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ON A FRACTIONAL WAVE EQUATION WITH SINGULAR INITIAL DATA

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Abstract. This paper focuses on the time fractional wave equation with the use of conformable derivative $D^{(\alpha)}$ for $1 < \alpha < 2$ which we will prove to be inside Colombeau algebra, the initial data are singular distibution. Nets of conformable cosine family $(C_\epsilon^\alpha)_\epsilon$ with polynomial development in ϵ as $\epsilon \to 0$ are defined for the first time and used for solving this irregular fractional problems.

Keywords: Fractional wave equation, Colombeau algebra, Conformable cosine family, Generalized function.

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1. Introduction

The wave equation is a fundamental equation in physics that describes the behavior of waves. It is a partial differential equation that relates the second derivative of a wave function with respect to time to the second derivative of the same function with respect to space. This equation applies to a wide range of physical phenomena, including sound waves, electromagnetic waves, and water waves. The solution to the wave equation can be used to predict the behavior of waves in different situations, such as reflection, refraction, and interference. It is a cornerstone of many fields of physics, including acoustics, optics, and fluid dynamics, among others.

In traditional calculus, derivatives are defined for integer orders only, such as the first derivative, second derivative, and so on. However, conformable calculus allows for derivatives of any real or complex order, including non-integer orders.

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The basic idea behind conformable calculus is to redefine the traditional difference operator by using the conformable fractional difference operator, which is a generalization of the traditional difference operator. The conformable fractional difference operator uses the concept of fractional calculus [12], which is a division of calculus that concerned with derivatives and integrals of non-integer orders.

Conformable calculus has applications in various fields, including physics, engineering, finance, and biology. It provides a new tool for modeling complex systems that cannot be accurately described using traditional calculus [20][16].

In the first time A. Benmerrous and al [4] were able to studied the non-homogeneous wave equation in Colombeau algebra, in their paper they deal with the following abstract problem, taking the initial values as generalized functions:

$$\begin{cases} \dfrac{d^2}{dt^2}u(t,x) - c^2\dfrac{d^2}{dx^2}u(t,x) = F(t,u(t,x)) \quad x \in \mathbb{R}, \quad t > 0 \\ \quad u(0,x) = a(x) \\ \partial_t u(0,x) = b(x) \end{cases} \tag{1}$$

With $a, b \in \mathcal{G}$. Then they studied the associations for this abstract problem.

In this paper we characterize a new method for solving the nonlinear fractional wave equations with initial data are generalized functions as we can see in the following

$$\begin{cases} D^{(\alpha)}f(t,y) + Af(t,y) = F(t,f(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f(0,y) = u_0(y), \quad \partial_t^{(\alpha)}f(0,y) = v_0(y) \end{cases} \tag{2}$$

Where $A = -c^2\frac{d^2}{dx^2}$, $D^{(\alpha)}$ is the conformable derivation with $1 < \alpha < 2$, the linear operator $A : D(A) \subset \mathcal{G} \to \mathcal{G}, \quad F : [0,T] \times \mathcal{G} \to \mathcal{G}, \quad \mathcal{G}$ is the Colombeau algebra.

The paper is organized as follows, in section 2 we mention some notions of Colombeau's algebra and some notion concerning the new derivative, in section 3 we will prove the existence and uniqueness of conformable fractional derivative of order α in Colombeau algebra, in section 4 we will deal with the basic definition of conformable cosine family and some properties, in section 5, we provided the existence and uniqueness of generalized solution.

2. Preliminaries

2.1. **Colombeau algebra.** Here we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in[0;1]}$ of elements of a set X will be denoted by $X^{[0;1]}$, such sequences will also be called nets and simply written as u_ε.
Let $\mathcal{D}(\mathbb{R}^n)$ be the space of all smooth functions $\varphi : \quad \mathbb{R}^n \longrightarrow \mathbb{C}$ with compact support.
For $q \in \mathbb{N}$ we denote:

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{\varphi \in \mathcal{D}\left(\mathbb{R}^n\right) / \int \varphi(x)dx = 1 and \int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \leq \alpha \leq q\right\}.$$

The elements of the set $\mathcal{A}_q$ are called test functions.

It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \ldots$. Colombeau in his books has proved that the sets $\mathcal{A}_k$ are non empty for all $k \in \mathbb{N}$.

For $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ and $\epsilon > 0$ it is denoted as $\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$ and $\check{\varphi}(x) = \varphi(-x)$.

We denote by:

$$\mathcal{E}\left(\mathbb{R}^n\right) = \left\{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/ \text{ with } u(\varphi, x) \text{ is } \mathcal{C}^\infty \text{ to the second variable } x\right\},$$

$$u\left(\varphi_\varepsilon, x\right) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1,$$

$$\begin{aligned}\mathcal{E}_M\left(\mathbb{R}^n\right) = \big\{&\left(u_\varepsilon\right)_{\varepsilon>0} \subset \mathcal{E}\left(\mathbb{R}^n\right)/\forall K \subset \mathbb{R}^n, \forall a \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that} \\ &\sup_{x \in K} \|D^\alpha u_\varepsilon(x)\| = \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0\big\},\\ \mathcal{N}\left(\mathbb{R}^n\right) = \big\{&\left(u_\varepsilon\right)_{\varepsilon>0} \in \mathcal{E}\left(\mathbb{R}^n\right)/\forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \forall p \in \mathbb{N} \text{ such that} \\ &\sup_{x \in K} \|D^\alpha u_\varepsilon(x)\| = \mathcal{O}\left(\varepsilon^{p}\right) \text{ as } \varepsilon \to 0\big\},\end{aligned}$$

The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G}\left(\mathbb{R}^n\right) = \mathcal{E}_M\left[\mathbb{R}^n\right]/\mathcal{N}\left[\mathbb{R}^n\right]$, where the elements of the set $\mathcal{E}_M\left(\mathbb{R}^n\right)$ are moderate while the elements of the set $\mathcal{N}\left(\mathbb{R}^n\right)$ are negligible.

The meaning of the term 'association' in $\mathcal{G}(\mathbb{R})$ is given with the next two definitions.

Definition 1. *Generalized functions $f, g \in \mathcal{G}(\mathbb{R})$ are said to be associated, denoted $f \approx g$, if for each representative $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \|f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)\| \psi(x) dx = 0.$$

Definition 2. *Generalized functions $f \in \mathcal{G}(\mathbb{R})$ is said to admit some as $u \in \mathcal{D}'(\mathbb{R})$ 'associated distribution', denoted $f \approx u$, if for each representative $f(\varphi_\varepsilon, x)$ of f and any $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x) \psi(x) dx = \langle u, \psi \rangle.$$

2.2. **Conformable derivative.** The definition of conformable derivation is provided in the following part.

Definition 3. [12] *Let $n < \alpha \leq n+1$ and $u : \mathbb{R}^+ \to \mathbb{R}$ be n-differeniable, then the conformable fracrional derivauive of u of order α characterized by*

$$\begin{aligned} D^{(\alpha)} u(r) &= \lim_{\epsilon \to 0} \frac{u^{(n)}\left(r + \epsilon r^{n+1-\alpha}\right) - u^{(n)}(r)}{\epsilon} \\ D^{(\alpha)} u(0) &= \lim_{r \to 0} D^{(\alpha)} u(r) \end{aligned}$$

Remark 1. [12] *In light of the definition above, it is simple to demonstrate that*

$$D^{(\alpha)} u(r) = r^{n+1-\alpha} u^{(n+1)}(r)$$

with $n < \alpha \leq n+1$, and u is $(n+1)$-differeniable.

Definition 4. [12] *Let* $1 < \alpha \leq 2$,

$$\left(I^{(\alpha)}u\right)(r) = \int_0^t s^{\alpha-2}u(s)ds$$

.

Theorem 1. [12]

$$D^{(\alpha)}(I^{(\alpha)}u(r)) = u(r)$$

for $r \geq 0$

3. Generalized conformable derivative

Let $(f_\epsilon(t))_\epsilon$ be a representative of the function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $n-1 < \alpha < n$.

The generalized conformable fractional derivative of $(f_\epsilon(t))_\epsilon$, characterized by

$$D^{(\alpha)}f_\epsilon(y) = y^{1-\alpha}\frac{d}{dy}f_\epsilon(y) \tag{3}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$

Lemma 1. *Let* $(f_\epsilon(y))_\epsilon$ *be a representative of* $f(t) \in \mathcal{G}(\mathbb{R}^+)$. *Then,* $\forall \alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)}f_\epsilon(y) \mid$ *has a moderate bound.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]} \|D^{(\alpha)}f_\epsilon(y)\| = \sup_{y\in[0,T]} \|y^{1-\alpha}\frac{d}{dy}f_\epsilon(y)\| &\leq T^{1-\alpha}\sup_{y\in[0,T]} \|\frac{d}{dy}f_\epsilon(y)\| \\ &\leq T^{1-\alpha}C\epsilon^{-N} \\ &\leq C_{\alpha,T}\epsilon^{-N}\end{aligned}$$

Then, $\exists M \in \mathbb{N}$, such as

$$\sup_{y\in[0,T]} \|D^{(\alpha)}f_\epsilon(y)\| = \mathcal{O}\left(\epsilon^{-M}\right), \quad \epsilon \to 0$$

□

Lemma 2. *Let* $(f_{1\epsilon}(t))_\epsilon$, $(f_{2\epsilon}(t))_\epsilon$ *be two distinct representatives of* $f(t) \in \mathcal{G}(\mathbb{R}^+)$. *Then,* $\forall \alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)}f_{1\epsilon}(y) - D^{(\alpha)}f_{2\epsilon}(y) \mid$ *is negligible.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]} \|D^{(\alpha)}f_{1,\epsilon}(y) - D^{(\alpha)}f_{2,\epsilon}(y)\| &= \sup_{y\in[0,T]} \|y^{1-\alpha}\frac{d}{dy}f_{1,\epsilon}(y) - y^{1-\alpha}\frac{d}{dy}f_{2,\epsilon}(y)\| \\ &= \sup_{y\in[0,T]} \|y^{1-\alpha}\left(\frac{d}{dy}f_{1,\epsilon}(y) - \frac{d}{dy}f_{2,\epsilon}(y)\right)\| \\ &\leq T^{1-\alpha}\sup_{y\in[0,T]} \|\frac{d}{dy}f_{1,\epsilon}(y) - \frac{d}{dy}f_{2,\epsilon}(y)\|\end{aligned}$$

Since $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ represent the same Colombeau generalized function $f(y)$, so $\sup_{y\in[0,T]} \mid \frac{d}{dy}f_{1,\epsilon}(y) - \frac{d}{dy}f_{2,\epsilon}(y) \mid$ is negligible, then for all $p \in \mathbb{N}$

$$\sup_{y\in[0,T]} \|D^{(\alpha)}f_{1\epsilon}(y) - D^{(\alpha)}f_{2\epsilon}(y) \mid = \mathcal{O}\left(\epsilon^{-p}\right), \quad \epsilon \to 0$$

Therefore, $\sup_{y\in[0,T]} \|D^{(\alpha)}f_{1\epsilon}(y) - D^{(\alpha)}f_{2\epsilon}(y)\|$ is negligible. □

We may now initiate the generalized conformable fractional derivative of a Colombeau generalized function on $\mathbb{R}^+$ after establishing the first two lemmas.

Definition 5. *Let $f(y) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau function on $\mathbb{R}^+$.*
The generalized conformable fractional derivative of $f(y)$, using the notation $D^{(\alpha)}f(t) = \left[\left(D^{(\alpha)}f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfying (3).

Remark 2. *For $\alpha \in (0,1]$ the first-order derivative of $D^{(\alpha)}f_\epsilon(y)$ is*

$$\frac{d}{dy}D^{(\alpha)}f_\epsilon(y) = (1-\alpha)y^{-\alpha}\frac{d}{dy}f_\epsilon(y) + y^{1-\alpha}\frac{d^2}{dy^2}f_\epsilon(y)$$

and it fails to reach its limit.
Generally, the p-th order derivative $\frac{d^p}{dy^p}D^{(\alpha)}f_\epsilon(y)$ it fails to reach its limit on $\mathbb{R}^+$.

Then if we wants $D^{(\alpha)}$ to be in $\mathcal{G}(\mathbb{R}^+)$, thus the fractional derivative must be regularized.

Definition 6. *Let $(f_\epsilon)_\epsilon$ be a representative of a Colombesu generalized $f \in \mathcal{G}([0,\infty))$. The regularized of new fractional derivative of $(f_\epsilon)_{\epsilon\infty}$, is characterized by :*

$$\bar{D}^{(\alpha)}f_\epsilon(y) = \begin{cases} \left(D^{(\alpha)}f_\epsilon * \varphi_\epsilon\right)(y), & n-1 < \alpha < n \\ f_\epsilon^{(n)}(y) = (\frac{d}{dy})^n f_\epsilon(y), & \alpha = n, \end{cases} \tag{4}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.
where (3) gives $D^{(alpha)}f_\epsilon(y)$ and the first section gives $\varphi_\epsilon(y)$.

The convolution in (4) is $\left(D^{(\alpha)}f_\epsilon(y) * \varphi_\epsilon\right)(y) = \int_0^\infty D^{(\alpha)}f_\epsilon(y)\varphi_\epsilon(y-s)ds$.

Lemma 3. *Let $(f_\epsilon(y))_\epsilon$ be a representative of $f(y) \in \mathcal{G}(\mathbb{R}^+)$.*
So, $\forall\alpha > 0$, $k \in \{0,1,\ldots\}$, $\sup_{y\in[0,T]} \| \left(d^k/dy^k\right)\tilde{D}^{(\alpha)}f_\epsilon(y)\|$ has a moderate limit.

Proof. Let $0 < \epsilon < 1$.
For $\alpha \in \mathbb{N}, \tilde{D}^{(\alpha)}f_\epsilon(y)$ is the normal derivative of order α of $f_\epsilon(y)$ and the assertion follows immediately .
In the event that $n-1 < \alpha \leq n$, We've got

$$\begin{aligned}
\sup_{y\in[0,T]}\|\bar{D}^{(\alpha)}f_\epsilon(y)\| &= \sup_{y\in[0,T]}\|\left(D^{(\alpha)}\ f_\epsilon * \varphi_\epsilon\right)(y)\| \\
&\leq \sup_{y\in[0,T]}\|\int_0^\infty D^{(\alpha)}f_\epsilon(s)\varphi_\epsilon(y-s)ds\| \\
&\leq \sup_{r\in K}\|D^{(\alpha)}f_\epsilon(r)\| \sup_{y\in[0,T]}\|\int_K \varphi_\epsilon(y-s)ds\| \\
&\leq C\sup_{y\in K}\|D^{(\alpha)}f_\epsilon(y)\|
\end{aligned}$$

With C is a strictly positive constant.
Using the Lemma 1, $\sup_{y\in[0,T]} \mid D^{(\alpha)}f_\epsilon(y) \mid$ has a moderate bound, $\forall\alpha > 0$, as a result of this, $\sup_{y\in[0,T]} \mid \bar{D}^{(\alpha)}f_\epsilon(y) \mid$ has a moderate bound, too.

□

Lemma 4. *Let $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ be two different representatives of $f(y) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall\alpha > 0$, $k \in \{0,1,2,\ldots\}$, $\sup_{t\in[0,T]} \mid \left(\mathrm{d}^k/\mathrm{d}t^k\right)\left(\tilde{D}^{(\alpha)}f_{1\epsilon}(t) - \tilde{D}^{(\alpha)}f_{2\epsilon}(t)\right) \mid$ is negligible.*

Proof.

$$\sup_{y\in[0,T]} \mid \frac{d^k}{dy^k}\left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) \| =$$
$$\sup_{y\in[0,T]} \| \frac{d^k}{dy^k}\left(\left(D^{(\alpha)} f_{1\epsilon} * \varphi_\epsilon\right)(y) - \left(D^{(\alpha)} f_{2\epsilon} * \varphi_\epsilon\right)(y)\right) \mid$$
$$= \sup_{y\in[0,T]} \| \frac{d^k}{dy^k}\left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \varphi_\epsilon\right)(y)\|$$
$$= \sup_{y\in[0,T]} \| \left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \frac{d^k}{dy^k}\varphi_\epsilon\right)(y)\|$$
$$\leq \sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\| \sup_{y\in[0,T]} \| \int_K \frac{d^k}{dy^k}\varphi_\epsilon(y-r)dr\|$$
$$\leq C \sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\|$$

Using the Lemma 2, we have $\sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\|$ is negligible, so $\sup_{y\in[0,T]} \mid \frac{d^k}{dy^k}\left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) \mid$ is negligible. □

The regularized generalized conformable fractional derivative $D^{(alpha)}$ is now introduced in the following manner.

Definition 7. *Let $f(t) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau generalized function. The regularized generalized conformable fractional derivative of $f(t)$, writing $\bar{D}^{(\alpha)} f(t) = \left[\left(\tilde{D}^{(\alpha)} f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfy (4).*

4. Generalized conformable Cosine family

Let $(X, \|.\|)$ denote a Banach space, and $\mathcal{C}(X)$ denote the space of all linear continuous mappings.
Before we define the generalized conformable cosine family, we will state that an application from $\mathcal{G} \longrightarrow \mathcal{G}$ must be linear.

Definition 8. *Let X be a locally convex space with a semi-norm familly $(q_i)_{i\in I}$.*

We define $\mathcal{E}_M$ by the set of $(y_\epsilon)_\epsilon \subset X$ such that $\exists n \in \mathbb{N}$ and $\forall i \in I \subset \mathbb{N}$, $q_i(y_\epsilon) = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-n})$.

And
$\mathcal{N}(X)$ by $(y_\epsilon)_\epsilon \subset X$ such that $\forall m \in \mathbb{N}$ and $\forall i \in I \subset \mathbb{N}$, $q_i(y_\epsilon) = \mathcal{O}_{\epsilon\to 0}(\epsilon^n)$.

Then the Colombeau generalized function type by:

$$\overline{X} = \mathcal{E}_M(X)/\mathcal{N}(X)$$

Initially, using a provided family $(A_\epsilon)_{\epsilon\in[0,1]}$ of maps $A\epsilon : X \longrightarrow X$ we want to see if we can define a map $A : \overline{X} \longrightarrow \overline{X}$, $A_\epsilon \in \mathcal{L}(X)$.

The next lemma expresses the basic requirement:

Lemma 5. *Let $(A_\epsilon)_\epsilon$ represent a family of maps $A_\epsilon : X \longrightarrow X$.*

For each $(x_\epsilon)_\epsilon \in \mathcal{E}_M(X)$ and $(y_\epsilon)_\epsilon \in \mathcal{N}(X)$, suppose that:
1) $(A_\epsilon x_\epsilon)_\varepsilon \in \mathcal{E}_M(X)$
2) $(A_\epsilon (x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$

So

$$A : \begin{cases} \overline{X} \longrightarrow \overline{X} \\ x = [x_\epsilon] \longmapsto Ax = [A_\epsilon x_\epsilon] \end{cases}$$

is clearly stated.

Proof. The first attribute reveals that the class $[(A_\epsilon x_\epsilon)_\epsilon] \in \overline{X}$.
Let $x_\epsilon + y_\epsilon$ should serve as another example of $x = [x_\epsilon]$, we have from the second property:

$$(A_\epsilon (x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$$

and

$$[(A_\epsilon (x_\epsilon + y_\epsilon))_\epsilon] = [(A_\epsilon x_\epsilon))_\epsilon] \text{ in } \overline{X}$$

So A is well defined. □

We shall now introduce the idea of the generalized conformable cosine family(Convolution-type cosine family).

Definition 9.

$$E_{M,\alpha}\left(\mathbb{R}^+, \mathcal{C}(X)\right) := \Big\{ C_\epsilon^{\frac{1}{\alpha}} : \mathbb{R}^+ \to \mathcal{C}(X), \epsilon \in]0,1[/\forall T > 0, \exists a \in \mathbb{R} \;\; \textit{such that} \\ \sup_{t\in[0,T]} \|C_\epsilon^{\frac{1}{\alpha}}(t)\| = \mathcal{O}\left(\varepsilon^a\right) \;\; , \;\; \epsilon \to 0 \} \tag{5}$$

$$N_\alpha(\mathbb{R}^+, \mathcal{C}(X)) := \Big\{ N_\epsilon^{\frac{1}{\alpha}} : \mathbb{R}^+ \to \mathcal{C}(X), \epsilon \in]0,1[/\forall T > 0, \forall b \in \mathbb{R} \;\; \textit{such that} \\ \sup_{t\in[0,T]} \|N_\epsilon^{\frac{1}{\alpha}}(t)\| = \mathcal{O}\left(\epsilon^b\right) \;\; , \;\; \epsilon \to 0 \} \tag{6}$$

With the following characteristics:
1) $\exists s > 0$ and $\exists a \in \mathbb{R}$ such that

$$\sup_{t<s} \left\| \frac{N_\epsilon\left(t^{\frac{1}{\alpha}}\right)}{t} \right\| = O_{\epsilon\to 0}\left(\epsilon^a\right),$$

2) $\exists (H_\epsilon)_\epsilon$ in $\mathcal{C}(X)$ and $\epsilon \in]0,1[$ such that

$$\lim_{s\to 0} \frac{N_\epsilon\left(s^{\frac{1}{\alpha}}\right)}{s} e = H_\epsilon e, \quad e \in X,$$

For every $b > 0$,

$$\|H_\epsilon\| = O_{\epsilon\to 0}\left(\epsilon^b\right),$$

Proposition 1. *$N_\alpha(\mathbb{R}^+, \mathcal{C}(X))$ is an ideal of $E_{M,\alpha}(\mathbb{R}^+, \mathcal{C}(X))$ and $E_{M,\alpha}(\mathbb{R}^+, \mathcal{C}(X))$ is an algebra with respect to composition.*

Proof. Let $(C_\epsilon)_\epsilon \in E_{M,\alpha}\left([0,+\infty[,\mathcal{C}(X)\right)$ and $(N_\epsilon)_\epsilon \in \mathcal{N}_\alpha\left([0,+\infty[,\mathcal{C}(X)\right)$.
We shall simply establish the second statement, specifically,

$$\left(C_\epsilon\left(s^{\frac{1}{\alpha}}\right)N_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right)_\epsilon,\left(N_\epsilon\left(s^{\frac{1}{\alpha}}\right)C_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right)_\epsilon \in \mathcal{N}_\alpha\left([0,+\infty[,\mathcal{C}(X)\right)$$

Where $C_\epsilon\left(s^{\frac{1}{\alpha}}\right)N_\epsilon\left(s^{\frac{1}{\alpha}}\right)$ represents the composition.
By (1) and the definition of $\mathcal{N}_\alpha$ from the previous definition, we have:

$$\left\|C_\epsilon\left(s^{\frac{1}{\alpha}}\right)N_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right\| \leq \left\|C_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right\|\left\|N_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right\| = O_{\epsilon\to 0}\left(\epsilon^{a+b}\right),$$

The same is also true for $\left\|N_\epsilon\left(s^{\frac{1}{\alpha}}\right)C_\epsilon\left(s^{\frac{1}{\alpha}}\right)\right\|$.
Furthermore, (1) and (2) provide

$$\begin{aligned}\sup_{r<s}\left\|\frac{C_\epsilon\left(r^{\frac{1}{\alpha}}\right)N_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| &\leq \sup_{r<s}\left\|C_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|\sup_{r<s}\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| \\ &= O_{\epsilon\to 0}\left(\epsilon^{a}\right),\end{aligned}$$

In some situations $s>0$. We have,

$$\sup_{r>s}\left\|\frac{N_\epsilon\left(r^{\frac{1}{\alpha}}\right)C_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| = O_{\epsilon\to 0}\left(\epsilon^{a}\right),$$

For some $s>0$ and $a\in\mathbb{R}$. Let now $\epsilon\in]0,1[$ be fixed. We have

$$\begin{aligned}\left\|\frac{C_\epsilon(r^{\frac{1}{\alpha}})N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - C_\epsilon(0)H_\epsilon x\right\| &= \Big\|C_\epsilon(r^{\frac{1}{\alpha}})\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x + \\ &\quad C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - C_\epsilon(0)H_\epsilon x\Big\| \\ &\leq \left\|C_\epsilon(r^{\frac{1}{\alpha}})\right\|\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon x\right\| + \\ &\quad \left\|C_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - C_\epsilon(0)H_\epsilon x\right\|.\end{aligned}$$

According to (1) and (2), in addition to the continuity of $r\mapsto C_\epsilon(r^{\frac{1}{\alpha}})(H_\epsilon x)$ at 0, the final expression becomes zero as $r\to 0$, we have:

$$\begin{aligned}\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})C_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon C_\epsilon(0)x\right\| &= \Big\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(r^{\frac{1}{\alpha}})x - \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(0)x + \\ &\quad \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}C_\epsilon(0)x - H_\epsilon C_\epsilon(0)x\Big\| \\ &\leq \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\right\|\left\|C_\epsilon(r^{\frac{1}{\alpha}})x - H_\epsilon(r)C_\epsilon(0)x\right\| + \\ &\quad \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\left(C_\epsilon(0)x\right) - H_\epsilon\left(C_\epsilon(0)x\right)\right\|\end{aligned}$$

Assertions (1) and (2) require that the final expression goes to zero since $t\mapsto 0$. As a result, the proposition is proven in both circumstances. $\square$

Definition 10. *The Colombenu type algebra define by:*

$$G(\mathbb{R}^+, \mathcal{C}(X)) = E_{M,\alpha}(\mathbb{R}^+, \mathcal{C}(X))/N_\alpha(\mathbb{R}^+, \mathcal{C}(X))$$

Now we will define the concept of generelized conformable cosine family.

Definition 11. $C^\alpha = [(C_\epsilon^\alpha)]$ *with* $C_\epsilon \in E_{M,\alpha}(\mathbb{R}^+, \mathcal{C}(X))$ *say the generalized conformable cosine family if:*

1. $C^\alpha(0) = Id$
2. $C^\alpha\left((r+1)^{\frac{1}{\alpha}}\right) + C^\alpha\left((r-1)^{\frac{1}{\alpha}}\right) = 2C^\alpha\left(r^{\frac{1}{\alpha}}\right)C^\alpha\left(r^{\frac{1}{\alpha}}\right)$
3. *The mapping* $r \to C^\alpha(r)x$ *is a continuous mapping for each* $x \in \bar{X}$.

If $C^\alpha(r), r \in \mathbb{R}$ *is a strongly continuous conformable cosine family in* $\bar{X}$*, then:* $S^\alpha(r), r \in \mathbb{R}$ *is the one parameter family of operators in* $\bar{X}$ *defined by*

$$S^\alpha(r) = \int_0^r C^\alpha(\tau)d\tau.$$

Exemple 1. *Let* A *be a bounded linear operasor on* X*. Define* $C^\alpha(r) = \frac{e^{2r\alpha}+e^{-2r\alpha}}{2}$*. Then* $T(r)_r \geq 0$ *is a* $\frac{1}{2}$ *semigroup. Indeed:*
1. $C^\alpha(0) = 1$.
3. The conrinuiry is clear.

Proposition 2. *The family* $\{C^\alpha(r), r \in \mathbb{R}\}$ *is a srongly conformable cosine family if only if* $\left\{C(r) = C^\alpha(r)\left(r^{\frac{1}{\alpha}}\right), t \in \mathbb{R}\right\}$ *is a srongly conrinuous conformable cosine family.*

Proof. 1. It is clear that $C(0) = I$.
2. For all $s, t \in \mathbb{R}$, we have

$$\begin{aligned} C(s+s) + C(s-s) &= C^\alpha(t+s)\left(s^{\frac{1}{\alpha}}\right) + C^\alpha(t-s)\left(s^{\frac{1}{\alpha}}\right) \\ &= 2C^\alpha(t)\left(s^{\frac{1}{\alpha}}\right)C^\alpha(s)\left(s^{\frac{1}{\alpha}}\right) \\ &= 2C(s)C(s) \end{aligned}$$

3.Further the continuity of $r \to C_\epsilon^\alpha\left(r^{\frac{1}{\alpha}}\right)y$ and the continuity of $r \to r^\alpha$ implies that $r \to C(r)y$ is continuous.

It is sufficient to mention that for the necessary requirement $C^\alpha = C^\alpha(r)$, if $\{C^\alpha(r), r \in \mathbb{R}\}$ is a strongly continuous conformable cosine family in $\bar{X}$, then $\{S^\alpha(r), r \in \mathbb{R}\}$ is the one parameter family of operators in $\bar{X}$ defined by

$$S^\alpha(r)y = (IC^\alpha)(r)y, \quad \forall r \in \mathbb{R}, y \in X.$$

$\square$

Remark 3. *As the previous proposision* $\{S^\alpha(r), r \in \mathbb{R}\}$ *is a conformable sine family iff* $\left\{S(r) = S^\alpha\left(r^{\frac{1}{\alpha}}\right), r \in \mathbb{R}\right\}$ *is conformable sine family.*

Proposition 3. *Let* $\{C^\alpha(r), r \in \mathbb{R}\}$ *be a strongly continuous conformable cosine family in* $\bar{X}$*. The following statements are correct:*
1. $C^\alpha(r) = C^\alpha(-r) \quad \forall r \in \mathbb{R}$

2. $C^{\alpha}(r), S^{\alpha}(r), C^{\alpha}(s)$, *and* $S^{\alpha}(s)$ *commute for all* $r, s \in \mathbb{R}$
3. $S^{\alpha}(r)y$ *is continuous in* r *on* $\mathbb{R}$ *for each fixed* $y \in X$
4. $S^{\alpha}(r+s)+S^{\alpha}(r-s)=2S^{\alpha}(r)C^{\alpha}(s)$ *for all* $r, s \in \mathbb{R}$
5. $S^{\alpha}(r+s)=S^{\alpha}(r)C^{\alpha}(s)+S^{\alpha}(s)C^{\alpha}(r)$ *for all* $r, s \in \mathbb{R}$
6.$S^{\alpha}(t)=-S^{\alpha}(-t)$ *for all* $t \in \mathbb{R}$
7. There exist constant $M>1$ *and* $\omega \geq 0$ *such that* $C^{\alpha}(r) \leq M e^{\omega^{\alpha}}$ *for all* $r \in \mathbb{R}$ *and*

$$\left\|S^{\alpha}\left(r_1\right)-S^{\alpha}\left(r_2\right)\right\| \leq \frac{M}{\omega}\left(e^{\omega_1^{\alpha}}-e^{\omega_2^{\alpha}}\right)$$

Proof. The proposition 1-6 are consequence of the proposition 3.
For 7 , we have

$$\begin{aligned}\left\|S^{\alpha}\left(r_1\right)-S^{\alpha}\left(r_2\right)\right\| &= \int_{t_2}^{t_1} \frac{C^{\alpha}(s)}{s^{1-\alpha}} ds \\ &\leq M \int_{t_2}^{l_1} \frac{e^{\omega s^{\alpha}}}{s^{1-\alpha}} ds=\frac{M}{\omega}\left[e^{\omega s^{\alpha}}\right]_{t_2}^{t_1}\end{aligned}$$

□

Definition 12. *The conformable infinitesimal generator of a strongly continuous conformable cosine families* $C^{\alpha}(r), r \in \mathbb{R}$ *is the operator* $A: X \rightarrow X$ *defined by*

$$Ax=\lim_{r \rightarrow 0} D^{(\alpha)} C^{\alpha}(r)$$

$$D(A)=\left\{y, r \rightarrow D^{(\alpha)} C^{\alpha}(r) y, \text{ is continuous in } r\right\}$$

Lemma 6.

$$C(r)=\lim_{\alpha \rightarrow 2^{+}} C^{\alpha}(r) \text{ is a cosine family}$$

Proof. It suffice to note that $C^{\alpha}\left(r^{\frac{1}{\alpha}}\right)$ is a cosine families, $r \rightarrow r^{\frac{1}{\alpha}}$ is continuous. □

Proposition 4. *Let* $C^{\alpha}(r), r \in \mathbb{R}$, *be a srongly continuous conformable cosine family in* $\bar{X}$ *with conformable infinitesimal generator* A. *Then,*
1. $D(A)$ *is dense in* X *and* A *is a closed operasor in* $\bar{X}$.
2. if $x \in \bar{X}$ *and* $r, s \in \mathbb{R}$, *then* $z=\int_r^5 \frac{S^{a}(u)}{u^{1-\alpha}} x du \in D(A)$ *and* $Az=C^{\alpha}(s)x-C^{\alpha}(r)x$
3. if $x \in \bar{X}$, *then* $S^{\alpha}(t)x \in \bar{X}$
4. if $x \in \bar{X}$, *shen* $S^{\alpha}(t)x \in D(A)$ *and* $\left(D^{(\alpha)} C^{\alpha}\right)(t)x=AS^{\alpha}(t)x$
5 if $x \in D(A)$, *then* $C^{\alpha}(t)x \in D(A)$ *and* $D^{(\alpha)} C^{\alpha}(t)x=AC^{\alpha}(t)x=C^{\alpha}(t)Ax$
6. if $x \in D(A)$, *then* $\lim_{x \rightarrow 0} AS^{\alpha}(x)x=0$
7 if $x \in D(A)$, *then* $S^{\alpha}(t)x \in D(A)$ *and* $D^{(\alpha)} S^{\alpha}(t)x=AS^{\alpha}(t)x$
8. if $x \in D(A)$, *then* $S^{\alpha}(t)x \in D(A)$ *and* $AS^{\alpha}(t)x=S^{\alpha}(t)Ax$
9. $C^{\alpha}(r+s)-C^{\alpha}(s-s)=2AS^{\alpha}(t)S^{\alpha}(s)$ *for all* $s, t \in \mathbb{R}$.

Proof. For 1 it just to use the previous definition 17 and proposition 3.
For $2-9$ By change s by $s^{\frac{1}{\alpha}}$ and t by $t^{\frac{1}{\alpha}}$ and use proposition 2.2 in [23]. □

5. Generalized solution of the fractional wave equation

we consider the following problem :

$$\left\{\begin{array}{l} D^{(\alpha)} f(t, y)+A f(t, y)=F(t, f(t, y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f(0, y)=u_0(y), \quad D^{(\alpha)} f(0, y)=v_0(y) \end{array}\right. \tag{7}$$

with $A = -c^2 \frac{d^2}{dx^2}$, $u_0(y), v_0(y) \in \mathcal{G}(\mathbb{R}^n)$.

Now we will transform the problem (7) in the Colombeau algebra using the first section.

$$\begin{cases} D^{(\alpha)} f_\epsilon(t,y) + A_\epsilon f_\epsilon(t,y) = F_\epsilon(t, f_\epsilon(t,y)) \quad y \in \mathbb{R}, \quad t \geq 0 \\ f_\epsilon(0,y) = u_{0,\epsilon}(x), \quad D^{(\alpha)} f_\epsilon(0,y) = v_{0,\epsilon}(y) \end{cases} \tag{8}$$

With $A = -c^2 \frac{d^2}{dx^2}$ and $1 < \alpha < 2$, $u_{0,\epsilon}(y), v_{0,\epsilon}(y)$ are regularized of $a_0(x)$ and $b_0(x)$ respectively and by definition 18 $A = [(A_\varepsilon)]$ is the infinitesimal generator of generalized conformable cosine family $C = [(C_\epsilon^\alpha)_\epsilon]$.

The folowing definition is the definition of mild solution.

Definition 13. *A funcrion $f_\epsilon \quad : [0,\infty) \to X$ is a mild soluion of (8) if*
1. f_ϵ is continuous differential on $[0,\infty)$.
2. f_ϵ is continuously α-differeniable on $(0,\infty)$.
3. $f_\epsilon(r) \in D(A)$ for $r > 0$.
4. $f_\epsilon(s) = C_\epsilon^\alpha(s) u_{0,\epsilon} + S_\epsilon^\alpha(s) v_{1,\epsilon} + \int_0^t \frac{s^\alpha (t-s) F(s, f_\epsilon(s))}{s^{2-\alpha}} ds$.

Definition 14. *An element $F \in \mathcal{G}[\mathbb{R}^n]$ is L^∞ logarithmic type if it has a representative $(F_\epsilon)_\epsilon \in \mathcal{E}_M[\mathbb{R}^n]$ such that*

$$\|F_\epsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\log(\epsilon)) \quad \text{as} \quad \epsilon \to 0$$

Theorem 2. *Let ∇F_ϵ is L^∞ log-type and the conformable generalized sine family $S_\epsilon = [(S_\epsilon^\alpha)_\epsilon]$ is the associated of the conformable generalized cosine family $C = [(C_\epsilon^\alpha)_\epsilon]$ verify the properties of the previous section. Then the problem (8) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.*

Proof. **Existence.**

The integral solution of the problem 8 is:

$$f_\epsilon(t,y) = C_\epsilon^\alpha(t) u_{0,\epsilon}(y) + S_\epsilon^\alpha(t) v_{0,\epsilon}(y) + \int_0^t s^{\alpha-2} S_\epsilon^\alpha(t-s) F_\epsilon(s, f_\epsilon(s)) ds$$

Which implies that:

$$\begin{aligned} \|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \|C_\epsilon^\alpha(t)\| \|u_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)} + \|S_\epsilon^\alpha(t)\| \|v_{0,\epsilon}(x)\|_{L^\infty(\mathbb{R}^n)} \\ & + \int_0^t s^{\alpha-2} \|S_\epsilon^\alpha(t)\| \|F_\epsilon(s, f_\epsilon(s,.))\|_{L^\infty(\mathbb{R}^n)} ds, \end{aligned}$$

Then:

$$\begin{aligned} \|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \sup_{\tau \in [0,T]} \|C_\epsilon^\alpha(\tau)\| \|u_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau \in [0,T]} \|S_\epsilon^\alpha(\tau)\| \|v_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)} \\ & + \sup_{\tau \in [0,T]} \|S_\epsilon^\alpha(\tau)\| \int_0^t s^{\alpha-2} \|F_\epsilon(s, f_\epsilon(s,.))\|_{L^\infty(\mathbb{R}^n)} ds. \end{aligned}$$

The first approximation of F_ϵ is

$$F_\epsilon(s, f_\epsilon(s,.)) = F_\epsilon(s,0) + \nabla F_\epsilon f_\epsilon(s,.) + N_\epsilon(s)$$

with $N_\epsilon \in \mathcal{N}(\mathbb{R}^+)$
Then

$$\begin{aligned}\|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \|u_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|v_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha\| \int_0^t s^{\alpha-2}\|F_\epsilon(s,0)\| ds \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2}\|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \int_0^t s^{\alpha-2} N_\epsilon(s) ds\end{aligned}$$

We get

$$\begin{aligned}&\|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \|u_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha\| \|v_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|F_\epsilon(\tau,0)\| \\ &+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2}\|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds \\ &+ \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(\tau)\|\end{aligned}$$

So,

$$\begin{aligned}\|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \sup_{\tau\in[0,T]} \|C_\epsilon(\tau)\| \|u_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|v_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ & + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|F_\epsilon(\tau,0)\| \\ & + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(\tau)\| \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \; \|\nabla F_\epsilon\| \int_0^t s^{\alpha-2}\|f_\epsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds.\end{aligned}$$

By the Granwall's inequality:

$$\begin{aligned}\|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \Bigg(\sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \|u_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ & + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|v_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \\ & + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|F_\epsilon(\tau,0)\| \\ & + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon\| \Bigg) \\ & \times \exp\left(\frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\| \right).\end{aligned}$$

Since $C^{\alpha}_{\epsilon} \in G(\mathbb{R}^{+}, \mathcal{C}(X)), S^{\alpha}_{\epsilon} \in G([0,+\infty[, \mathcal{C}(X)), u_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n), v_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n) \quad (N_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^{+})$ and ∇F_ϵ is $L^\infty-$ logtype there exist $M \in \mathbb{N}$ such that
$\sup_{t\in[0,T]} \|f_\epsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}\left(\epsilon^{-M}\right), \quad \epsilon \to 0$
Then

$$f_\epsilon \in \mathcal{G}([0,+\infty), \mathbb{R}^n).$$

Uniqueness.

Let's say there are two solutions $f_{1,\epsilon}(t,.)$, $f_{2,\epsilon}(t,.)$ to problem (8), consequently :

$$\begin{cases} D^{(\alpha)} f_{1,\epsilon}(t,y) - A_\epsilon f_{1,\epsilon}(t,y) - D^{(\alpha)} f_{2,\epsilon} & (t,y) + A_\epsilon f_{2,\epsilon}(t,y) \\ & = F_\epsilon\left(t, f_{1,\epsilon}(t,y)\right) - F_\epsilon\left(t, f_{2,\epsilon}(t,y)\right) \\ & y \in \mathbb{R}^n, \quad t \geq 0 \\ f_{1,\epsilon}(0,y) - f_{2,\epsilon}(0,y) = N_{0,\epsilon}(y) & \\ D^{(\alpha)} f_{1,\epsilon}(0,y) - D^{(\alpha)} f_{2,\epsilon}(0,y) = \bar{N}_{0,\epsilon}(y) & \end{cases} \tag{9}$$

Then:

$$\begin{cases} D^{(\alpha)}\left(f_{1,\epsilon}(t,y) - f_{2,\epsilon}(t,y)\right) - A_\epsilon\left(f_{1,\epsilon}(t,y) + f_{2,\epsilon}(t,y)\right) = F_\epsilon\left(t, f_{1,\epsilon}(t,y)\right) \\ \qquad - F_\epsilon\left(t, f_{2,\epsilon}(t,y)\right) \\ \qquad y \in \mathbb{R}^n, \quad t \geq 0 \\ f_{1,\epsilon}(0,y) - f_{2,\epsilon}(0,y) = N_{0,\epsilon}(y) \\ D^{(\alpha)} f_{1,\epsilon}(0,y) - D^{(\alpha)} f_{2,\epsilon}(0,y) = \bar{N}_{0,\epsilon}(y) \end{cases} \tag{10}$$

With $(N_{0,\epsilon})_\epsilon, \left(\bar{N}_{0,\epsilon}\right)_\epsilon \in \mathcal{N}(\mathbb{R}^{+})$.
The integral solution of the equation (10) is:

$$\begin{aligned} f_{1,\epsilon}(t,y) - f_{2,\epsilon}(t,y) &= C^{\alpha}_{\epsilon}(t) N_{0,\epsilon}(y) + S^{\alpha}_{\epsilon}(t) \bar{N}_{0,\epsilon}(y) \\ &+ \int_0^t s^{\alpha-2} S^{\alpha}_{\epsilon}(t)\left(F_\epsilon\left(s, f_{1,\epsilon}(s,x)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,x)\right)\right) ds \end{aligned}$$

Then:
$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \|C^{\alpha}_{\epsilon}(t)\| \, \|N_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$
$+ \|S^{\alpha}_{\epsilon}(t)\| \left\|\tilde{N}_{0,\epsilon}(\cdot)\right\|_{L^\infty(\mathbb{R}^n)}$
$+ \int_0^t s^{\alpha-2} \|S^{\alpha}_{\epsilon}(t)\| \, \|F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right)\|_{L^\infty(\mathbb{R}^n)} ds.$
Which implies that:

$$\begin{aligned} \|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} &\leq \sup_{\tau\in[0,T]} \|C^{\alpha}_{\epsilon}(\tau)\| \, \|N_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{\tau\in[0,T]} \|S^{\alpha}_{\epsilon}(\tau)\| \left\|\tilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} \\ &+ \sup_{\tau\in[0,T]} \|S^{\alpha}_{\epsilon}(\tau)\| \\ &\int_0^t s^{\alpha-2} \|F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right)\|_{L^\infty} ds. \end{aligned}$$

The initial estimate of $F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right)$ is provided by

$$F_\epsilon\left(s, f_{1,\epsilon}(s,.)\right) - F_\epsilon\left(s, f_{2,\epsilon}(s,.)\right) = \|\nabla F_\epsilon\| \left(f_{1,\epsilon}(s,.) \; - f_{2,\epsilon}(s,.)\right) + N_\epsilon(s),$$

With $(N_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^+)$.
So

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \, \|N_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \left\|\tilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\|$$
$$\int_0^t s^{\alpha-1} \|\nabla F_\epsilon\| \|f_{1,\epsilon}(s,.) - f_{2,\epsilon}(s,.)\|_{L^\infty(\mathbb{R}^n)} ds$$
$$+ \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \, \|N_\epsilon(s)\|$$

So,

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \, \|N_{0,\epsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \left\|\widetilde{N}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} + \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(s)\|$$
$$+ \sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\|$$
$$\int_0^t s^{\alpha-2} \|\nabla F_\epsilon\| \, \|f_{1,\epsilon}(s,.) - f_{2,\epsilon}(s,.)\|_{L^\infty(\mathbb{R}^n)} ds$$

Using the Granwall's inequality:

$$\|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq (\sup_{\tau\in[0,T]} \|C_\epsilon^\alpha(\tau)\| \|N_{0,\epsilon}(.)\|_{L^\infty} + \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha\| \|\widetilde{N}_{0,\varepsilon}(.)\|_{L^\infty}$$
$$+ \frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \sup_{\tau\in[0,T]} \|N_\epsilon(s)\|)$$

$$\times \exp\left(\frac{T^{\alpha-1}}{\alpha-1} \sup_{\tau\in[0,T]} \|S_\epsilon^\alpha(\tau)\| \|\nabla F_\epsilon\|\right).$$

Since:
$C_\epsilon^\alpha \in G(\mathbb{R}^+, \mathcal{L}(X)), S_\epsilon^\alpha \in G(\mathbb{R}^+, \mathcal{L}(X)), (N_{0,\epsilon})_\epsilon, \quad (\widetilde{N}_{0,\epsilon})_\epsilon \in \mathcal{N}(\mathbb{R}^+)(N_\epsilon)_\epsilon$
$\in \mathcal{N}(\mathbb{R}^+)$ and ∇F is L^∞ - logtype and for every $q \in \mathbb{N}$ such that:

$$\sup_{t\in[0,T]} \|f_{1,\epsilon}(t,.) - f_{2,\epsilon}(t,.)\|_{L^\infty} = \mathcal{O}(\epsilon^q) \quad \epsilon \to 0$$

□

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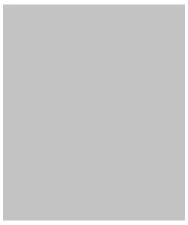

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