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EDGE INCIDENT 2-EDGE COLORING SUM OF GRAPHS

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Abstract. The edge incident 2-edge coloring number, $\psi'_{ein2}(G)$, of a graph G is the highest coloring number used in an edge coloring of a graph G such that the edges incident to an edge $e = uv$ in G is colored with at most two distinct colors. The edge incident 2-edge coloring sum of a graph G, denoted as $\sum\limits_{ein2'}(G)$, is the greatest sum among all the edge incident 2-edge coloring of graph G which receives maximum $\psi'_{ein2}(G)$ colors. The main objective of this paper is to study the edge incident 2-edge coloring sum of graphs and find the exact values of this parameter for some known graphs.

Keywords: Edge incident 2-edge coloring, edge incident 2-edge coloring number, edge incident 2-edge coloring sum.

AMS Subject Classification: 05C15

1. Introduction

An edge coloring is an assignment of colors to the edges of a graph G such that no two adjacent edges of G receive the same color. The minimum number of colors required in a proper edge coloring of a graph G is called the chromatic index of G and is denoted by $\chi'(G)$. Ewa Kubicka and Allen J Schwenk in [12] introduced the notion of chromatic sum of a graph G, denoted as $\sum(G)$, and is defined as the smallest possible sum of colors among all possible proper vertex coloring of a graph G with natural numbers. A few research articles on this topic can be seen in [5, 8]. There is another graph invariant called the minimum edge-chromatic sum (MECS) as defined in [7]. An edge coloring of a graph $G = (V, E)$ is a mapping $\phi : E \longrightarrow \mathbb{N}$ such that no two adjacent edges of G receive the same color. MECS is the smallest possible sum of colors among all possible proper edge coloring of a graph G with natural numbers of G.

Recently, a lot of studies have been made towards the maximization of the coloring numbers under certain constraints. The study on the 3-consecutive vertex coloring number was a concept introduced by E. Sampathkumar in [14] to find the maximum number of

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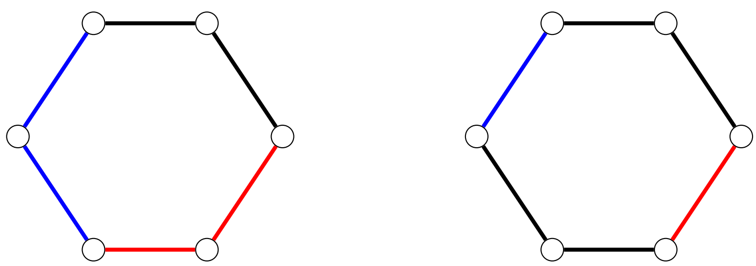
colors in a vertex coloring of a graph G. Later, the edge analog to the 3-consecutive vertex coloring of a graph was studied in [2]. These coloring concepts have their applications in the network sciences and strong signed graph structures [15]. The 3-sequent achromatic number of a graph G as mentioned in [4], $\psi_{3s}(G)$, is the maximum number of colors that can be used in a vertex coloring of G such that if xy and yz are any two sequent edges in G, then either the vertex x or the vertex z is assigned with the same color as given to vertex y. C. Dominic and J.V. Devassia defined the concept of 3-sequent achromatic sum of graphs, $\sum\limits_{3s}(G)$, as the greatest sum of colors among all proper 3s-coloring that requires 3−sequent achromatic number of a graph G. Later, A. Joseph and C. Dominic in [11], introduced the vertex induced 2−edge coloring sum and vertex incident 2−edge coloring sum of graphs. The findings in this paper have been inspired by the concepts studied in [4, 11, 10]. We are mainly interested in the study of the maximum sum of colors among all the edge incident 2-edge coloring of a graph G having the highest number of colors.

Let $V(G)$ be the finite vertex set, and $E(G)$ be the finite edge set of a simple connected graph $G = (V, E)$. Two vertices $u, v \in V(G)$ are said to be adjacent if there is an edge between them. This implies that the two vertices u and v in a graph G are incident with an edge $e = uv$. Two edges are said to be adjacent or incident if there is a common vertex between them.

An edge coloring $\psi : E \longrightarrow \mathbb{N}$ of a graph G is said to be an edge incident 2-edge coloring (or $ein2$−edge coloring) if for every adjacent vertex u and v in $V(G)$, all the edges incident to the vertices u and v cannot receive more than two distinct colors. The edge incident 2-edge coloring number of a graph G denoted as $\psi'_{ein2}(G)$, is the maximum number of colors permitted in such a coloring. The edge incident 2-edge coloring sum (or $ein2$−edge coloring sum) of G, $\sum\limits_{ein2'}(G)$, is the maximum sum attained among all the edge incident 2-edge coloring of G which receives the maximum ψ'_{ein2} colors.

Consider the edge incident 2-edge coloring of a graph G as shown in figure 1. At most, three colors, namely 1 (blue), 2 (red), and 3 (black), are required to color the edges of the graph C_6. In figure 1.(a), $\psi'_{ein2}(C_6) = 3$ and sum $= 12$. In figure 1.(b), $\psi'_{ein2}(C_6) = 3$ and sum $= 15$. Therefore, $\sum\limits_{ein2'}(C_6) = 15$.



(A) $\psi'_{ein2}(C_6) = 3$; sum = 12

(B) $\psi'_{ein2}(C_6) = 3$; sum = 15

FIGURE 1. $ein2$−edge coloring sum of graph C_6
■ Color 1 ■ Color 2 ■ Color 3

We use the following definitions and notations for the further development of this paper.

- A (n, m)−graph is a graph G with order n and size m.

- The distance between two vertices in a graph is the number of edges in the shortest or minimal path. It gives the available minimum distance between two edges. There can exist more than one shortest path between two vertices.
- The diameter of a graph G denoted as $diam(G)$ or simply $d(G)$, is the maximum distance between the pair of vertices in G.

Throughout this paper, we deal with simple and connected graphs G of order n and size m unless otherwise mentioned. For more definitions of graph theory, refer to [9].

Theorem 1.1. *Let G be a simple connected graph with order n and size m. If $d(G) \geq 3$, then $\sum_{ein2'}(G) < \frac{m(m+1)}{2}$, where $d(G)$ denotes the diameter of the graph G.*

Proof. Let $d(G)$ be the diameter of a simple connected graph G with order n and size m. If $d(G) \geq 3$, then clearly the size m of G is strictly greater than 2. From theorem 2.3 in [10] it can be observed that if $d(G) \geq 3$ then $\psi'_{ein2}(G) < m$. This implies m distinct edges in G cannot be given m distinct colors. Thus, $\sum_{ein2'}(G) < \frac{m(m+1)}{2}$. □

This upper bound is sharp and is not attained for any G with size $m \geq 3$.

Theorem 1.2. *For a connected graph G of order $n \geq 2$ and size $1 \leq m < 3$, $\sum_{ein2'}(G) = \frac{m(m+1)}{2}$ if and only if G is either K_2 or P_3.*

The proof of the above theorem is evident and is omitted for the reader.

Definition 1.1. *An independent edge set M of a graph G is a subset of the edges set $E(G)$ such that no two edges in the subset M share a common vertex of G. A maximum independent edge set is an independent edge set containing the largest possible number of edges among all independent edge sets for a given graph. The size of a maximum independent edge set is known as the matching number or the edge independence number denoted as $\nu(G)$* [1].

Theorem 1.3. *Let G be a simple connected $(n,m)-$graph and let $\nu(G)$ denote the edge independent number of the graph G. Then, $\sum_{ein2'}(G) \leq \frac{(\nu(G)+1)(2m-\nu(G))}{2}$.*

Proof. Let $S = \{e_1, e_2, \dots, e_k\}$, where $1 \leq k < m$, be the largest set of independent edges in a graph G. It is evident that $|S| = \nu(G)$. From theorem 2.10 in [10] it can be seen that for a simple connected graph G, $\psi'_{ein2}(G) \leq \nu(G) + 1$. As discussed in theorem 2.10 of [10], there exists at least one edge $e \in G$ and $e \notin S$ such that the edge e is incident to two edges in S, say e_1 and e_2. Thus, at most three colors are required to color the edges $\{e_1, e_2\}$ and all the edges incident to $\{e_1, e_2\}$. In a similar manner at most one new color can be given to each edge selected from the set S. This implies each edge from the set S can be assigned a color from the color set $\{1, 2, \dots, \nu(G)\}$ and all the remaining uncolored edges in G can be colored with the color $\nu(G) + 1$. Hence, the upper bound is given by,

$$\begin{aligned}\sum_{ein2'}(G) &\leq (m - \nu(G))(\nu(G) + 1) + \frac{\nu(G)(\nu(G) + 1)}{2} \\ &= \frac{(\nu(G) + 1)(2m - \nu(G))}{2}.\end{aligned}$$

□

The equality holds for the complete graph K_n, the star graph $K_{1,n}$, the cycle graph C_3, the path graph P_2, P_3, P_5, and a few other graphs. It remains an open problem to characterize the graphs G for which $\sum\limits_{ein2'}(G) = \frac{(\nu(G)+1)(2m-\nu(G))}{2}$.

2. $ein2-$EDGE COLORING SUM OF CERTAIN GRAPH CLASSES

In this section, we compute the $ein2-$edge coloring sum of the sun graph, closed sun graph, antiprism graph, double wheel graph, friendship graph, generalized friendship graph, and $H-$graph.

Definition 2.1. *Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of a complete graph K_n and $\{v_1v_2, v_2v_3, \ldots, v_nv_1\}$ be the edges of the outer rim in K_n. Then, the sun graph S_n, where $n \geq 3$, is a graph obtained by taking the complete graph K_n and the vertices $U = \{u_1, u_2, \ldots, u_n\}$ corresponding to each vertex of K_n and by adding edges $u_1v_1, u_1v_2, u_2v_2, u_2v_3, \ldots, u_nv_n, u_nv_1$ (see* [3]*).*

Theorem 2.1. *Let $n \geq 3$. Then, the edge incident 2-edge coloring sum of the sun graph S_n is $\sum\limits_{ein2'}(S_n) = n^2 + 3n - 1$, where n is the order of the complete graph K_n in S_n.*

Proof. From [10], it is well known that the $ein2-$edge coloring number of the complete graph K_n is 2. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n and let $\{u_1, u_2, \ldots, u_n\}$ be a copy of $V(K_n)$ such that each u_i corresponds to v_i in the sun graph S_n. Suppose the coloring procedure is initiated by giving two colors to the edges of K_n such that exactly one edge, say edge v_1v_2, is colored with the color 1 and all the remaining other edges of K_n is colored with the color 2. Then, it is impossible to use any new color for the remaining uncolored edges of the sun graph S_n; else, there will exist a path P_4 with three distinct colors. For instance, if the edge u_4v_5 in the graph S_9 is assigned with the color 3, then the edges of the path $v_1v_2v_5u_4$ receive three distinct colors, a contradiction to the definition of $ein2-$edge coloring.

Again, consider if all the edges of the complete graph K_n in S_n are colored with one color, say color 1. Suppose that the edge u_1v_1 and u_5v_5 in the graph S_9 are colored with the color 2 and color 3, respectively. Then, there exists a path $u_1v_1v_5u_5$ with three distinct colors, a contradiction to the definition of $ein2-$edge coloring. Thus, the $ein2-$edge coloring number of the sun graph S_n is 2. This implies the maximum coloring sum is obtained if exactly one edge of S_n is colored with color 1 and the remaining colorless edges are assigned with color 2. Therefore,

$$\begin{aligned}\sum_{ein2'}(S_n) &= 2\left(\frac{n^2+3n}{2} - 1\right) + 1\\ &= n^2 + 3n - 1.\end{aligned}$$

□

Definition 2.2. *A closed sun graph CS_n is the graph obtained from the sun graph S_n by adding the edges $u_1u_2, u_2u_3, \ldots, u_nu_1$ (see* [3]*).*

Theorem 2.2. *Let $n \geq 3$. Then, the edge incident $2-$edge coloring number and $ein2-$edge coloring sum of the closed sun graph CS_n, where n is the order of the complete graph K_n in CS_n, is given by,*

$$\psi'_{ein2}(CS_n) = \begin{cases} \frac{n+3}{3}, & n \equiv 0 \ (mod\ 3) \\ \frac{n+2}{3}, & n \equiv 1 \ (mod\ 3) \\ \frac{n+4}{3}, & n \equiv 2 \ (mod\ 3) \end{cases}$$

and

$$\sum_{ein2'}(CS_n) = \begin{cases} \frac{3n^3+23n^2+42n}{18}, & n \equiv 0 \ (mod\ 3) \\ \frac{3n^3+20n^2+29n+2}{18}, & n \equiv 1 \ (mod\ 3) \\ \frac{3n^3+26n^2+55n-4}{18}, & n \equiv 2 \ (mod\ 3) \end{cases}$$

Proof. Consider CS_n to be the closed sun graph of order $2n$ and size $\frac{n(n+5)}{2}$. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of the complete graph K_n in the closed sun graph CS_n. Let $u_1, u_2, \ldots, u_n$ be the n vertices corresponding to each $v_i; 1 \leq i \leq n$ vertex of K_n. The closed sun graph CS_n has a clique of order n, and the $ein2-$edge coloring number of K_n is 2 (refer to Theorem 3.2 in [10]). Also, $\psi'_{ein2}(S_n) = 2$. Suppose the edges of K_n are assigned two different colors. Then the $ein2-$edge coloring number of the graph CS_n need not be a maximum edge coloring. So, all the edges of the complete graph K_n in the graph CS_n are colored with one color. The variation in the $ein2-$edge coloring number depends on the number of vertices. Thus, the edge coloring sum of the graph CS_n is mentioned below in three different cases.

Case 1: Assume that $n \equiv 0 \ (mod\ 3)$. The $ein2-$edge coloring number of S_n is 2. The edges of the sun subgraph in CS_n graph can be colored in such a way that exactly one edge, say edge u_1v_1, is colored with color 1, and all remaining edges of subgraph S_n in the graph CS_n are colored with color 2. Now, the edges u_iu_{i+1}, which form the outer cycle of the graph CS_n, are assigned color in the following manner. The edges incident to the vertices $\{u_1, v_1, u_2, u_n\}$ are colored with color 2; else, the $ein2-$edge coloring condition fails at the edge receiving the new color. So, the edges u_1u_2, u_2u_3, u_nu_1, and $u_{n-1}u_n$ are colored with color 2. The edge u_3u_4 can be colored with a new color, say color 3. The edges $u_{3n}u_{3n+1}$ in the outer rim of CS_n receive new colors. In this case, there are $\frac{n}{3} - 1$ edges in the outer edge of the closed sun graph CS_n, which can be given $\frac{n}{3} - 1$ distinct colors, whereas the remaining uncolored edges of CS_n are all colored with the color 2. This implies, $\psi'_{ein2}(CS_n) = \frac{n}{3} - 1 + 2 = \frac{n+3}{3}$. In order to get the highest edge coloring sum, as mentioned above, there are $\frac{n(n+5)}{2}$ edges in the graph CS_n. Out of which $\frac{n}{3}$ edges are given one color each from the color set $\{1, 2, \ldots, \frac{n}{3}\}$ and the remaining edges of the graph CS_n are assigned with the $\left(\frac{n+3}{3}\right)^{\text{rd}}$ color. Thus,

$$\begin{aligned} \sum_{ein2'}(CS_n) &= \left(\frac{n(n+5)}{2} - \frac{n}{3}\right)\left(\frac{n+3}{3}\right) + \frac{\left(\frac{n}{3}\right)\left(\frac{n+3}{3}\right)}{2} \\ &= \left(\frac{3n^2+13n}{6}\right)\left(\frac{n+3}{3}\right) + \frac{n^2+3n}{18} \\ &= \frac{3n^3+23n^2+42n}{18}. \end{aligned}$$

Case 2: Assume that $n \equiv 1 \ (mod\ 3)$. As discussed in case 1, $\frac{n-1}{3} - 1$ edges in the outer cycle of the graph CS_n are colored with $\frac{n-1}{3} - 1$ distinct colors. The edges in the subgraph S_n of the graph CS_n are colored in such a way that exactly one edge, say edge u_1v_1, is colored with the $\left(\frac{n-1}{3}\right)^{\text{rd}}$ color whereas the remaining edges are colored with $\left(\frac{n+2}{3}\right)^{\text{rd}}$ color. Therefore, $\psi'_{ein2}(CS_n) = \frac{n-1}{3} - 1 + 2 = \frac{n+2}{3}$. Thus, in this case, the $ein2-$edge coloring sum of CS_n is given by,

$$\begin{aligned}\sum_{ein2'}(CS_n) &= \left(\frac{n(n+5)}{2} - \frac{n-1}{3}\right)\left(\frac{n+2}{3}\right) + \frac{\left(\frac{n-1}{3}\right)\left(\frac{n+2}{3}\right)}{2} \\ &= \left(\frac{3n^2+13n+2}{6}\right)\left(\frac{n+2}{3}\right) + \frac{n^2+n-2}{18} \\ &= \frac{3n^3+20n^2+29n+2}{18}.\end{aligned}$$

Case 3: Assume that $n \equiv 2\ (mod\ 3)$. As discussed in case 1, there are $\frac{n+1}{3} - 1$ edges in the outer rim of the graph CS_n that can be given one color each from the color set $\{1, 2, \dots, \frac{n+1}{3} - 1\}$. The edges in the subgraph S_n of the graph CS_n are colored in such a way that exactly one edge, say edge u_1v_1, is colored with the $\left(\frac{n+1}{3}\right)^{\text{rd}}$ color whereas the remaining edges are colored with $\left(\frac{n+4}{3}\right)^{\text{rd}}$ color. Therefore, $\psi'_{ein2}(CS_n) = \frac{n+1}{3} - 1 + 2 = \frac{n+4}{3}$. Thus,

$$\begin{aligned}\sum_{ein2'}(CS_n) &= \left(\frac{n(n+5)}{2} - \frac{n+1}{3}\right)\left(\frac{n+4}{3}\right) + \frac{\left(\frac{n+1}{3}\right)\left(\frac{n+4}{3}\right)}{2} \\ &= \frac{(3n^2+13n-2)(n+4)}{18} + \frac{n^2+5n+4}{18} \\ &= \frac{3n^3+26n^2+55n-4}{18}.\end{aligned}$$

□

Definition 2.3. *Let $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of two cycles C_n and C'_n respectively. The antiprism graph, denoted by A_n, is obtained by joining the vertices of these two cycles and adding the edges in the form $u_1v_1, u_1v_2, u_2v_2, u_2v_3, \dots, u_nv_n, u_nv_1$ (see* [3]*).*

Theorem 2.3. *Let $n \geq 3$. Then, the edge incident $2-$edge coloring number and $ein2-$edge coloring sum of the antiprism graph A_n, where n is the order of cycle C_n in A_n, is given by,*

$$\psi'_{ein2}(A_n) = \begin{cases} \frac{n+2}{2}, & n \text{ is even} \\ \frac{n+1}{2}, & n \text{ is odd.} \end{cases}$$

and

$$\sum_{ein2'}(A_n) = \begin{cases} \frac{15n^2+30n}{8}, & \text{if } n \text{ is even} \\ \frac{15n^2+16n+1}{8}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The antiprism graph A_n is a graph of order $2n$ and size $4n$. Let C_n and C'_n be the two cycles of A_n with vertices $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ respectively (here C_n is considered as the inner cycle whereas C'_n as the outer cycle). The $ein2-$edge coloring number and $ein2-$edge coloring sum of the graph A_n are discussed in the following two cases.

Case 1: Assume that n is even. Suppose the coloring procedure is initiated by assigning the edges of the inner cycle (or the outer cycle) in A_n with $\frac{n}{2}$ different colors (as $\psi'_{ein2}(C_n) = \lfloor \frac{n}{2} \rfloor$, refer Theorem 1.5 in [10]). Then, it can be observed that all the remaining uncolored edges of the graph A_n cannot be given more than $\frac{n}{2}$ colors. This coloring approach will not give the maximum edge coloring number for A_n. Again, if the coloring procedure of the graph A_n is initiated by assigning a new edge color to every third edge of the inner cycle C_n (or the outer cycle C'_n), whereas, all the other edges in C_n (or C'_n) are colored with the same color say, color 1. (Note that here the edge u_1u_2 is considered as the zeroth edge, thus making u_4u_5 the third edge and so on in the cycle graph. This implies the zeroth edge, third edge, sixth edge, etc., each is given a new color, whereas the first edge, second edge, fourth edge, etc., are all colored with the same color.) Then, the remaining uncolored edges of the graph A_n have to be colored with color 1, or else the $ein2-$edge coloring condition fails. This coloring approach will also not give the highest $ein2-$edge coloring number. Hence, the edges of the inner cycle and outer cycle are all colored with one single color. Every v_iu_i edge of the graph A_n receives a new color, where $i \equiv -1 \ mod \ 2$. Thus, $\frac{n}{2}$ edges of A_n are assigned with $\frac{n}{2}$ different colors, and remaining all the colorless edges of the antiprism graph A_n are colored with the color $\left(\frac{n}{2}+1\right)$. This implies, the $ein2-$edge coloring number of A_n is $\frac{n}{2}+1 = \frac{n+2}{2}$. Therefore, the greatest $ein2-$edge coloring sum of A_n, in this case, is given by,

$$\begin{aligned}\sum_{ein2'}(A_n) &= \left(4n-\frac{n}{2}\right)\left(\frac{n+2}{2}\right)+\frac{\left(\frac{n}{2}\right)\left(\frac{n+2}{2}\right)}{2}\\ &= \frac{15n^2+30n}{8}.\end{aligned}$$

Case 2: Assume that n is odd. As discussed above in case 1, every v_iu_i edge, where $i \equiv -1 \ mod \ 2$, of the graph A_n receives a new color. This implies, $\frac{n-1}{2}$ edges of A_n receives one color each from the color set $\{1, 2, \dots, \frac{n-1}{2}\}$. All the remaining uncolored edges of the antiprism graph A_n are colored with the color $\frac{n-1}{2}+1$. Therefore, $\psi'_{ein2}(A_n) = \frac{n-1}{2}+1 = \frac{n+1}{2}$. This coloring approach gives the maximum sum with the highest ψ'_{ein2} colors. Thus,

$$\begin{aligned}\sum_{ein2'}(A_n) &= \left(4n-\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)+\frac{\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)}{2}\\ &= \frac{15n^2+16n+1}{8}.\end{aligned}$$

□

Definition 2.4. *A double wheel graph DW_n is a graph defined by $2C_n + K_1$. That is, a double wheel graph is a graph obtained by joining all vertices of the two disjoint cycles to an external vertex* [13].

Theorem 2.4. *Let $n \geq 3$. Then, the edge incident $2-$edge coloring number and $ein2-$edge coloring sum of the double wheel graph DW_n, where n is the order of cycle C_n in DW_n, is given by,*

$$\psi'_{ein2}(DW_n) = \begin{cases} \frac{2n+3}{3}, & n \equiv 0 \ (mod \ 3) \\ \frac{2n+1}{3}, & n \equiv 1 \ (mod \ 3) \\ \frac{2n-1}{3}, & n \equiv 2 \ (mod \ 3) \end{cases}$$

and

$$\sum_{ein2'}(DW_n) = \begin{cases} \frac{44n^2+66n}{18}, & n \equiv 0 \ (mod \ 3) \\ \frac{44n^2+26n+2}{18}, & n \equiv 1 \ (mod \ 3) \\ \frac{44n^2-14n-4}{18}, & n \equiv 2 \ (mod \ 3) \end{cases}$$

Proof. Consider DW_n to be the double wheel graph of order $2n+1$ and size $4n$. The double wheel graph DW_n is a graph obtained by joining all the vertices of two disjoint cycles, say C_n and C'_n, to a universal vertex v_0. Suppose if all the edges incident the vertex v_0 is colored with two distinct colors, then the highest $ein2-$edge coloring number used in the graph DW_n is restricted to 2 colors. So, all the edges incident to the universal vertex must be assigned a single color, as we aim to maximize the edge coloring number. Thus, the edge incident $2-$edge coloring number depends on the coloring pattern that is given to the edges in the outer cycle of each wheel subgraph in DW_n. This is discussed below in three cases.

Case 1: Assume that $n \equiv 0 \ (mod\ 3)$. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of two disjoint cycles C_n and C'_n respectively. Every third edge of the outer cycle in a wheel subgraph of the graph DW_n is colored with a new color (here, the edges u_1u_2 and v_1v_2 are considered to be the zeroth edge which makes the edges u_4u_5, v_4v_5 as the third edge and so on in each cycle). That is, a maximum of $\frac{2n}{3}$ colors are required to color every third edge in the cycles C_n and C'_n of DW_n. The remaining uncolored edges in each disjoint cycle of the double wheel graph DW_n and the edges that are incident to the vertex v_0 are colored with the color $\frac{2n}{3}+1$. This implies, $\psi'_{ein2}(DW_n) = \frac{2n}{3}+1 = \frac{2n+3}{3}$. This coloring gives the greatest $ein2-$edge coloring sum of DW_n, hence,

$$\sum_{ein2'}(DW_n) = \left(4n - \frac{2n}{3}\right)\left(\frac{2n+3}{3}\right) + \frac{\left(\frac{2n}{3}\right)\left(\frac{2n+3}{3}\right)}{2}$$
$$= \frac{44n^2+66n}{18}.$$

Case 2: Assume that $n \equiv 1 \ (mod\ 3)$. As discussed in case 1, every third edge is given a new color. So, at most $\frac{2n-2}{3}$ colors are required to color every third edge in both cycles of the graph DW_n. The remaining uncolored edges in each disjoint cycle of DW_n and all the edges that are incident to the vertex v_0 are colored with the color $\frac{2n-2}{3}+1$. This implies, in this case, the $ein2-$edge coloring number of DW_n is $\frac{2n+1}{3}$. The above-mentioned coloring itself gives the greatest coloring sum, thus,

$$\sum_{ein2'}(DW_n) = \left(4n - \frac{2n-2}{3}\right)\left(\frac{2n+1}{3}\right) + \frac{\left(\frac{2n-2}{3}\right)\left(\frac{2n+1}{3}\right)}{2}$$
$$= \frac{44n^2+26n+2}{18}.$$

Case 3: Assume that $n \equiv 2 \ (mod\ 3)$. As discussed in case 1, every third edge is given a new color. So, at most $\frac{2n-4}{3}$ colors are required to color every third edge in both cycles of the graph DW_n. The remaining uncolored edges in each disjoint cycle of DW_n and all the edges that are incident to the vertex v_0 are colored with the color $\frac{2n-4}{3}+1$. Hence, in this case, $\psi'_{ein2}(DW_n) = \frac{2n-4}{3}+1 = \frac{2n-1}{3}$ and the $ein2-$edge coloring sum is given by,

$$\sum_{ein2'}(DW_n) = \left(4n - \frac{2n-4}{3}\right)\left(\frac{2n-1}{3}\right) + \frac{\left(\frac{2n-4}{3}\right)\left(\frac{2n-1}{3}\right)}{2}$$
$$= \frac{44n^2-14n-4}{18}.$$

□

Definition 2.5. *The friendship graph F_n is obtained by taking $n-$copies of the cycle graph C_3 with a common vertex. The generalized friendship graph $F_{n,r}$ is a collection of $n-$copies of the cycle graph C_r of order r, meeting at a common vertex (see* [6]*).*

Theorem 2.5. *Let $n \geq 2$. Then, the edge incident 2-edge coloring sum of the friendship graph F_n is $\sum\limits_{ein2'}(F_n) = \frac{5n(n+1)}{2}$, where F_n is a graph obtained by taking $n-$copies of the cycle C_3, meeting at a common vertex v.*

Proof. The friendship graph F_n is a graph of order $2n+1$ obtained by attaching n triangles to the central vertex v. The $ein2-$edge coloring number of the graph F_n is $n+1$ (see [10]). All the edges incident to the vertex v cannot be colored with two colors as the maximum number of colors that can be used to color the edges will be restricted to 2. So, the edges incident to the universal vertex of F_n is colored using the color $n+1$. The remaining colorless edges of the friendship graph are assigned with one of the colors from the color set $\{1, 2, \dots, n\}$. Therefore,

$$\begin{aligned}\sum_{ein2'}(F_n) &= 2n(n+1) + \frac{n(n+1)}{2} \\ &= \frac{5n(n+1)}{2}.\end{aligned}$$

□

Corollary 2.1. *For the generalized friendship graph $F_{n,r}$, where $n \geq 2$ and $r = 4, 5$,*

(1) $\sum\limits_{ein2'}(F_{n,4}) = \frac{7n(n+1)}{2}$.
(2) $\sum\limits_{ein2'}(F_{n,5}) = \frac{9n(n+1)}{2}$.

The proof of the above result is similar to the theorem 2.5.

Theorem 2.6. *Let $F_{n,r}$ be the generalized friendship graph having $n-$copies of the cycle graph C_r (of order $r \geq 4$), meeting at a common vertex v. Then, for $n \geq 2$ and $r \geq 6$,*

$$\sum_{ein2'}(F_{n,r}) = \begin{cases} \frac{n^2(r^2-4)+2n(r+2)}{4}, & \text{when } r \text{ is even} \\ \frac{n^2(r^2-9)+2n(r+3)}{4}, & \text{when } r \text{ is odd.} \end{cases}$$

Proof. Let $v \in V(F_{n,r})$ be a vertex with maximum degree, that is, $deg(v) = \Delta(F_{n,r})$. The size of the graph $F_{n,r}$ is nr. From [10] it can be observed that the $ein2-$edge coloring number of $F_{n,r}$ is $n\left\lfloor\frac{r-2}{2}\right\rfloor + 1$. The edge coloring sum of the generalized friendship graph is discussed in the following two cases.

Case 1: Assume that r is even. All the edges incident to the universal vertex of the graph $F_{n,r}$ are colored with the same color. In order to get the greatest sum, these $2n$ edges are colored with the color $n\left(\frac{r-2}{2}\right)+1$. It is to be noted that in a generalized friendship graph, two edges in each copy of the cycle C_r are already colored. Since, $\psi'_{ein2}(C_r) = \frac{r}{2}$. So, the colorless edges in $n-$copies of cycle C_r in the graph $F_{n,r}$ will be assigned with one color from the color set $\{1, 2, \ldots, \frac{n(r-2)}{2}\}$ such that each color will appear exactly twice. Thus,

$$\begin{aligned}\sum_{ein2'}(F_{n,r}) &= 2\frac{\left(\frac{n(r-2)}{2}\right)\left(\frac{n(r-2)}{2}+1\right)}{2} + 2n\left(\frac{n(r-2)}{2}+1\right)\\ &= \frac{(nr-2n)(nr+2n+2)+8n}{4}\\ &= \frac{n^2(r^2-4)+2n(r+2)}{4}.\end{aligned}$$

Case 2: Assume that r is odd. As discussed in case 1, all the edges incident to the central vertex v are assigned with the color $\frac{n(r-3)}{2}+1$. Since each cycle C_r of the graph $F_{n,r}$ is odd length. so, the remaining uncolored edges in the $n-$copies of cycle C_r in the graph $F_{n,r}$ are colored with one color from the color set $\{1, 2, \ldots, \frac{n(r-3)}{2}\}$ such that each color will appear exactly twice except the last edge. The last edge in each C_r of $F_{n,r}$ is colored with the color $\frac{n(r-3)}{2}+1$. Thus,

$$\begin{aligned}\sum_{ein2'}(F_{n,r}) &= 2\frac{\left(\frac{n(r-3)}{2}\right)\left(\frac{n(r-3)}{2}+1\right)}{2} + 3n\left(\frac{n(r-3)}{2}+1\right)\\ &= \frac{(nr-3n)^2+2n(r+3)+6n^2(r-3)}{4}\\ &= \frac{n^2(r^2-9)+2n(r+3)}{4}.\end{aligned}$$

□

Definition 2.6. *The H-graph $H(r)$, $r \geq 2$, is the $3-$regular graph of order $6r$, with vertex set $V(H(r)) = \{u_i, v_i, w_i : 0 \leq i \leq 2r-1\}$ and edge set (subscripts are taken modulo $2r$) $E(H(r)) = \{(u_i, u_{i+1}), (w_i, w_{i+1}),$ $(u_i, v_i), (v_i, w_i) : 0 \leq i \leq 2r-1\} \cup \{(v_{2i}, v_{2i+1}) : 0 \leq i \leq r-1\}$ (see* [16]*).*

Theorem 2.7. *Let $H(r)$, $r \geq 2$ be a H-graph. Then,*

$$\sum_{ein2'}(H(r)) = \begin{cases} \frac{100r^2+75r}{9}, & r \equiv 0 \ (mod\ 3)\\ \frac{100r^2+121r-5}{9}, & r \equiv 1 \ (mod\ 3)\\ \frac{100r^2+98r-2}{9}, & r \equiv 2 \ (mod\ 3).\end{cases}$$

Proof. Let $V(H(r)) = \{u_i, v_i, w_i : 0 \leq i \leq 2r-1\}$ be the vertex set of the H-graph $H(r), r \geq 2$. Clearly, $H(r)$ is a $3-$regular graph with the order $6r$ and size $9r$. It can be observed that a new color is assigned to every third edge (we consider the edge u_0u_1 as the zeroth edge whereas the edges u_3u_4 and w_0w_1 as the third edge from the edge u_0u_1). That is, if the edge u_0u_1 is colored with the color 1, then a new color, say color 2 and color 3, can be assigned to the edges u_3u_4 and w_0w_1 respectively, whereas the edges $u_1u_2, u_2u_3, u_0v_0, u_1v_1, v_0v_1, v_0w_0$, and v_1w_1, etc. are colored with the color $\psi'_{ein2}H(r)$ (refer theorem 4.11 in [10] for more clarity). This implies the variation of $ein2-$edge coloring of the H-graph depends on the number of vertices. Hence, we have the following three cases.

Case 1: When $r \equiv 0 \ (mod\ 3)$, $\psi'_{ein2}(H(r)) = \frac{4r}{3}+1 = \frac{4r+3}{3}$ (see [10]). Assume that the edge u_0u_1 of the graph $H(r)$ is colored with color 1. Then, as discussed above, every third edge receives a new color from the color set $\{2, 3, \dots, \frac{4r}{3}\}$. There are exactly $\frac{4r}{3}$ edges in the graph $H(r)$ that are colored with $\frac{4r}{3}$ distinct colors. The remaining uncolored edges are all colored with the color $\frac{4r+3}{3}$ to get the highest edge coloring sum. Thus,

$$\begin{aligned}\sum_{ein2'}(H(r)) &= \frac{\left(\frac{4r}{3}\right)\left(\frac{4r+3}{3}\right)}{2} + \left(9r - \frac{4r}{3}\right)\left(\frac{4r+3}{3}\right)\\ &= \frac{16r^2+12r}{18} + \frac{23r(4r+3)}{9}\\ &= \frac{100r^2+75r}{9}.\end{aligned}$$

Case 2: When $r \equiv 1 \ (mod\ 3)$, $\psi'_{ein2}(H(r)) = \frac{4r+2}{3} + 1 = \frac{4r+5}{3}$ (see [10]). As discussed earlier, if the edge u_0u_1 of the graph $H(r)$ is colored with color 1, then every third edge receives a new color from the color set $\{2, 3, \dots, \frac{4r+2}{3}\}$. Thus, $\frac{4r+2}{3}$ edges of the graph $H(r)$ receives $\frac{4r+2}{3}$ different colors. The remaining uncolored edges of the graph $H(r)$ are colored with the color $\frac{4r+5}{3}$. Hence, the greatest sum is given by,

$$\begin{aligned}\sum_{ein2'}(H(r)) &= \frac{\left(\frac{4r+2}{3}\right)\left(\frac{4r+5}{3}\right)}{2} + \left(9r - \frac{4r+2}{3}\right)\left(\frac{4r+5}{3}\right)\\ &= \frac{16r^2+28r+10}{18} + \frac{(23r-2)(4r+5)}{9}\\ &= \frac{100r^2+121r-5}{9}.\end{aligned}$$

Case 3: When $r \equiv 2 \ (mod\ 3)$, $\psi'_{ein2}(H(r)) = \frac{4r+1}{3} + 1 = \frac{4r+4}{3}$ (see [10]). As discussed above, $\frac{4r+1}{3}$ edges of the graph $H(r)$ are colored with one color each from the color set $\{1, 2, \dots, \frac{4r+1}{3}\}$. The remaining uncolored edges of the graph $H(r)$ are colored with the color $\frac{4r+4}{3}$. Thus,

$$\begin{aligned}\sum_{ein2'}(H(r)) &= \frac{\left(\frac{4r+1}{3}\right)\left(\frac{4r+4}{3}\right)}{2} + \left(9r - \frac{4r+1}{3}\right)\left(\frac{4r+4}{3}\right)\\ &= \frac{16r^2+20r+4+(46r-2)(4r+4)}{18}\\ &= \frac{100r^2+98r-2}{9}.\end{aligned}$$

□

Conclusion and Further scopes

In this paper, the concept of edge incident 2−edge coloring sum has been introduced. In section 2, we found the edge incident 2−edge coloring sum of the sun graph, closed sun graph, antiprism graph, double wheel graph, friendship graph, generalized friendship graph, and the $H-$graph. This study can be further extended to find the edge incident 2−edge coloring sum of some graph products and cubic graphs.

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