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PAIR DIFFERENCE CORDIALITY OF SOME PRODUCT RELATED GRAPHS

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Abstract. In this paper we investigate the pair difference cordial labeling behaviour of some product related graphs.

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1. Introduction

In this paper we consider only finite, undirected and simple graphs.The concept of cordial labeling was introduced by Cahit [1]. Various types of cordial related labeling was studied in [3, 4, 5, 7, 8, 9, 18, 19, 20, 21]. In the similar line the notion of pair difference cordial labeling of a graph was introduced in [10]. The pair difference cordial labeling behaviour of several graphs like path, cycle, star, wheel,triangular snake,alternate triangular snake, butterfly etc have been investigated in [10, 11, 12, 13, 14, 15, 16, 17]. In this paper we investigate the pair difference cordial labeling behaviour of some product related graphs.

2. Preliminaries

Definition 2.1. [6]. *The product graph $G_1 \times G_2$ is defined as follows: Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Definition 2.2. [6]. *The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in*

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the i^{th} copy G_2, where G_1 is graph of order n.

Definition 2.3. [6]. *The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uvw.*

Definition 2.4. [2]. *The graph $C_m \times C_n$ is called a Torus grid.*

Definition 2.5. [2]. *The Book graph B_m is the graph $S_m \times P_2$ where S_m is the star with $m + 1$ vertices.*
Let $V(B_m) = \{x, y, x_i, y_i : 1 \leq i \leq m\}$ and $E(B_m) = \{xy, xx_i, yy_ix_iy_i : 1 \leq i \leq m\}$.

Definition 2.6. [2]. *The Mangolian tent graph $M_{m,n}$ is obtained from the grid $P_m \times P_n$, n is odd by joinging one extra vertex above the grid and joining the vertex a with the vertices $a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n}$.*

Definition 2.7. [2]. *For even $n > 2$, define a plus graph of size n denoted by Pl_n as the graph obtained by starting with paths $P_2, P_4, P_6, \cdots, P_n, P_n, P_{n-2}, , \cdots, P_4, P_2$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows.*

Definition 2.8. [2]. *For $n \geq 3$, define a step grid as the graph obtained by starting with paths $P_n, P_n, P_{n-1}, P_{n-2}, \cdots, P_3, P_2$ arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows.Step grid graph is denoted by St_n.*

Definition 2.9. [2]. *The graph $M^*_{n,n}$ is obtained from the Mangolian Tent $M_{n,n}$ by deleting the vertices $a_{3,n}, a_{4,n}, a_{4,n-1}, a_{5,n}, a_{5,n-1}, a_{5,n-2}, \cdots, a_{n,n}, a_{n,n-1},$ $a_{n,n-2}, \cdots, a_{n,3}$.*

Definition 2.10. [10]. *Let $G = (V, E)$ be a (p, q) graph.*
Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1.A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

Theorem 2.1. [10]. *If G is a (p, q) pair difference cordial graph then*

$$q \leq \begin{cases} 2p-3 & \text{if } p \text{ is even} \\ 2p-1 & \text{if } p \text{ is odd} \end{cases}$$

Theorem 2.2. [10]. *The path P_n is pair difference cordial for all values of $n \neq 3$.*

Corollary 2.1. [10]. *The cycle C_n is pair difference cordial for all valus of $n \neq 3$.*

Theorem 2.3. [11]. *The wheel W_n is pair difference cordial if and only if n is even.*

Theorem 2.4. [10]. *$C_n \odot K_1$ is pair difference cordial for all values of $n \geq 3$.*

3. Main results

Theorem 3.1. *If G_1 and G_2 are (p_1, q_1) and (p_2, q_2) graphs respectively with $q_1 \geq p_1$ and $q_2 \geq p_2$ then $G_1 \times G_2$ is not pair difference cordial.*

Proof. Clearly $G_1 \times G_2$ is $(p_1p_2, p_1q_2 + p_2q_1)$ graph. Suppose $G_1 \times G_2$ is pair difference cordial. Then by theorem 2.12, there are two cases arises.
Case 1. p_1p_2 is even.
$2p_1p_2 - 3 \geq p_1q_2 + p_2q_1$
$\Rightarrow 2p_1p_2 - 3 \geq p_1p_2 + p_2p_1$
$\Rightarrow -3 \geq 0$, *a contradiction*

Case 2. p_1p_2 is odd.
$2p_1p_2 - 1 \geq p_1q_2 + p_2q_1$
$\Rightarrow 2p_1p_2 - 1 \geq p_1p_2 + p_2p_1$
$\Rightarrow -1 \geq 0$, *a contradiction*
Hence $G_1 \times G_2$ is not pair difference cordial. □

Corollary 3.1. *Torus grid $C_m \times C_n, m \geq 3$ and $n \geq 3$ is not pair difference cordial.*

Proof. Follows from Theorem 3.1.
□

Theorem 3.2. *The prism $C_n \times P_2$ is pair difference cordial for all values of $n \geq 3$.*

Proof. Let $V(C_n \times P_2) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(C_n \times P_2) = \{u_1u_n, v_1v_n\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}$.There are three cases arises.

Case 1. $3 \leq n \leq 5$.
A pair difference cordial labeling of $C_n \times P_2, 3 \leq n \leq 5$ is shown in Table 1.

n	u_1	u_2	u_3	u_4	u_5	v_1	v_2	v_3	v_4	v_5
3	1	2	3			-1	-2	-3		
4	1	2	3	4		-1	-2	-3	-4	
5	1	2	3	4	5	-1	-2	-3	-4	-5

Table 1

Case 2. $n \geq 6$.
Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_n$. Now we

assign the labels $-1, -2, -3$ to the vertices v_1, v_2, v_3 respectively and assign the labels $-4, -6, -5, -7$ respectively to the vertices v_4, v_5, v_6, v_7. Next assign the labels $-8, -10, -9, -11$ respectively to the vertices v_8, v_9, v_{10}, v_{11}. Proceeding like this until we reach the vertex v_n.
Note that i) When $n \equiv 0(mod4), v_n$ gets the label $-n$.
ii) When $n \equiv 1(mod4)$, v_n gets the label $-n$.
iii) When $n \equiv 2(mod4)$, v_n gets the label $-n+1$.
iv) When $n \equiv 3(mod4)$, v_n gets the label $-n$.
The Table 2 given below establish that this vertex labeling f is a pair difference cordial labeling of $C_n \times P_2$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n \equiv 0(mod4)$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1(mod4)$	$\frac{3n-2}{2}$	$\frac{3n+2}{2}$
$n \equiv 2(mod4)$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3(mod4)$	$\frac{3n-2}{2}$	$\frac{3n+2}{2}$

TABLE 2

□

Theorem 3.3. *The graph $K_m \times P_2$ is pair difference cordial if and only if $m \leq 3$.*

Proof. **Case 1.** $m \leq 3$.
Since $K_1 \times P_2 \cong P_2$, by theorem 2.13, $K_1 \times P_2$ is pair difference cordial. We know that $K_2 \times P_2 \cong C_4$, by corollary 2.14, $K_2 \times P_2$ is pair difference cordial. Next $K_3 \times P_2$ is a prism , by theorem 3.3, $K_3 \times P_2$ is pair difference cordial.
Case 2. $m \geq 4$.
Suppose $K_m \times P_2$ is pair difference cordial. Clearly $K_m \times P_2$ has $2m$ vertices and m^2 edges.
By theorem 2.12, $m^2 \leq 4m - 3$, this implies that $m^2 - 4m + 3 \leq 0$ hence $m \leq 3$.

□

Theorem 3.4. *The book graph B_m is pair difference cordial if and only if $m \geq 5$.*

Proof. Take the vertex set and edge set from definition 2.5.

When $m = 1, B_m \cong C_4$. By corolary 2.14, B_1 is pair difference cordial.
A pair difference cordial labeling of the book $B_m, 2 \leq m \leq 5$ is given in Table 3.

□

n	x	y	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
2	1	−	2	−3				3	−2			
3	1	4	2	−1	−3			3	−2	−4		
4	1	4	2	−1	−3	−5		−3	−2	−4	5	
5	1	4	2	6	−1	−3	−5	3	5	−2	−4	−6

TABLE 3

Theorem 3.5. *The graph $C_4 \times P_n$ is pair difference cordial for all values of n.*

Proof. Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i : 1 \leq i \leq n\}$ and $E(C_4 \times P_n) = \{a_ib_i, b_ic_i, c_id_i, d_ia_i : 1 \leq i \leq n\} \cup \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} : 1 \leq i \leq n-1\}$.
Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_n$ and assign the labels $-1, -2, -3, \cdots, -n$ respectively to the vertices $c_1, c_2, c_3, \cdots, c_n$. Secondly assign the labels $n+1, n+2, n+3, \cdots, 2n$ to the vertices $b_n, b_{n-1}, b_{n-2}, \cdots, b_1$ respectively. Next assign the labels $-(n+1), -(n+2), -(n+3), \cdots, -2n$ to the vertices $d_n, d_{n-1}, d_{n-2}, \cdots, d_1$ respectively.

Pair difference cordial labeling of $C_4 \times P_5$ is given in figure 1.

□

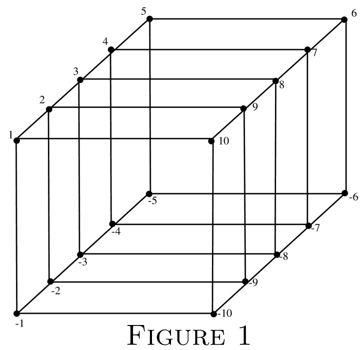


FIGURE 1

Theorem 3.6. *The graph $(C_4 \times P_n) \odot K_1$ is pair difference cordial for all values of n.*

Proof. Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i : 1 \leq i \leq n\}$ and $E(C_4 \times P_n) = \{a_ib_i, b_ic_i, c_id_i, d_ia_i : 1 \leq i \leq n\} \cup \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} : 1 \leq i \leq n-1\}$ and let $V((C_4 \times P_n) \odot K_1) = V(C_4 \times P_n) \cup \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E((C_4 \times P_n) \odot K_1) = E(C_4 \times P_n) \cup \{a_iu_i, b_ix_i, c_iy_i, d_iv_i : 1 \leq i \leq n\}$.

Case 1. n is odd.
Subcase 1. $n = 1$.

Since $(C_4 \times P_n) \odot K_1 \cong C_4 \odot K_1$. By using theorem 2.16, $(C_4 \times P_n) \odot K_1$ is pair difference cordial.
Subcase 2. $n \geq 3$.

Assign the labels $1, 5, 9 \cdots, (2n-1)$ respectively to the vertices $u_1, u_3, u_5, \cdots, u_n$ and assign the labels $4, 8, 12, \cdots, (2n-2)$ respectively to the vertices $u_2, u_4, u_6, \cdots, u_{n-1}$. Secondly assign the labels $2, 6, 10 \cdots, 2n$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_n$ respectively and assign the labels $3, 7, 11, \cdots, 2n-3$ to the vertices $a_2, a_4, a_6, \cdots, a_{n-1}$ respectively.

Next assign the labels $(2n+1), (2n+5), (2n+9), \cdots, (4n-1)$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_n$ and assign the labels $(2n+2), (2n+6), (2n+10), \cdots, (4n)$ respectively to the vertices $d_1, d_3, d_5, \cdots, d_n$. Now we assign the labels $(2n+3), (2n+7), (2n+11), \cdots, (4n-3)$ respectively to the vertices $d_2, d_4, d_6, \cdots, d_{n-1}$ and assign the labels $(2n+4), (2n+8), (2n+12), \cdots, (4n-2)$ respectively to the vertices $v_2, v_4, v_6, \cdots, v_{n-1}$.

Assign the labels $-1, -5, -9 \cdots, -(2n-1)$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_n$ and assign the labels $-4, -8, -12, \cdots, -(2n-2)$ respectively to the vertices $x_2, x_4, x_6, \cdots, x_{n-1}$. We assign the labels $-2, -6, -10 \cdots, -2n$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_n$ respectively and assign the labels $-3, -7, -11, \cdots, -(2n-3)$ to the vertices $b_2, b_4, b_6, \cdots, b_{n-1}$ respectively.

Next assign the labels $-(2n+1), -(2n+5), -(2n+9), \cdots, -(4n-1)$ respectively to the vertices $y_1, y_3, y_5, \cdots, y_n$ and assign the labels $-(2n+2), -(2n+6), -(2n+10), \cdots, -(4n)$ respectively to the vertices $c_1, c_3, c_5, \cdots, c_n$. Now we assign the labels $-(2n+3), -(2n+7), -(2n+11), \cdots, -(4n-3)$ respectively to the vertices $y_2, y_4, y_6, \cdots, y_{n-1}$ and assign the labels $-(2n+4), -(2n+8), -(2n+12), \cdots, -(4n-2)$ respectively to the vertices $c_2, c_4, c_6, \cdots, c_{n-1}$.

Case 2. n is even.

Subcase 1. $n = 2$.

Assign the labels $1, 2, 3, 4, 5, 6, 7, 8$ respectively to the vertices $u_1, a_1, a_2, u_2, u_3, a_3, a_4, u_4$ and assign the labels $-1, -2, -3, -4, -5, -6, -7, -8$ to the vertices $x_1, b_1, b_2, x_2, x_3, c_3, c_4, x_4$ respectively. Secondly assign the labels $9, 10, 11, 12, 13, 14, 15, 16$ to the vertices $v_1, d_1, d_2, v_2, v_3, d_3, d_4, v_4$ respectively and assign the labels $-9, -10, -11, -12, -13, -14, -15, -16$ respectively to the vertices $c_1, y_1, c_2, y_2, c_3, y_3, c_4, y_4$.

Subcase 2. $n = 4$.
Assign the labels $1, 2, 3, 4, 5, 6, 7, 8$ respectively to the vertices $u_1, a_1, a_2, u_2, d_1, v_1, d_2, v_2$ and assign the labels $-1, -2, -3, -4, -5, -6, -7, -8$ to the vertices $x_1, b_1, b_2, x_2, c_1, y_1, c_2, y_2$ respectively.

Subcase 3. $n \geq 6$.

Assign the labels $1, 5, 9 \cdots, (2n-3)$ respectively to the vertices $u_1, u_3, u_5, \cdots, u_{n-1}$ and assign the labels $4, 8, 12, \cdots, (2n)$ respectively to the vertices $u_2, u_4, u_6, \cdots, u_n$. Secondly assign the labels $2, 6, 10 \cdots, 2n-2$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{n-1}$ respectively and assign the labels $3, 7, 11, \cdots, 2n-1$ to the vertices $a_2, a_4, a_6, \cdots, a_n$ respectively.

Next assign the labels $(2n+1), (2n+5), (2n+9), \cdots, (4n-3)$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_{n-1}$ and assign the labels $(2n+2), (2n+6), (2n+10), \cdots, (4n-2)$ respectively to the vertices $d_1, d_3, d_5, \cdots, d_{n-1}$. Now we assign the labels $(2n+3), (2n+7), (2n+11), \cdots, (4n-1)$ respectively to the vertices $d_2, d_4, d_6, \cdots, d_n$ and assign the labels $(2n+4), (2n+8), (2n+12), \cdots, (4n)$ respectively to the vertices $v_2, v_4, v_6, \cdots, v_n$.

Assign the labels $-1, -5, -9 \cdots, -(2n-3)$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{n-1}$ and assign the labels $-4, -8, -12, \cdots, -(2n)$ respectively to the vertices $x_2, x_4, x_6, \cdots, x_n$. We assign the labels $-2, -6, -10 \cdots, -(2n-2)$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{n-1}$ respectively and assign the labels $-3, -7, -11, \cdots, -(2n-1)$ to the vertices $b_2, b_4, b_6, \cdots, b_n$ respectively.

Next assign the labels $-(2n+1), -(2n+5), -(2n+9), \cdots, -(4n-11)$ respectively to the vertices $y_1, y_3, y_5, \cdots, y_{n-5}$ and assign the labels $-(2n+2), -(2n+6), -(2n+10), \cdots, -(4n-10)$ respectively to the vertices $c_1, c_3, c_5, \cdots, c_{n-5}$. Now we assign the labels $-(2n+3), -(2n+7), -(2n+11), \cdots, -(4n-9)$ respectively to the vertices $y_2, y_4, y_6, \cdots, y_{n-4}$ and assign the labels $-(2n+4), -(2n+8), -(2n+12), \cdots, -(4n-8)$ respectively to the vertices $c_2, c_4, c_6, \cdots, c_{n-4}$.

Finally assign the labels $4n-7, 4n-5, 4n-3, 4n-1$ respectively to the vertices $c_{n-3}, c_{n-2}, c_{n-1}, c_n$ and assign the abels $4n-6, 4n-4, 4n-2, 4n$ to the vertices $y_{n-3}, y_{n-2}, y_{n-1}, y_n$ respectively.

In both cases $\Delta_{f_1^c} = \Delta_{f_1} = 6n-2$.

□

Theorem 3.7. *The graph $(C_4 \times P_n) \odot 2K_1$ is pair difference cordial for all values of n.*

Proof. Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i : 1 \le i \le n\}$ and $E(C_4 \times P_n) = \{a_ib_i, b_ic_i, c_id_i, d_ia_i : 1 \le i \le n\} \cup \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} : 1 \le i \le n-1\}$ and let $V((C_4 \times P_n) \odot 2K_1) = V(C_4 \times P_n) \cup \{u_i, v_i, x_i, y_i, l_i, m_i, o_i, p_i : 1 \le i \le n\}$ and $E((C_4 \times P_n) \odot 2K_1) = E(C_4 \times P_n) \cup \{a_iu_i, b_iv_i, c_ix_i, d_iy_i, a_il_i, b_im_i, c_io_i, d_ip_i : 1 \le i \le n\}$.

Assign the labels $1, 4, 7, \cdots, (3n-2)$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_n$ and assign the labels $3, 6, 9, \cdots, 3n$ respectively to the vertices $l_1, l_2, l_3, \cdots, l_n$. Now assign the labels $2, 5, 8, \cdots, 3n-1$ respectively to the vertices $a_1, a_2,$
$a_3, \cdots, a_n$ respectively .

Secondly assign the labels $(3n+1), (3n+4), (3n+7), \cdots, (6n-2)$ respectively to the vertices $v_1, v_2, v_3, \cdots, v_n$ and assign the labels $(3n+2), (3n+5), (3n+8), \cdots, (6n-1)$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_n$. Now assign the labels $(3n+3), (3n+6), (3n+9), \cdots, (6n)$ respectively to the vertices $m_1, m_2, m_3, \cdots, m_n$ respectively.

Thirdly assign the labels $-1, -4, -7, \cdots, -(3n-2)$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-3, -6, -9, \cdots, -3n$ respectively to the vertices $o_1, o_2, o_3, \cdots, o_n$. Now assign the labels $-2, -5, -8, \cdots, -(3n-1)$ respectively to the vertices $c_1, c_2, c_3, \cdots, c_n$ respectively .

Next we assign the labels $-(3n+1), -(3n+4), -(3n+7), \cdots, -(6n-8)$ respectively to the vertices $y_1, y_2, y_3, \cdots, y_{n-2}$ and assign the labels $-(3n+2), -(3n+5), -(3n+8), \cdots, -(6n-7)$ respectively to the vertices $d_1, d_2, d_3, \cdots, d_{n-2}$. Now assign the labels $-(3n+3), -(3n+6), -(3n+9), \cdots, -(6n-6)$ respectively to the vertices $p_1, p_2, p_3, \cdots, p_{n-2}$ respectively.

Finally assign the labels $-(6n-5), -(6n-4), -(6n-3), -(6n-1), -6n$ respectively to the vertices $d_{n-1}, y_{n-1}, p_{n-1}, d_n, y_n, p_n$.

□

Theorem 3.8. *The subdivision of $C_4 \times P_n$, $S(C_4 \times P_n)$ is pair difference cordial for all values of n.*

Proof. Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i : 1 \le i \le n\}$ and $E(C_4 \times P_n) = \{a_ib_i, b_ic_i, c_id_i, d_ia_i : 1 \le i \le n\} \cup \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} : 1 \le i \le n-1\}$.

Let u_i, v_i, x_i and y_i be the vertices which subdivide the edges a_ib_i, b_ic_i, c_id_i and $d_ia_i, 1 \leq i \leq n$ respectively.
Let l_i, m_i, o_i and p_i be the vertices which subdivide the edges $a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}$ and $d_id_{i+1}, 1 \leq i \leq n-1$ respectively.

Case 1. $n = 1$.
When $n = 1, S(Q_1) \cong C_8$, By corollary 2.14, $S(Q_1)$ is pair difference cordial.

Case 2. $n \geq 2$.

Assign the labels $1, 3, 5, \cdots, (2n-1)$ to the vertices $d_1, d_2, d_3, \cdots, d_n$ respectively and assign the labels $-1, -3, -5, \cdots, -(2n-1)$ respectively to the vertices $c_1, c_2, c_3, \cdots, c_n$. Next we assign the labels $2n+1, 2n+3, 2n+5, \cdots, (4n-3)$ to the vertices $a_1, a_2, a_3, \cdots, a_{n-1}$ respectively and assign the labels $-(2n+1), -(2n+3), -(2n+5), \cdots, -(4n-3)$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{n-1}$.

Now we assign the labels $4n+1, 4n+2, 4n+3, \cdots, 5n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$ respectively and assign the labels $-(4n+1), -(4n+2), -(4n+3), \cdots, -(5n)$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_n$. We assign the labels $5n+1, 5n+2, 5n+3, \cdots, (6n-2)$ to the vertices $y_2, y_3, y_4, \cdots, y_{n-1}$ respectively and assign the labels $-(5n+1), -(5n+2), -(5n+3), \cdots, -(6n-2)$ respectively to the vertices $v_2, v_3, v_4, \cdots, v_{n-1}$.

Assign the labels $2, 4, 6, \cdots, (2n-2)$ to the vertices $p_1, p_2, p_3, \cdots, p_{n-1}$ respectively and assign the labels $-2, -4, -6, \cdots, -(2n-2)$ respectively to the vertices $o_1, o_2, o_3, \cdots, o_{n-1}$. Next we assign the labels $2n+2, 2n+4, 2n+6, \cdots, (4n-2)$ to the vertices $l_1, l_2, l_3, \cdots, l_{n-1}$ respectively and assign the labels $-(2n+2), -(2n+4), -(2n+6), \cdots, -(4n-2)$ respectively to the vertices $l_1, l_2, l_3, \cdots, l_{n-1}$.

Finally assign the labels $4n, -4n, 2n, 4n-1, -2n, -(4n-1)$ respectively to the vertices $a_n, b_n, y_n, y_1, v_n, v_1$. Clearly $\Delta_{f_1^c} = \Delta_{f_1} = 8n-4$.

□

Theorem 3.9. *Let G be a $(p, q)-$ connected graph. Then $G \times K_n$ is not pair difference cordial if $n > 3$.*

Proof. The order and size of $G \times K_n$ are np and $nq + p\frac{n(n-1)}{2}$ respectively. Suppose $G \times K_n$ is pair difference cordial with $p \geq 1$. Then by theorem 2.12, there are two cases arises.

Case 1. np is even.
$nq + p\frac{n(n-1)}{2} \leq 2np - 3$
$\Rightarrow 2np - 3 \geq 2nq + p\frac{n(n-1)}{2}$
$\Rightarrow 4np - 6 \geq 2nq + n^2p - np$
$\Rightarrow -6 \geq 2n(p-1) + n^2p - np - 4np$
$\Rightarrow -6 \geq 2np - 2n + n^2p - 5np$
$\Rightarrow -6 \geq n^2p - 3np - 2n,$ *a contradiction*

Case 2. np is odd.
$nq + p\frac{n(n-1)}{2} \leq 2np - 1$
$\Rightarrow 2np - 1 \geq 2nq + p\frac{n(n-1)}{2}$

$\Rightarrow 4np - 2 \geq 2nq + n^2p - np$
$\Rightarrow -2 \geq 2n(p-1) + n^2p - np - 4np$
$\Rightarrow -2 \geq 2np - 2n + n^2p - 5np$
$\Rightarrow -2 \geq n^2p - 3np - 2n$, *a contradiction*
Hence $G \times K_n$ is not pair difference cordial. □

Theorem 3.10. *Let $p > 1$ and G be a $(p,q)-$ connected graph. Then the graph $G \times W_n, n \geq 3$ is not pair difference cordial.*

Proof. The order and size of $G \times K_n$ are $(n+1)p$ and $2np + (n+1)q$ respectively. Suppose $G \times W_n$ is pair difference cordial with $p \geq 1$. Then by theorem 2.12, there are two cases arises.

Case 1. $(n+1)p$ is even.
$2np + (n+1)q \leq 2(n+1)p - 3$
$\Rightarrow 2np + (n+1)q \leq 2np + 2p - 3$
$\Rightarrow (n+1)q \leq 2p - 3$
$\Rightarrow 2p - 3 \geq (n+1)q$
$\Rightarrow 2p - 3 \geq (n+1)(p-1)$
$\Rightarrow 2p - 3 \geq np - n + p - 1$
$\Rightarrow -2 + n \geq np - p$ *a contradiction*

Case 2. $(n+1)p$ is odd.
$2np + (n+1)q \leq 2(n+1)p - 1$
$\Rightarrow 2np + (n+1)q \leq 2np + 2p - 1$
$\Rightarrow (n+1)q \leq 2p - 1$
$\Rightarrow 2p - 1 \geq (n+1)q$
$\Rightarrow 2p - 1 \geq (n+1)(p-1)$
$\Rightarrow 2p - 1 \geq np - n + p - 1$
$\Rightarrow n \geq np - p$ *a contradiction*

□

Remark 3.1. *If $p = 1$ then the graph $G \times W_n$ is pair difference cordial if and only if n is even.*

Proof. When $p = 1$, $G \cong K_1$, $K_1 \times W_n \cong W_n$ by theorem 2.15, $K_1 \times W_n$ is pair difference cordial only when n is even. □

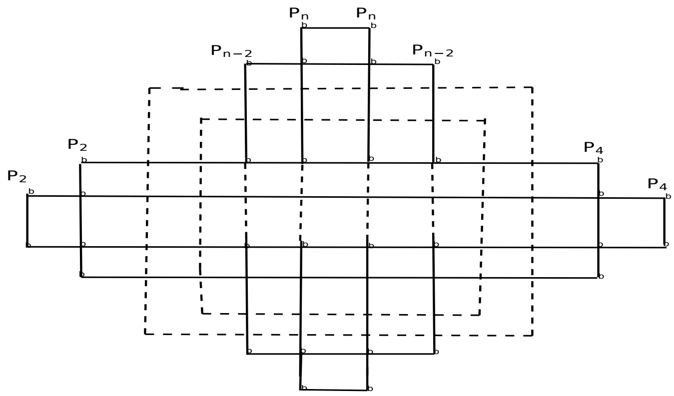
Theorem 3.11. *The Plus graph Pl_n is pair difference cordial for all values of n.*

Proof. Consider the paths $P_2, P_4, P_6, \cdots, P_n$, (from left to right).
Let P_n be the path ${u_1}^{(n)}, {u_2}^{(n)}, {u_3}^{(n)}, \cdots, {u_n}^{(n)}$.

Consider the paths $P_2, P_4, P_6, \cdots, P_n$, (from right to left).
Let P_n be the path ${v_1}^{(n)}, {v_2}^{(n)}, {v_3}^{(n)}, \cdots, {v_n}^{(n)}$.

The plus grpah Pl_n is given below .

Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices ${u_1}^{(n)}, {u_2}^{(n)}, {u_3}^{(n)}, \cdots, {u_n}^{(n)}$ and assign the labels $(n+1), (n+2), (n+3), \cdots, (2n-2)$ to the vertices ${u_1}^{(n-2)}, {u_2}^{(n-2)}, {u_3}^{(n-2)}, \cdots, {u_{n-2}}^{(n-2)}$. Next assign the labels $(2n-1), (2n), (2n+1), \cdots, (3n-6)$ respectively to the vertices ${u_1}^{(n-4)}, {u_2}^{(n-4)}, {u_3}^{(n-4)}, \cdots, {u_{n-4}}^{(n-4)}$ and assign the labels $(3n-5), (3n-$



4), $(3n-3), \cdots, (4n-10)$ to the vertices $u_1{}^{(n-6)}, u_2{}^{(n-6)}, u_3{}^{(n-6)}, \cdots, u_{n-6}{}^{(n-6)}$. Proceeding this process until we reach the vertices $u_1{}^{(2)}, u_2{}^{(2)}$ of the path P_2.

Secondly assign the labels $-1, -2, -3, \cdots, -n$ respectively to the vertices $v_1{}^{(n)}$, $v_2{}^{(n)}, v_3{}^{(n)}, \cdots, v_n{}^{(n)}$ and assign the labels $-(n+1), -(n+2), -(n+3), \cdots, -(2n-2)$ to the vertices $v_1{}^{(n-2)}, v_2{}^{(n-2)}, v_3{}^{(n-2)}, \cdots, v_{n-2}{}^{(n-2)}$. Next assign the labels $-(2n-1), -(2n), -(2n+1), \cdots, -(3n-6)$ respectively to the vertices $v_1{}^{(n-4)}, v_2{}^{(n-4)}$, $v_3{}^{(n-4)}, \cdots, v_{n-4}{}^{(n-4)}$ and assign the labels $-(3n-5), -(3n-4), -(3n-3), \cdots, -(4n-10)$ to the vertices $v_1{}^{(n-6)}, v_2{}^{(n-6)}, v_3{}^{(n-6)}, \cdots, v_{n-6}{}^{(n-6)}$. Proceeding this process until we reach the vertices $v_1{}^{(2)}, v_2{}^{(2)}$ of the path P_2.

□

Theorem 3.12. *The Step grid graph St_n is pair difference cordial for all values of n.*

Proof. Let $u_{1i}, u_{2i}, u_{3i}, \cdots, u_{ni}$ be the vertices of the path P_i.

Case 1. $n \equiv 0, 1 \pmod 4$.

Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices $u_{11}, u_{21}, u_{31}, \cdots, u_{n1}$ and assign the labels $n+1, n+2, n+3, \cdots, 2n$ to the vertices $u_{12}, u_{22}, u_{32}, \cdots, u_{n2}$ respectively. Next assign the labels $2n+1, 2n+2, 2n+3, \cdots, 3n-1$ respectively to the vertices $u_{13}, u_{23}, u_{33}, \cdots, u_{(n-1)3}$. Proceeding like this until we use the label $\frac{n^2+3n-4}{4}$. There are two cases arises.

Subcase 1. When u_{nx} receive the label $\frac{n^2+3n-4}{4}$.

Now we assign the labels $-1, -2, -3, \cdots,$ to the vertices of $(x+1)^{th}$ column consecutively. Proceeding like this until we use the label $-(\frac{n^2+3n-4}{4})$ to the vertex u_{n1}. Finally assign the label $-(\frac{n^2+3n-8}{4})$ to the vertex u_{n2}.

Subcase 2. When u_{nx} receive the label $\frac{n^2+3n-4}{4} - r$.

In this case we move to the r^{th} column. Assign the labels $\frac{n^2+3n-4}{4}-r+1$, $\frac{n^2+3n-4}{4}-r+2$, $\frac{n^2+3n-4}{4}-r+, \cdots, \frac{n^2+3n-4}{4}$ to the vertices $u_{1r}, u_{2r}, u_{3r}, \cdots, u_{xr}$. Next we assign the labels $-1, -2, -3, \cdots,$ to the vertices of $(x+1)^{th}$ column consecutively. Proceeding like this until we use the label $-(\frac{n^2+3n-4}{4})$ to the vertex u_{n1}. Finally assign the label $-(\frac{n^2+3n-8}{4})$ to the vertex u_{n2}.

Case 2. $n \equiv 2, 3 \pmod 4$.

Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices $u_{11}, u_{21}, u_{31}, \cdots, u_{n1}$ and assign the labels $n+1, n+2, n+3, \cdots, 2n$ to the vertices $u_{12}, u_{22}, u_{32}, \cdots, u_{n2}$ respectively. Next assign the labels $2n+1, 2n+2, 2n+3, \cdots, 3n-1$ respectively to the vertices $u_{13}, u_{23}, u_{33}, \cdots, u_{(n-1)3}$. Proceeding like this until we use the label $\frac{n^2+3n-2}{4}$. There are two cases arises.

Subcase 1. When u_{nx} receive the label $\frac{n^2+3n-2}{4}$.

Now we assign the labels $-1, -2, -3, \cdots,$ to the vertices of $(x+1)^{th}$ column consecutively. Proceeding like this until we use the label $-(\frac{n^2+3n-2}{4})$.

Subcase 2. When u_{nx} receive the label $\frac{n^2+3n-2}{4} - r$.

In this case we move to the r^{th} column. Assign the labels $\frac{n^2+3n-2}{4}-r+1$, $\frac{n^2+3n-2}{4}-r+2$, $\frac{n^2+3n-2}{4}-r+, \cdots, \frac{n^2+3n-2}{4}$ to the vertices $u_{1r}, u_{2r}, u_{3r}, \cdots, u_{xr}$. Next we assign the labels $-1, -2, -3, \cdots,$ to the vertices of $(x+1)^{th}$ column consecutively. Proceeding this Proces until we use the label $-(\frac{n^2+3n-2}{4})$.

□

Theorem 3.13. *The graph $M^*_{n,n}$ is pair difference cordial for all values of n.*

Proof. Assign the label 1 to the vertex a. Next assign the labels $2, 3, 4, \cdots, n+1$ respectively to the vertices $a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n}$ (from left to right) in the first row. Now we move to the second row. Now we assign the labels $-1, -2, -3, \cdots, -n$ respectively to the vertices $a_{2,n}, a_{2,n-1}, a_{2,n-2}, \cdots, a_{2,1}$ (from right to left) in the second row. Next we assign the labels $-(n+1), -(n+2), -(n++3), \cdots, -(2n-1)$ respectively to the vertices $a_{3,1}, a_{3,2}, a_{3,3}, \cdots, a_{3,n-1}$ (from left to right) in the third row. $n+2, n+3, n+4, \cdots, 2n-1$ to the vertices $a_{4,n-2}, a_{4,n-3}, a_{4,n-4}, \cdots, a_{4,1}$ (from right to left) in the fourth row.

Next assign the labels to the vertices of the fifth row from left to right and assign the labels to the vertices of the sixth row frm right to left. Proceedin like this until we reach the vertices $a_{n-2,1}, a_{n-2,2}, a_{n-2,3}, a_{n-2,4}$. Now there are four cases arises.

Case 1. $n \equiv 0 \pmod 4$.

Note that the vertices $a_{n-2,1}, a_{n-2,2}, a_{n-2,3}, a_{n-2,4}$ gets the labels $-(\frac{n^2+3n}{4}-3)$,$-(\frac{n^2+3n}{4}-4)$,$-(\frac{n^2+3n}{4}-5)$,$-(\frac{n^2+3n}{4}-6)$. Next assign the labels $-(\frac{n^2+3n}{4}-2)$,$-(\frac{n^2+3n}{4}-1)$,$-(\frac{n^2+3n}{4})$ respectively to the vertices $a_{n-1,1}, a_{n-1,2}, a_{n-1,3}$ and assign the labels $(\frac{n^2+3n}{4}-1)$,$(\frac{n^2+3n}{4})$ respectively to the vertices $a_{n,1}, a_{n,2}$.

Case 2. $n \equiv 1 \pmod 4$.

In this case the vertices $a_{n-2,1}, a_{n-2,2}, a_{n-2,3}, a_{n-2,4}$ receive the labels $-(\frac{n^2+3n}{4} - 4)$, $-(\frac{n^2+3n}{4} - 3)$,$-(\frac{n^2+3n}{4} - 2)$,$-(\frac{n^2+3n}{4} - 1)$. Now we assign the labels $(\frac{n^2+3n}{4} - 1)$,$(\frac{n^2+3n}{4} - 2)$,$(\frac{n^2+3n}{4} - 3)$ respectively to the vertices $a_{n-1,1}, a_{n-1,2}, a_{n-1,3}$ and assign the labels $(\frac{n^2+3n}{4})$,$-(\frac{n^2+3n}{4})$ respectively to the vertices $a_{n,1}, a_{n,2}$.

Case 3. $n \equiv 2 \pmod 4$.

Note that the vertices $a_{n-2,1}, a_{n-2,2}, a_{n-2,3}, a_{n-2,4}$ get the labels $-(\frac{n^2+3n-2}{4} - 2)$, $-(\frac{n^2+3n-2}{4} - 3)$,$-(\frac{n^2+3n-2}{4} - 4)$,$-(\frac{n^2+3n-2}{4} - 5)$. Finally assign the labels $(\frac{n^2+3n-2}{4} - 1)$,$(\frac{n^2+3n-2}{4})$,$(\frac{n^2+3n-2}{4} - 1)$ respectively to the vertices $a_{n-1,1}, a_{n-1,2}, a_{n-1,3}$ and assign the labels $-(\frac{n^2+3n-2}{4} - 1)$,$-(\frac{n^2+3n-2}{4})$ respectively to the vertices $a_{n,1}, a_{n,2}$.

Case 4. $n \equiv 3 \pmod 4$.

In this type the vertices $a_{n-2,1}, a_{n-2,2}, a_{n-2,3}, a_{n-2,4}$ receive the labels $(\frac{n^2+3n-2}{4} - 3)$, $(\frac{n^2+3n-2}{4} - 2)$,$(\frac{n^2+3n-2}{4} - 1)$,$(\frac{n^2+3n-2}{4})$. Lastly assign the labels $-(\frac{n^2+3n-2}{4} - 1)$,$-(\frac{n^2+3n-2}{4} - 2)$,$-(\frac{n^2+3n-2}{4} - 3)$ respectively to the vertices $a_{n-1,1}, a_{n-1,2}, a_{n-1,3}$ and assign the labels $-(\frac{n^2+3n-2}{4})$,$-(\frac{n^2+3n-2}{4})$ respectively to the vertices $a_{n,1}, a_{n,2}$.

□

4. Conclusions

In this paper, we have studied about the pair difference cordial labeling behavior of some product related graphs. Investigation of the pair difference cordiality of product of some special graphs are the open problems.

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