v_i, a), (v_j), b\}$. Take $u = (v_i, a), v = (v_j, b)$.
Define $\phi_{2_M}(u) = u', \phi_{2_M}(v) = v', \phi_{1_M}(u') = u'', \phi_{1_M}(v') = v''$.

$$\begin{aligned}\tau_M(\phi_{1_M}\phi_{2_M})(w_{ij}, a) &= \tau_M(\phi_{1_M}\phi_{2_M})\{u, v\} = \phi^I_{1_M}\phi^I_{2_M}\{u, v\} \\ &= \{(\phi_{1_M}\phi_{2_M})(u), \phi_{1_M}\phi_{2_M}(v)\} = \{\phi_{1_M}(u'), \phi_{1_M}(v')\} = \{u'', v''\}.\end{aligned}$$

$$\begin{aligned}\tau_M(\phi_{1_M})\tau_M(\phi_{2_M})(w_{ij}, a) &= \tau_M(\phi_{1_M})\tau_M(\phi_{2_M})(\{u, v\}) = \tau_M(\phi_{1_M})(\{\phi_{2_M}(u), \phi_{2_M}(v)\}) \\ &= \tau_M(\phi_{1_M})(\{u', v'\}) = \{\phi_{1_M}(u'), \phi_{1_M}(v')\} = \{u'', v''\}.\end{aligned}$$

Hence τ_M is an isomorphism. □

Proposition 4.1. *Let $M(G_1^{S_1})$ and $M(G_2^{S_2})$ be middle graph of S-valued graphs. Then $M(G_2^{S_2})$ is a retract of $M(G_1^{S_1})$ if and only if there exist S-valued homomorphisms $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ and $g : M(G_2^{S_2}) \to M(G_1^{S_1})$ such that $f \circ g \in Aut(M(G_2^{S_2}))$.*

In particular, if $M(G_2^{S_2})$ is a subgraph of $M(G_1^{S_1})$, then $M(G_2^{S_2})$ is a retract of $M(G_1^{S_1})$ if and only if there is a S-valued homomorphism $f : M(G_1^{S_1}) \to M(G_2^{S_2})$ whose restriction to $M(G_2^{S_2})$ is an automorphism of $M(G_2^{S_2})$.

Proposition 4.2. *A middle graph of S-valued graph $M(G^S)$ is a core if and only if endomorphism of $M(G^S)$ is an automorphism.*

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